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# CONVEXITY-PRESERVING FLOWS OF TOTALLY COMPETITIVE PLANAR LOTKA–VOLTERRA EQUATIONS AND THE GEOMETRY OF THE CARRYING SIMPLEX

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*Abstract* We show that the flow generated by the totally competitive planar Lotka–Volterra equations deforms the line connecting the two axial equilibria into convex or concave curves, and that these curves remain convex or concave for all subsequent time. We apply the observation to provide an alternative proof to that given by Tineo in 2001 that the carrying simplex, the globally attracting invariant manifold that joins the axial equilibria, is either convex, concave or a straight-line segment.

*Keywords:* Lotka–Volterra equations; curvature-preserving flow; competitive dynamics; carrying simplex

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Secondary 34C12;35B41; 34C37

## 1. Introduction

Consider the planar autonomous Lotka–Volterra equations with both intra- and inter-specific competition written in the following form:

$$\dot{x} = f(x, y) = px(1 - x - ay), \qquad \dot{y} = g(x, y) = qy(1 - y - bx), \tag{1.1}$$

where $p, q, a, b > 0$. Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} \colon x > 0\}$ and $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \colon x \geqslant 0\}$. Then the flow of (1.1) leaves $\mathbb{R}^2_{\geqslant 0}$ invariant. The system (1.1) is also dissipative, as all orbits eventually enter the interior of the compact set $B = [0, 2]^2$, and thus the orbits of (1.1) are globally defined in time.

It is well known that (1.1) has a locally Lipschitz invariant manifold $\mathcal{S}_2$ that connects the so-called axial equilibria $(0, 1)$ and $(1, 0)$, and that this manifold attracts all points in $\mathbb{R}^2_{\geqslant 0} \setminus \{0\}$. This manifold is known as the *carrying simplex*, and its existence can be traced back to de Mottoni and Schiaffino [**2**] and Hirsch [**4**]. It is also known [**5**] that the carrying simplex of a planar strongly competitive system, that is

$$\dot{x} = f(x, y) = xF(x, y), \qquad \dot{y} = g(x, y) = yG(x, y) \tag{1.2}$$

 

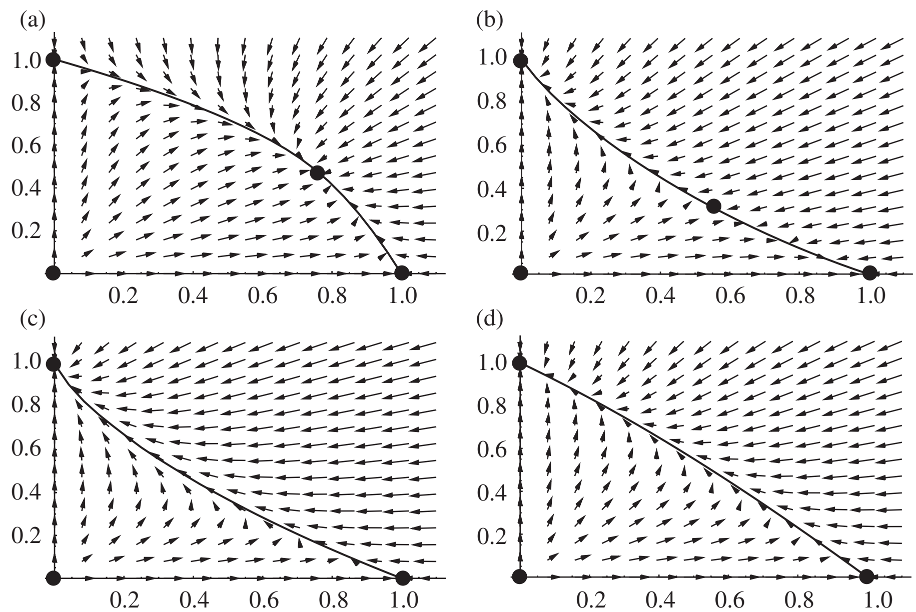


Figure 1. Examples of the carrying simplex (solid line) for four parameter sets for the flow generated by (1.1): (a) $a = 0.5$, $b = 0.7$ (concave), (b) $a = 1.5$, $b = 1.2$ (convex), (c) $a = 1.9$, $b = 0.8$ (convex), (d) $a = 1.1$, $b = 0.7$ (concave); $p = 1$, $q = 1$ in all cases. Steady states are denoted by the solid dots. Note that the vector field has been rescaled in order to show the flow directions more clearly.

with $F_x, F_y, G_x, G_y < 0$ and $F(0,0) > 0, G(0,0) > 0$ (and normalized so that the axial equilibria are at $(1,0), (0,1)$), is the graph of a decreasing function $u\colon [0,1] \to \mathbb{R}$ that is at least $C^1([0,1])$, and in [**7**] conditions are found for $u$ to be $C^2([0,1])$. Hence, we know that for (1.1) the carrying simplex $\mathcal{S}_2$ is unique, globally attracts $\mathbb{R}^2_{\geqslant 0} \setminus \{0\}$ and is the graph of a decreasing $C^1([0,1])$ function.

Examples of the carrying simplex are shown in Figure 1, where in each case, for simplicity, $p = q = 1$. Part (a) shows the case $a = 0.5$, $b = 0.7$, which has four equilibria. The carrying simplex for this case, where $\alpha = -0.8$, is concave.* Part (b) shows the case $a = 1.5$, $b = 1.2$, which gives $\alpha = 0.7$ and a convex carrying simplex. In parts (c) and (d), we show examples where there is no interior equilibrium. The carrying simplex in each case was computed as the graph of the converged solution $u_\infty$ of the first-order quasilinear partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = -f\frac{\partial u}{\partial x} + g, \quad (x,t) \in (0,1) \times \mathbb{R}_{>0}, \tag{1.3}$$

with initial data $u(x,0) = 1 - x$. Notice that in all cases in Figure 1 the curvature of the carrying simplex appears to be one-signed. This non-vanishing of the curvature

* Here 'concave' means that for two points on the carrying simplex, the chord joining them lies below the simplex. Similarly, 'convex' means that for two points on the carrying simplex, the chord joining them lies above the simplex. This follows the terminology used in [**7**], but is opposite to that used in [**8**].

has been investigated recently in [**7**, **8**]. For equations (1.1) it was shown [**7**] that, with $\alpha := p(a-1)+q(b-1)$, the carrying simplex of (1.1) is strictly convex if $\alpha > 0$, strictly concave if $\alpha < 0$ and linear if $\alpha = 0$.

In computing the carrying simplex using the PDE (1.3) we were led to the unit simplex $\Sigma_2 = \{(x,y) \in \mathbb{R}^2_{\geqslant 0} \colon x+y=1\}$ as the natural choice of initial curve. The constant $\alpha$ has one interpretation as determining the uniform sign of the normal component of the vector field along $\Sigma_2$, and hence whether the competitive flow moves the unit simplex above or below its initial position. However, we also found that, at least for small times, the curvature of the image of $\Sigma_2$ under the flow has the same sign as $\alpha$ *uniformly along* $\Sigma_2$. More precisely, if $\alpha > 0$, the image of the simplex is strictly convex for small enough time, and if $\alpha < 0$, it is strictly concave for small enough time. If $\alpha = 0$, the carrying simplex $\mathcal{S}_2 = \Sigma_2$. In fact, a similar observation is true for all lines in the plane: due to the quadratic form of the vector field generating the flow, straight lines are rendered strictly convex or strictly concave by the flow, at least for small enough time, or they are invariant.

The aims of this paper are two-fold. The first purpose is to show that the carrying simplex $\mathcal{S}_2$ is the identical limit of upper and lower solution sequences of a first-order quasilinear PDE. That is, the carrying simplex is the common limit of two bounded and monotone sequences of graphs: one that is non-increasing in time and the other that is non-decreasing in time. This provides a variation on the general method of proof of the existence of carrying simplices (where they are graphs of functions) found in [**2**, **4**], where it is shown, by considering individual orbits, that the carrying simplex $\mathcal{S}_n$ is the upper boundary of the basin of repulsion of the origin in $\mathbb{R}^n_{\geqslant 0}$. By comparison, the proof given here for $\mathcal{S}_2$ follows a line of orbits, and an advantage of this approach is that the curvature of the image of the line under the flow can be tracked. In other words, we solve a time-dependent PDE for the carrying simplex. A similar method establishes concavity of solutions of a non-homogeneous Burger equation occurring in the study of Fréedericksz transitions in liquid crystals [**1**, Remark 1].

The second purpose of the paper is to show that the convexity or concavity of $\mathcal{S}_2$ can be established by squeezing a sequence of convex or concave graphs that are the image of the unit simplex under the flow between these upper and lower solution sequences. The methods used here are of interest since the majority of them can be applied to higher-dimensional competitive Lotka–Volterra systems. We shall deal with the higher-dimensional case in a separate paper.

## 2. Preliminaries

Let $f, g \colon \mathbb{R}^2 \to \mathbb{R}^2$ be smooth functions. The ordinary differential equations

$$\dot{x} = f(x,y) = xF(x,y), \qquad \dot{y} = g(x,y) = yG(x,y) \tag{2.1}$$

give rise to a flow $\varphi_t \colon \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}^2_{\geqslant 0}$. For simplicity, we suppose that the flow exists for all $t \in \mathbb{R}$. The flow of (2.1) leaves $\mathbb{R}^2_{\geqslant 0}$ invariant, and also the two coordinate axes $\{(x,0) \colon x \geqslant 0\}$ and $\{(0,y) \colon y \geqslant 0\}$ are invariant. Let $u_0 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be a given smooth

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function, and denote by $\Sigma = \{(x, u_0(x)) \colon x \geqslant 0\}$ the graph of $u_0$. Then, under the flow $\varphi_t$ of (2.1), $\Sigma$ is mapped to a new set $\Sigma_t = \varphi_t(\Sigma)$, which is the graph of a smooth function $u(\cdot, t) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ provided that $\Sigma_t$ projects in a one-to-one way onto the non-negative $x$-axis. The function $u(\cdot, t) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ satisfies the first-order quasilinear PDE

$$u_t + f(x, u)u_x = g(x, u), \quad (x, t) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \tag{2.2}$$

with initial data $u(x, 0) = u_0(x)$, $x \in \mathbb{R}_{\geqslant 0}$. If solutions $u(\cdot, t)$ exist for all time (no-shocks form), and converge to a continuous function $u_\infty \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, the graph of $u_\infty$ is an invariant manifold for (2.1). We shall therefore seek the carrying simplex as the steady solution of (2.2).

In what follows we will use $\mathrm{d}/\mathrm{d}t$ to denote the material (Lie, Lagrangian) derivative following a trajectory of (2.1), that is $\mathrm{d}/\mathrm{d}t = \partial/\partial t + f\partial/\partial x$. Thus, $\mathrm{d}u_x/\mathrm{d}t$ measures the rate of change of the gradient of the curve following a material point on the curve as it moves with the flow (2.1). It is worth noting that, since in most cases we will be working with functions (e.g. $u_x, u_{xx}$) that are non-positive or non-negative everywhere, it will not be necessary to identify material points, since typically our aim is to show that the functions cannot change sign anywhere on $\mathbb{R}_{\geqslant 0}$.

The first lemma below shows that non-increasing graphs remain non-increasing under the competitive flow (i.e. the normal to the curve lies in the first quadrant), and the second and third lemmas are useful for determining how the flow component normal to a surface changes as the surface evolves under the flow. Finally, the fourth lemma shows that the graph of a non-increasing function is a Lipschitz manifold.

**Lemma 2.1.** *Suppose $u_0 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is smooth, and $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ are smooth functions of the form given in (2.1). For small enough $t$, the solution $u(\cdot, t) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ of (2.2) satisfying $u(\cdot, 0) = u_0$ exists, is smooth in $x$, and we have*

$$\frac{\mathrm{d}u_x}{\mathrm{d}t} = g_x + (g_y - f_x - f_y u_x)u_x. \tag{2.3}$$

**Proof.** By differentiating (2.2) we obtain $u_{xt} + fu_{xx} + (f_x + f_y u_x)u_x = g_x + g_y u_x$ and hence (2.3) via identifying $\mathrm{d}u_x/\mathrm{d}t = (u_x)_t + f(x, u)(u_x)_x$. □

Now take $(f, g)$ to be a smooth totally competitive vector field, as in (1.2), so that $g_x \leqslant 0$ and $f_y \leqslant 0$. Recall that the flow leaves the first quadrant invariant.

**Corollary 2.2.** *If the smooth initial data $u_0$ satisfies $u_0 \geqslant 0$ and $(u_0)_x \leqslant 0$, then $u(\cdot, t)$ is defined and smooth for all $t \geqslant 0$ and $u(\cdot, t) \geqslant 0, u_x(\cdot, t) \leqslant 0$ for all $t \geqslant 0$.*

**Proof.** If $u_x(s, t) = 0$ for some material point labelled by $s \in \mathbb{R}_{>0}$, then since $g_x \leqslant 0$ we have $\mathrm{d}u_x(s, t)/\mathrm{d}t \leqslant 0$ from (2.3). Moreover, the solution $u(\cdot, t)$ remains defined for all $t \geqslant 0$, since $u_x$ cannot grow unbounded in finite time. To see this, simply note that the coefficient of $u_x^2$ in (2.3) is $-f_y = -xF_y > 0$ if $x > 0$. □

**Remark 2.3.** For the $f$, $g$ in (1.1), if $u_0 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is decreasing, then $u_x(x, t) < 0$ whenever $x$ is such that $u(x, t) > 0$, since then $g_x = -bqu$.

**Remark 2.4.** We recall that $\mathbb{R}^2$ can be made into an ordered space $(\mathbb{R}^2, \leqslant)$ through the ordering defined by $x \leqslant y$ if and only if $x_i \leqslant y_i$ for $i = 1, 2$, $x < y$ if $x \leqslant y$, and $x \neq y$ and $x \ll y$ if and only if $x_i < y_i$ for $i = 1, 2$. Hirsch uses the fact that, backwards in time, (1.2) is strongly monotone on $\mathbb{R}^2_{>0}$. A flow $\psi_t \colon U \to U$, $U \subset \mathbb{R}^2$ open, is strongly monotone if, whenever $x < y$, $\psi_t(x) \ll \psi_t(y)$ for all $t > 0$. Thus, if $\psi_t = \varphi_{-t}$, where $\varphi_t$ is the flow of (1.2), then $\psi_t$ is a strongly monotone flow on $\mathbb{R}^2_{>0}$.

For convenience, write $\mathcal{F} = (f, g)^{\mathrm{T}}$.

**Lemma 2.5.** *Let $n$ denote the upward unit normal to the graph of $u(\cdot, t)$. Then $\dot{n} = -D\mathcal{F}^{\mathrm{T}} n + (n^{\mathrm{T}} D\mathcal{F} n)n$.*

**Proof.** Pick a point $x$ on the graph of $u$ and $v$ a tangent vector at $x$. Then $n^{\mathrm{T}} v = 0$ so that $0 = \dot{n}^{\mathrm{T}} v + n^{\mathrm{T}} \dot{v} = \dot{n}^{\mathrm{T}} v + n^{\mathrm{T}} D\mathcal{F} v$. Hence, $\dot{n} + D\mathcal{F}^{\mathrm{T}} n = \theta n$ for some $\theta$. Since $n$ is a unit normal, we find $\theta = n^{\mathrm{T}} D\mathcal{F} n$ as required. □

**Lemma 2.6.** *Let $n$ denote the upward unit normal to the graph of $u(\cdot, t)$ and $q = n^{\mathrm{T}} \mathcal{F}$. Then $\dot{q} = (n^{\mathrm{T}} D\mathcal{F} n)q$. Hence, $q$ is either zero or has constant sign for all $t$.*

**Proof.** $\dot{q} = \dot{n}^{\mathrm{T}} \mathcal{F} + n^{\mathrm{T}} D\mathcal{F}\mathcal{F} = \mathcal{F}^{\mathrm{T}}(-D\mathcal{F}^{\mathrm{T}} n + (n^{\mathrm{T}} D\mathcal{F} n)n) + n^{\mathrm{T}} D\mathcal{F}\mathcal{F} = (n^{\mathrm{T}} D\mathcal{F} n)q$. □

This lemma says that the component of the flow normal to the curve always has the same sign.

The following establishes directly that decreasing functions have Lipschitz graphs. For an alternative approach, see [**4**].

**Lemma 2.7.** *If $u \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is smooth and non-increasing, then the graph of $u$ is a Lipschitz manifold with Lipschitz constant unity.*

**Proof.** Consider the transformation of coordinates $X = \frac{1}{2}(x - y)$, $Y = \frac{1}{2}(x + y)$. Then in the new coordinates $y = u(x)$ transforms to $Y = X + u(X + Y)$. Without loss of generality we may suppose that $\lim_{x\to\infty} u(x) = 0$. Given $X \in J = [-u(0)/2, \infty)$, $Y = X + u(X + Y)$ has a unique solution $Y = \Phi(X) \in \mathbb{R}_{\geqslant 0}$, which defines a single-valued function $\Phi \colon J \to \mathbb{R}_{\geqslant 0}$. The function $\Phi$ is continuous. Suppose, on the contrary, it is not. Then, if $\Phi(X+) = Y_1$, $\Phi(X-) = Y_2$ and $\Delta Y = Y_2 - Y_1$, then $\Delta Y = u(X + \Delta Y + Y_1) - u(X + Y_1)$ and since $u$ is non-increasing we must have $\Delta Y = 0$. Formally, we compute

$$\Phi'(X) = \frac{1 + u'(X + \Phi(X))}{1 - u'(X + \Phi(X))},$$

which shows that $\Phi$ is continuously differentiable on $J$ since $u' \leqslant 0$ and $\Phi$ is continuous, and we have

$$|\Phi'(X)| = \frac{|1 - |u'(X + \Phi(X))||}{1 + |u'(X + \Phi(X))|}$$

and so $|\Phi'| \leqslant 1$ on $J$. □

## 3. Evolution of the curvature

**Lemma 3.1.** *We have*

$$\frac{\mathrm{d}u_{xx}}{\mathrm{d}t} = g_{xx} + u_x(2g_{xy} - f_{xx}) + u_x^2(g_{yy} - 2f_{xy}) - f_{yy}u_x^3 + u_{xx}(g_y - 2f_x - 3f_y u_x). \quad (3.1)$$

**Proof.** This is similar to that of Lemma 2.1. □

**Corollary 3.2.** *For the functions $f(x,y) = px(1-x-ay)$, $g(x,y) = qy(1-y-bx)$ we obtain*

$$\frac{\mathrm{d}u_{xx}}{\mathrm{d}t} = 2u_x(p-bq+(ap-q)u_x) + u_{xx}(3apxu_x + 2(ap-q)u + (4p-bq)x + q - 2p), \quad (3.2)$$

*and, at $t=0$, $\mathrm{d}u_{xx}/\mathrm{d}t = 2\alpha$ when $u_0(x) = 1-x$, where*

$$\alpha = p(a-1) + q(b-1). \quad (3.3)$$

The corollary shows that, with the initial data $u_0(x) = 1-x$, the solution $u(\cdot,t)$ to (2.2) is strictly convex (concave) for small enough $t>0$ when $\alpha > 0$ ($\alpha < 0$). Equivalently, if $\alpha > 0 (< 0)$, then $\Sigma_t = \varphi_t(\Sigma)$ is strictly convex (concave) for $t>0$ sufficiently small. We will show later that $\alpha > 0 (<0)$ is actually enough to ensure strict convexity (concavity) of $u(\cdot,t)$ and $\Sigma_t$ for all $t>0$.

## 4. Convexity of $u(\cdot,t)$ when $\alpha > 0$

Now we restrict our attention to the evolution of the unit simplex $\Sigma_2$ under the flow $\varphi_t$ of (1.1). We parametrize the unit simplex $\Sigma_2 = \{(s, 1-s) \colon s \in [0,1]\}$. We now show that once the manifold $\Sigma_t = \varphi_t(\Sigma_2)$ is strictly convex, it remains strictly convex. Note that since $(1,0)$ and $(0,1)$ are equilibria, $\Sigma_t$ can be considered as the graph of a smooth function over the interval $[0,1]$.

First we prove a lemma which will enable us to track the curvature following a material point.

**Lemma 4.1.** *Let $\theta, \sigma \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be continuous functions. Suppose that $\psi \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ satisfies*

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \sigma\psi + \theta,$$

*with $\psi(t_0) = \psi_0$, for some $t_0 \geqslant 0$. Then if*

(a) *$\psi_0 \geqslant 0$ and*

(b) *$\theta(T) > 0$ whenever $\psi(T) = 0$ ($T \geqslant t_0$),*

*then $\psi(t) > 0$ for all $t > t_0$.*

**Proof.** We simply compute, for $t \geqslant w$,

$$\psi(t) = \exp\biggl(\int_w^t \sigma(\tau)\,\mathrm{d}\tau\biggr)\biggl\{\psi(w) + \int_w^t \theta(\nu)\exp\biggl(-\int_w^\nu \sigma(\tau)\,\mathrm{d}\tau\biggr)\mathrm{d}\nu\biggr\}. \tag{4.1}$$

If $\psi_0 = 0$, then since we are told in (b) that $\theta(t_0) > 0$, by right continuity of $\theta$ we have $\theta(t) > 0$ for $t \in [t_0, t_0 + \delta_0)$ for some $\delta_0 > 0$. But then by (4.1) we have $\psi(t) > 0$ for $t \in (t_0, t_0 + \delta_0)$. If $\psi(t_0) > 0$, then there exists $\delta_1 > 0$ such that $\psi(t) > 0$ for all $t \in [t_0, t_0 + \delta_1)$. Thus, when $\psi(t_0) \geqslant 0$ there exists some smallest $\delta_2 > 0$ such that $\psi(t) > 0$ for $t \in (t_0, t_0 + \delta_2)$ and $\psi(t_0 + \delta_2) = 0$. Either $\delta_2 = \infty$ or $\delta_2$ is finite and $\psi(t_0 + \delta_2) = 0$. But then $\theta(t_0 + \delta_2) > 0$, and by continuity $\theta(t) > 0$ for $t \in (t_0 + \delta_2 - \eta, t_0 + \delta_2 + \eta)$ for some $\eta > 0$. From (4.1) this then implies that $\psi(t_0 + \delta_2) > 0$: a contradiction. □

We now apply this lemma as follows: in equation (3.2), set

$$\begin{aligned}\theta &= 2u_x(p - bq + (ap - q)u_x),\\ \sigma &= 3apxu_x + 2(ap - q)u + (4p - bq)x + q - 2p.\end{aligned}$$

Thus, the idea is to show that whenever the curvature $u_{xx} = 0$ we must necessarily have $\theta > 0$. We actually achieve this by showing that, in fact, $\theta(\cdot, t) > 0$ on $[0, 1]$ for $t \geqslant 0$.

Recall that $\Sigma_t$ is the graph of a $C^\infty([0, 1])$ function $u(\cdot, t)$ for each $t \geqslant 0$. From Corollary 2.2 and Remark 2.3, competitiveness of the flow ensures that $u(\cdot, t)$ is a decreasing function on $[0, 1]$. Notice also that $u_x(1, t) < 0$ for all $t \geqslant 0$, since $z = u_x(1, t)$ is a solution of $\dot{z} = z(q(1 - b) + p + apz)$ with $z(0) = -1$. Hence, $u_x(\cdot, t) < 0$ on $[0, 1]$ for $t \geqslant 0$.

We continue to work with the case $\alpha > 0$ (which means $\theta(\cdot, 0) > 0$ on $[0, 1]$). Since $u_x < 0$ for $x \in [0, 1]$ for $t \geqslant 0$, we need only consider the expression

$$\xi = (p - qb) + (pa - q)u_x.$$

We will show that $\xi < 0$ for all $t \geqslant 0$ and hence that $\theta > 0$ for all $t \geqslant 0$.

We know that, since $\alpha > 0$, $\xi < 0$ on $[0, 1]$ at $t = 0$. Suppose that $\xi$ first vanishes at some $t = T > 0$. From equation (4.1) with $\psi = u_{xx}$ and $\theta$, $\sigma$ as defined above, we see that $u_{xx} > 0$ for all $t \in (0, T]$, so that $\Sigma_t$ is strictly convex for $t \in (0, T]$ and thus we know that if $\xi$ vanishes at $t = T$ it must be for $x = 0$ or $x = 1$ (since by convexity $u_x$ takes its extreme values at these end points). First we dispense with the case $pa = q$. If $pa - q = 0$, then $\xi = (p - qb) < 0$, since we are assuming $0 < \alpha = p(a - 1) + q(b - 1) = qb - p$. Thus, for $\xi$ to vanish we need $pa - q \neq 0$. Thus, now suppose that $pa \neq q$. If $qb = p$, then $pa - q = \alpha > 0$ and at a point where $\xi = 0$ we must have $u_x = 0$, which contradicts the notion that $u_x < 0$ for $t \geqslant 0$. If $qb \neq p$ and $pa \neq q$, then at a point where $\xi = 0$ we have $u_x = -(p - qb)/(pa - q)$, which is only possible if $(p - qb)/(pa - q) > 0$. In other words, we have shown that $\xi$ cannot vanish unless $p > qb$, $pa > q$ or $p < qb$, $pa < q$. We compute

$$\dot{\xi} = (ap - q)\frac{\mathrm{d}u_x}{\mathrm{d}t} = (ap - q)\{g_x + g_y u_x - (f_x + f_y u_x)u_x\}.$$

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We find at $x = 0$, $y = 1$ that

$$\begin{aligned}\dot{\xi} &= (p-q)(p-bq) - (ap^2 + bq^2 - 2pq) \\ &= p(p(1-a) + q(1-b)) \\ &= -p\alpha < 0.\end{aligned}$$

Similarly, one finds that at $x = 1$, $y = 0$ we have

$$\dot{\xi} = q\alpha\left(\frac{qb-p}{pa-q}\right) < 0$$

since here $(qb-p)/(pa-q) = u_x < 0$. Thus, we find that, whenever $\xi = 0$, $\dot{\xi} < 0$, so that in fact, since initially $\xi < 0$, $\xi(\cdot, t) < 0$ for all $t \geqslant 0$. By Lemma 4.1, $u_{xx} > 0$ for all $t > 0$.

Hence, when $\alpha > 0$ the solutions $u(\cdot, t)$ (equivalently, the manifolds $\Sigma_t$) are strictly convex for all $t > 0$. When $\alpha < 0$ a similar argument shows that $u(\cdot, t)$ is strictly concave for all $t > 0$.

At this point, we can easily conclude the existence of a convex invariant manifold for (1.1) that connects the axial equilibria. To see this, note that we have a bounded and uniformly Lipschitz sequence of strictly convex manifolds $\Sigma_t$, from which we may extract a convergent subsequence that converges to an invariant and convex Lipschitz manifold $\mathcal{S}_2$ which passes through the axial equilibria by construction. However, we will now improve on this by showing that $\mathcal{S}_2$ is the graph of a locally Lipschitz function and that it attracts all points in $\mathbb{R}^2_{\geqslant 0} \setminus \{0\}$.

## 5. Construction of upper and lower solution sequences

We do the following for the strongly competitive system (1.2) which includes the Lotka–Volterra system (1.1). The system (1.2) has unique axial equilibria and we suppose that the coordinates are scaled so that they lie at $E_1 = (0, 1)$ and $E_2 = (1, 0)$.

Since $F(0, 0) > 0$, $G(0, 0) > 0$, the origin is an unstable node for (1.2) and there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ all points with $x \in [0, \infty)$ on the non-increasing, piecewise linear curve $y_{\mathrm{L}} \colon \mathbb{R} \to \mathbb{R}$ given by $y_{\mathrm{L}}(x) = \max\{\delta - x, 0\}$ move in the upwards direction or along it. We smooth $y_{\mathrm{L}}$ using a suitable mollifier to produce a new non-increasing $C^\infty(\mathbb{R}_{\geqslant 0})$ function $\tilde{y}_{\mathrm{L}} \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ (i.e. now restrict to the invariant first quadrant) such that $n(r) \cdot \mathcal{F}(r) \geqslant 0$ along its graph, where $n(r)$ is the upward unit normal at the point $r$ on the graph. Let $\Gamma_{\mathrm{L},t}$ be the image of the graph of $\tilde{y}_{\mathrm{L}}$ under the competitive flow, and let $y_{\mathrm{L}}(\cdot, t) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be the function whose graph is $\Gamma_{\mathrm{L},t}$. The functions $y_{\mathrm{L}}(\cdot, t)$ are solutions of the PDE (1.3), and we call them lower solutions. By Lemma 2.6 we have that $\partial y_{\mathrm{L}}/\partial t = (-\partial y_{\mathrm{L}}/\partial x, 1)^{\mathrm{T}} \mathcal{F} \geqslant 0$, so that the sequence $y_{\mathrm{L}}(\cdot, t)$ is non-decreasing. By Corollary 2.2, since $\tilde{y}_{\mathrm{L}}$ is non-increasing, each $y_{\mathrm{L}}(\cdot, t)$ is a non-increasing function, and, moreover, $y_{\mathrm{L}}(\cdot, t)$ has compact support upon which it is decreasing. The non-decreasing sequence of Lipschitz manifolds $\Gamma_{\mathrm{L},t}$, which is bounded by dissipation, converges to a Lipschitz manifold $\Gamma^*_{\mathrm{L}}$ that is invariant under the flow. Let $y^*_{\mathrm{L}}(x) = \max\{y \geqslant 0 \colon (x, y) \in \Gamma^*_{\mathrm{L}}\}$. Then $y_{\mathrm{L}}(x, t) \to y^*_{\mathrm{L}}(x)$ for each $x \in \mathbb{R}_{\geqslant 0}$.

By Helly's Theorem for sequences of monotone functions (see, for example, [**3**]), $y_{\mathrm{L}}^*$ is non-increasing, with a countable number of points of discontinuity.

Now we follow [**2**] to show that $y_{\mathrm{L}}^*$ is continuous and decreasing. Suppose that $\tau > 0$ is a point of discontinuity of $y_{\mathrm{L}}^*$ (which is isolated since they are countable in number). Then the two points $A = (\tau, y_{\mathrm{L}}^*(\tau-))$ and $B = (\tau, y_{\mathrm{L}}^*(\tau+))$ are ordered: $A_1 = B_1$ and $B_2 < A_2$, so that $B < A$. Since the flow is strongly monotone backwards in time (see Remark 2.4) on $\mathbb{R}^2_{>0}$, we then have two points $\varphi_{-t}(A)$, $\varphi_{-t}(B)$ in the invariant graph $\Gamma_{\mathrm{L}}^*$ of $y_{\mathrm{L}}^*$ that satisfy $\varphi_{-t}(B) \ll \varphi_{-t}(A)$ for $t > 0$. On the other hand, since $A_2 > B_2$, for $\delta > 0$ small enough $(\varphi_{-\delta}(A))_2 - (\varphi_{-\delta}(B))_2 > 0$, whereas by invariance of $\Gamma_{\mathrm{L}}^*$ we must have $(\varphi_{-\delta}(A))_1 - (\varphi_{-\delta}(B))_1 \leqslant 0$. Hence, the line segment joining $\varphi_{-\delta}(A)$ and $\varphi_{-\delta}(B)$ has either finite and negative slope or is vertical. In either case we contradict the notion that $\varphi_{-t}(B) \ll \varphi_{-t}(A)$ for $t = \delta > 0$.

Hence, $y_{\mathrm{L}}^*$ is continuous on $(0, \infty)$. If $y_{\mathrm{L}}^*$ were not also continuous at $x = 0$, then since $\Gamma_{\mathrm{L}}^*$ is a Lipschitz manifold, $\Gamma_{\mathrm{L}}^*$ would have a vertical line segment at $x = 0$. But since $x = 0$ is invariant, and on the set $\{(0, y) \colon y > 0\}$ the dynamic is convergent to the axial equilibrium $E_1 = (0, 1)$, this line segment must be the single point $E_1$, and hence $y_{\mathrm{L}}^*$ is continuous on $\mathbb{R}_{\geqslant 0}$, with $y_{\mathrm{L}}^*(0) = 1$.

Now note that we could have done all the foregoing analysis with the roles of $x, y$ reversed, and hence $y_{\mathrm{L}}^*$ must be a locally Lipschitz function satisfying $y_{\mathrm{L}}^*(0) = 1$, $y_{\mathrm{L}}^*(1) = 0$ and decreasing on $[0, 1]$.

Similarly, by choosing $\delta_1 > 0$ large enough, for each $\delta > \delta_1$, we may construct a curve $y_{\mathrm{U}}(x) = \max\{\delta - x, 0\}$ such that in the first quadrant each point on the curve either moves along the curve or moves downwards under the flow. We then smooth this function to give the $C^\infty$ function $\tilde{y}_{\mathrm{U}}$. The image of the graph of $\tilde{y}_{\mathrm{U}}$ under the flow is then the graph of a smooth non-increasing function $y_{\mathrm{U}}(\cdot, t)$. The non-increasing sequence $y_{\mathrm{U}}(\cdot, t)$ of non-increasing functions converges to a locally Lipschitz function $y_{\mathrm{U}}^* \colon [0, \infty) \to \mathbb{R}$ with $y_{\mathrm{U}}^*(0) = 1$, $y_{\mathrm{U}}^*(1) = 0$, decreasing on $[0, 1]$, and whose graph $\Gamma_{\mathrm{U}}^*$ is invariant under the flow.

**Remark 5.1.** Hirsch's results in [**4**] show that $\mathcal{S}_2$ is balanced and its interior $\operatorname{int} \mathcal{S}_2$ is strongly balanced. By 'balanced' here we mean that no two points $p_1, p_2 \in \mathcal{S}_2$ can be strongly ordered (i.e. neither $p_1 \ll p_2$ or $p_2 \ll p_1$ is true), and by 'strongly balanced' we mean no two points can be ordered. In the present context, this amounts to showing that $\mathcal{S}_2$ is the graph of a decreasing function on $[0, 1]$.

## 6. Convergence of upper and lower solutions to the carrying simplex

We now demonstrate that these upper and lower solutions converge to the same limit: $y_{\mathrm{L}}^* = y_{\mathrm{U}}^*$. For ease of notation let $y_1 = y_{\mathrm{L}}^*$ and $y_2 = y_{\mathrm{U}}^*$. Let $\mathcal{G}_i$ denote the graph of $y_i$ restricted to $[0, 1]$ for $i = 1, 2$. By construction, $y_2 \geqslant y_1$. We also know that $y_1$, $y_2$ are continuous (in fact, locally Lipschitz) and decreasing and hence invertible with continuous inverses $y_1^{-1}$, $y_2^{-1}$. For $\varepsilon \in [0, 1]$ define $D_\varepsilon = \{(x, y) \in [\varepsilon, 1]^2 \colon y_1(x) \leqslant y \leqslant y_2(x)\}$. Then $D_\varepsilon$ is measurable and, if $y_1 \neq y_2$, has decreasing measure for increasing $\varepsilon \in [0, 1]$. We

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will work in generality here: to cover the strongly competitive system (1.2) take $f = xF$, $g = yG$, where $F_x < 0$, $G_y < 0$.

By Green's Theorem for smooth functions, and a region whose boundary is Lipschitz,

$$\int_{D_\varepsilon} \operatorname{div}\left(\frac{1}{xy}(f,g)\right) \mathrm{d}A = \int_{\partial D_\varepsilon} \frac{1}{xy}(f,g)\cdot n \,\mathrm{d}s,$$

where $\mathrm{d}A$ is the area element, $\mathrm{d}s$ the line element and $n$ the (outward) unit normal to the Lipschitz boundary $\partial D_\varepsilon$ of $D_\varepsilon$ taken anticlockwise. But the left-hand side $L$ is just

$$L(\varepsilon) = \int_{D_\varepsilon} \left(\frac{1}{x}F_x + \frac{1}{y}G_y\right) \mathrm{d}A < 0,$$

since $F_x < 0$, $G_y < 0$, and hence $L(\varepsilon)$ is decreasing with decreasing $\varepsilon \in [0,1]$.

On the other hand, for the right-hand side $R(\varepsilon)$ the line integrals along the sections of the graphs $\mathcal{G}_1$, $\mathcal{G}_2$ vanish by invariance under the flow. The right-hand side terms that remain are

$$R(\varepsilon) = \int_{y_2(\varepsilon)}^{y_1(\varepsilon)} -\frac{F(\varepsilon,y)}{y}\,\mathrm{d}y + \int_{y_1^{-1}(\varepsilon)}^{y_2^{-1}(\varepsilon)} -\frac{G(x,\varepsilon)}{x}\,\mathrm{d}x.$$

We have

$$\begin{aligned} |R(\varepsilon)| \leqslant (y_2(\varepsilon) - y_1(\varepsilon)) \max_{y\in[y_1(\varepsilon),y_2(\varepsilon)]} \left|\frac{F(\varepsilon,y)}{y}\right| \\ + (y_2^{-1}(\varepsilon) - y_1^{-1}(\varepsilon)) \max_{x\in[y_1^{-1}(\varepsilon),y_2^{-1}(\varepsilon)]} \left|\frac{G(x,\varepsilon)}{x}\right|. \end{aligned}$$

Now as $\varepsilon \to 0+$, $y_1(\varepsilon), y_2(\varepsilon) \to 1$ and $y_1^{-1}(\varepsilon), y_2^{-1}(\varepsilon) \to 1$, so that $R(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Since $L(\varepsilon) < 0$ and is decreasing with decreasing $\varepsilon \in [0,1]$, for $\varepsilon$ small enough we contradict the Green Theorem identity $R(\varepsilon) = L(\varepsilon)$. Hence, we must have that the measure of $D$ is zero, and thus $\mathcal{G}_1 = \mathcal{G}_2$ and $y^*_{\mathrm{L}} = y^*_{\mathrm{U}}$.

Now all we need to do is observe that the graphs $\Sigma_t = \varphi_t(\Sigma_2)$ are sandwiched between suitably chosen convergent upper and lower sequences that converge to the same Lipschitz and invariant manifold, which we will denote by $M^*$, and which is the graph of a locally Lipschitz function $\psi^*\colon [0,1] \to \mathbb{R}$. Each $\Sigma_t$ is the graph of a locally Lipschitz convex function and hence $\psi^*$ is also convex [**6**] and locally Lipschitz. Finally, by choosing $\delta > 0$ small enough in the lower solution or $\delta$ large enough in the upper solution, we may push any point in $\mathbb{R}^2_{\geqslant 0} \setminus \{0\}$ onto $M^*$ and so $M^* = \mathcal{S}_2$ attracts all points in $\mathbb{R}^2_{\geqslant 0} \setminus \{0\}$.

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