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ON ALGEBRAS STABLY EQUIVALENT TO AN HEREDITARY ARTIN ALGEBRA

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Introduction. Let Λ be an artin algebra, that is, an artin ring that is a finitely generated module over its center C which is also an artin ring. We denote by mod Λ the category of finitely generated left Λ-modules. We recall that the category $\underline{\text{mod}}\ \Lambda$ of finitely generated modules modulo projectives is the category given by the following data: the objects are the finitely generated Λ-modules. The set of morphisms $\underline{\text{Hom}}\ (A, B)$ between two Λ-modules A and B in $\underline{\text{mod}}\ \Lambda$ is $\text{Hom}_\Lambda(A, B)/P(A, B)$, where $P(A, B)$ is the C-submodule of $\text{Hom}_\Lambda(A, B)$ consisting of all the maps $f: A \to B$ that factor through a projective module. Two artin algebras Λ and Λ' are *stably equivalent* if $\underline{\text{mod}}\ \Lambda$ and $\underline{\text{mod}}\ \Lambda'$ are equivalent [**1**].

Artin algebras stably equivalent to an hereditary artin algebra have been characterized in [**1**] by M. Auslander and I. Reiten, who also gave a description of an hereditary ring Γ stably equivalent to Λ. The purpose of this paper is to give a different description of a hereditary ring Γ stably equivalent to Λ together with a functor F: mod $\Lambda \to$ mod Γ that induces a stable equivalence between Λ and Γ. The description is the following: let a denote the two sided ideal sum of the nonprojective simples of the socle of Λ and let b be the left annihilator of a in Λ. Then a is a $\Lambda/b - \Lambda/a$-bimodule and $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$. We consider the Γ-modules as triples (A, B, f), where A is a Λ/a-module, B is a Λ/b-module and $f: a \otimes A \to B$ is a Λ/b-homomorphism (see [**1**]). Then for a Λ-module M, $F(M) = (M/a\ M, aM, m)$, where $m: a \otimes M/a\ M \to aM$ is given by the multiplication map.

Many of the results about hereditary algebras obtained either by using representations of diagrams and K-species (see, for example, [**3**]), or by using the notions of almost split sequences and irreducible maps (see [**2**]), have been extended to artin algebras that are stably equivalent to an hereditary algebra in [**4**]. Most of the results obtained in [**4**] can be deduced from the hereditary case by using the functor F: mod $\Lambda \to$ mod Γ that induces the stable equivalence between Λ and the hereditary ring $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$. This is one of the reasons why we are interested in this particular description of Γ and F. On the other hand, another reason why we are interested in giving this description, different from that given in [**1**], is because it facilitates computations.

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We assume throughout the paper that Λ is an artin algebra and that all the modules are finitely generated. We recall from [**1**, ch. I, Th 1.6 and 2.1] that Λ is stably equivalent to an hereditary algebra if and only if the following conditions are satisfied:

1) Every indecomposable submodule of an indecomposable projective Λ-module is projective or simple.

2) If S is a nonprojective simple submodule of a projective module then S is a factor of an injective module.

We will prove here that when Λ satisfies the conditions 1) and 2), then Λ is stably equivalent to the hereditary ring $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$. This then gives a different proof of part of the characterization just mentioned, that is, it proves that conditions 1) and 2) imply that Λ is stably equivalent to an hereditary algebra.

Since some of the results that we prove are true when only one of the two conditions 1) or 2) is needed, we will study properties of artin algebras satisfying either 1) or 2). Then we will apply these results to our case, that is, to the case when both properties hold.

When Λ is an artin algebra of radical square zero the conditions 1) and 2) are satisfied, so Λ is stably equivalent to an hereditary ring. For this special case, a construction similar to that which is used in this paper has been given in [**1**, Ch. V]. It is proven there that if Λ is of radical square zero and r denotes the radical of Λ, then Λ is stably equivalent to the ring $\Gamma_1 = \begin{bmatrix} \Lambda/r & 0 \\ r & \Lambda/r \end{bmatrix}$. While the rings Γ_1 and $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ are not always Morita equivalent, they are closely related, since there is a semisimple ring V such that Γ_1 is Morita equivalent to $\Gamma \times V$.

We begin by proving that when Λ satisfies 1), then $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ is hereditary. Then we prove that if Λ also satisfies 2), then Γ is stably equivalent to Λ. It has been proven in [**1**, Ch. III, § 2] that each stable equivalence class contains essentially one hereditary algebra. More precisely, if Λ and Λ' are hereditary artin algebras, they are stably equivalent if and only if there are simisimple algebras V and V' such that mod $(\Lambda \times V)$ and mod $(\Lambda' \times V')$ are equivalent. So in the stable equivalence class of Λ there is only one (up to Morita equivalence) hereditary artin algebra Γ with no semisimple summands. We end the paper by proving that if Λ has no semisimple summands then Γ has no semisimple summands.

I would like to take this opportunity to thank Professor Maurice Auslander for many helpful conversations and suggestions.

1. We keep the notations of the introduction: a denotes the sum of the nonprojective simples in the socle of Λ, b is the left annihilator of a in Λ, and

$\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$. A Λ-module M is said to be *torsionless* if it is a submodule of a projective module. A Λ-module M is said to be *torsion* if the indecomposable submodules of M are not torsionless.

If M is a Λ-module and c an ideal in Λ, then $\tau_c(M)$ denotes the trace of c over M, that is, the submodule of M generated by the images of the maps from c to M. We observe that when M is projective, then the multiplication map $c \otimes M \to cM$ is an isomorphism.

We begin by recalling some results of [**4**] (Lemmas 5.1, 5.2, and Proposition 5.3) that will be used in this section.

LEMMA 1.1. *Let Λ be an artin algebra and P a projective Λ-module. Then $aP = \tau_a(P)$ = trace of a over P.*

LEMMA 1.2. *Assume that Λ satisfies* 1), *let M be in* mod (Λ), *let P be a projective Λ-module, $\pi: P \to M$ an epimorphism, and $K =$* Ker (π). *Then $K = V \amalg Q$, where $V \subseteq aP$ and Q is projective. For any decomposition of K of this type, the sequence*

$$0 \to Q/aQ \to P/aP \to M/aM \to 0$$

is exact.

COROLLARY 1.3. *If Λ satisfies the condition* 1), *then Λ/a is an hereditary ring.*

We also recall from [**1**, Ch. V, § 1] that if Λ_1 and Λ_2 are artin algebras and M is a $\Lambda_2 - \Lambda_1$-bimodule, then the category of modules over $\Gamma = \begin{bmatrix} \Lambda_1 & 0 \\ M & \Lambda_2 \end{bmatrix}$ is equivalent to the category $\mathscr{C}$ of triples (A, B, f), where A is a Λ_1-module, B is a Λ_2-module and $f: M \otimes_{\Lambda_1} A \to B$ is a Λ_2-homomorphism. A morphism $(A, B, f) \to (A', B', f')$ between two objects in $\mathscr{C}$ is a pair of maps (g_1, g_2), $g_1: A \to A'$, $g_2: B \to B'$, such that the diagram

$$\begin{tikzcd} M \otimes A \arrow[r, "1_M \otimes g_1"] \arrow[d, "f"'] & M \otimes A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"] & B' \end{tikzcd}$$

commutes.

Let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $M = (1 - e)\Gamma e$, and the equivalence G: mod $\Gamma \to \mathscr{C}$ is given by $G(X) = (eX, (1-e)X, f)$, where $f((1-e)\gamma e \otimes em) = (1-e)\gamma em$. We will often describe the modules over Γ as triples of the type that we have indicated.

Our next aim is to prove that when Λ satisfies 1) then the ring $\begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$

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is hereditary. This will follow as an immediate consequence of Corollary 1.3 and the following result.

PROPOSITION 1.4. *Assume that* Λ *satisfies* 1). *Then the ring* $\Gamma_1 = \begin{bmatrix} \Lambda & 0 \\ a & \Lambda/b \end{bmatrix}$ *also satisfies* 1), *and the sum of the nonprojective simples of the socle of* Γ_1 *is the ideal* $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. The indecomposable projective Γ_1-modules correspond to the triples $(P, a \otimes P, \mathrm{id})$ and (O, S, O), where P is a projective indecomposable Λ-module and S is a projective indecomposable, hence simple, module over the semisimple ring Λ/b.

Let now (N_1, N_2, f) be an indecomposable submodule of an indecomposable projective module. We will see that it is projective or simple. Since Λ/b is semisimple, $\mathrm{Im} f$ is a direct summand of N_2. Thus, we can write $(N_1, N_2, f) = (N_1, \mathrm{Im} f, f) \amalg (O, B, O)$, for some Λ/b module B. Since (N_1, N_2, f) is indecomposable, then either $B = 0$ and f is an epimorphism, or $N_1 = 0$ and B is simple. Since in the second case $(N_1, N_2, f) = (O, B, O)$ is projective, we may assume that f is an epimorphism. Let (g_1, g_2): $(N_1, N_2, f) \rightarrow (P, a \otimes P, \mathrm{id})$ be a monomorphism, with P indecomposable projective. Then g_1: $N_1 \rightarrow P$ is a monomorphism. Therefore N_1 is projective or simple torsionless nonprojective, because Λ satisfies 1). If N_1 is simple torsionless nonprojective, then we know by Lemma 1.1, that $g_1(N_1) \subseteq a\, P$. From the commutative diagram

$$\text{(1)} \qquad \begin{tikzcd} a \otimes N_1 \arrow[r, "1 \otimes g_1"] \arrow[d, "f"'] & a \otimes P \arrow[d, "\mathrm{id}"] \\ 0 \rightarrow N_2 \arrow[r, "g_2"] & a \otimes P \end{tikzcd}$$

we find that $(1 \otimes g_1)\ (a \otimes N_1) \subseteq a \otimes aP \subseteq a^2 \otimes P = 0$; so $g_2 f = 0$. But g_2 is a monomorphism and f is an epimorphism. Therefore $N_2 = 0$; so $(N_1, N_2, f) = (N_1, 0, 0)$ is simple torsionless nonprojective.

Assume now that N_1 is projective. We have the commutative diagram

$$\begin{tikzcd} a \otimes N_1 \arrow[r, "1 \otimes g_1"] \arrow[d] & a \otimes P \arrow[d] \\ aN_1 \arrow[r, "g_1 | aN_1"] & aP. \end{tikzcd}$$

N_1 and P are projective, so, as we observed before, the vertical maps are isomorphisms. Moreover, the map $g_1|aN_1$ is a monomorphism. Therefore $1 \otimes g_1$ is also a monomorphism. From the diagram (1) we have that $g_2 f = 1 \otimes g_1$, so the epimorphism f is also a monomorphism. Therefore $(N_1, N_2, f) \simeq (N_1, a \otimes N_1, \mathrm{id})$ is projective.

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This proves that $\begin{bmatrix} \Lambda & 0 \\ a & \Lambda/b \end{bmatrix}$ satisfies the property 1), and also that the non-projective torsionless simple modules are of the form (S, O, O) with $S \subseteq a$. This ends the proof of the proposition.

The result that interests us now follows as a corollary.

COROLLARY 1.5. *If Λ satisfies Property* 1) *then the ring* $\begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ *is hereditary.*

Assume now that Λ is an arbitrary artin algebra and c a two sided ideal in Λ such that $c^2 = 0$. Let N_1 be a Λ-module such that $cN_1 = 0$, $\alpha: P \to N_1$ a projective cover for N_1 and $K = \operatorname{Ker} \alpha$, We can then define a map $\epsilon: c \otimes N_1 \to K/cK$ by $\epsilon(\mathrm{c} \otimes \alpha(p)) = \mathrm{c}\bar{p} \in K/cK$.

LEMMA 1.6. *In the notations of the preceding paragraph, the sequence*

$$0 \to c \otimes N_1 \xrightarrow{\epsilon} K/cK \xrightarrow{\bar{\imath}} P/cP \xrightarrow{\bar{\alpha}} N_1 \to 0$$

is exact, where $\bar{\alpha}$ and $\bar{\imath}$ are the maps induced by α and the inclusion i: $K \to P$ respectively. If Λ/c is an hereditary ring then ϵ is a splitable monomorphism. Moreover, if $K/cK = \operatorname{Im}(\epsilon) \amalg M$, then M is projective.

Proof. It follows from the definition of ϵ that $\operatorname{Im}(\epsilon) = \operatorname{Ker}(\bar{\imath})$. We will prove now that ϵ is a monomorphism. The sequence

$$0 \to K \xrightarrow{i} P \xrightarrow{\alpha} N \to 0$$

induces an exact sequence

$$\begin{tikzcd} c \otimes K \arrow[r, "\mathrm{id} \otimes i"] & c \otimes P \arrow[r, "\mathrm{id} \otimes \alpha"] & c \otimes N_1 \arrow[r] \arrow[d, "\epsilon"] & 0 \\ & & K/cK. & \end{tikzcd}$$

If $\epsilon.(\mathrm{id} \otimes \alpha)\ (\sum \mathrm{c}_i \otimes p_i) = 0$ then $\sum \mathrm{c}_i p_i \in cK$, so $\sum \mathrm{c}_i p_i = \sum \mathrm{c}_i' k_i$, with $k_i \in K$, $\mathrm{c}_i' \in c$. Then the elements $\sum \mathrm{c}_i \otimes p_i$ and $\sum \mathrm{c}_i' \otimes i(k_i)$ have the same image under the product map $c \otimes P \to cP$. But this map is an isomorphism because P is projective, so $\sum \mathrm{c}_i \otimes p_i = \sum \mathrm{c}_i' \otimes i(k_i) \in \operatorname{Im}(\mathrm{id} \otimes i)$. Thus $\operatorname{Ket}(\epsilon.(\mathrm{id} \otimes \alpha)) \subseteq \operatorname{Im}(\mathrm{id} \otimes i)$, and this proves that ϵ is a monomorphism.

If Λ/c is hereditary then $\bar{\imath}(K/cK)$ is projective, since it is a submodule of the projective Λ/c-module $P/c\,P$. Then the map $\bar{\imath}$: $K/cK \to \bar{\imath}(K/cK)$ splits, and therefore ϵ splits. Moreover, a complement of $\operatorname{Im}(\epsilon)$ in K/cK is projective because it is isomorphic to $\bar{\imath}(K/cK)$.

Let c be a two sided ideal of Λ, such that $c^2 = 0$ and let d be the left annihilator of c in Λ. Then c is a $\Lambda/d - \Lambda/c -$ bimodule. Let $\mathscr{C}$ be the category

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defined above, equivalent to mod $\begin{bmatrix} \Lambda/c & 0 \\ c & \Lambda/d \end{bmatrix}$, whose objects are the triples (A, B, f), where A is a Λ/c-module, B is a Λ/d-module, and $f: c \otimes A \to B$ is a homomorphism. In all that follows, we will denote by $\mathscr{C}_1$ the full subcategory of $\mathscr{C}$ consisting of the triples (A, B, f) in $\mathscr{C}$ such that f is an epimorphism. Then we have the following result.

PROPOSITION 1.7. *Assume that c is a two-sided ideal in Λ such that $c^2 = 0$ and Λ/c is an hereditary ring. Then F: mod $\Lambda \to \mathscr{C}_1$ is dense.*

Proof. Let $(N_1, N_2, f) \in \mathscr{C}_1$ and let $\alpha: P \to N_1$ be a projective cover of the Λ-module N_1. Let $K = \operatorname{Ker}(\alpha)$. We know by Lemma 1.6 that there is an exact sequence

$$0 \to c \otimes N_1 \xrightarrow{\epsilon} K/cK \to P/cP \xrightarrow{\bar{\alpha}} N_1 \to 0,$$

where ϵ is a splittable monomorphism. Let $\delta: K/cK \to c \otimes N_1$ be such that $\delta\epsilon = \mathrm{id}$, and denote by $\tilde{f}: K \to N_2$ the composition

$$K \xrightarrow{p} K/cK \xrightarrow{\delta} c \otimes N_1 \xrightarrow{f} N_2,$$

where $p: K \to K/cK$ is the canonical epimorphism. $\tilde{f}$ is a composition of epimorphisms and is, therefore, an epimorphism. Let N be the pushout of $i: K \to P$ and $\tilde{f}: K \to N_2$. We have the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\tilde{f}"'] & P \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & N_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N_2 \arrow[r, "j"] & N \arrow[r, "\rho"] & N_1 \arrow[r] & 0. \end{tikzcd}$$

We will prove that $\rho: N \to N_1$ induces an isomorphism $\bar{\rho}: N/cN \to N_1$, this will prove that $j(N_2) = cN$. j induces a map $\bar{j}: N_2/cN_2 \to N/cN$, and the diagram

$$\begin{tikzcd} K/cK \arrow[r, "\bar{\imath}"] \arrow[d, "\bar{\tilde{f}}"'] & P/cP \arrow[r] \arrow[d, "\bar{\lambda}"'] & N_1 \arrow[r] \arrow[d, equal] & 0 \\ N_2/cN_2 \arrow[r, "\bar{j}"] \arrow[d] & N/cN \arrow[r, "\bar{\rho}"] & N_1 \arrow[r] & 0 \\ 0 & & & \end{tikzcd}$$

commutes. We know that $K/cK \simeq \operatorname{Im}(\epsilon) \amalg \operatorname{Ker}(\delta)$. By the definition of $\bar{\tilde{f}}$, we know that $\bar{\tilde{f}}|\operatorname{Ker}(\delta) = 0$. Then $\operatorname{Im} \bar{j} = \operatorname{Im} \bar{j}\bar{\tilde{f}} = \bar{j}\bar{\tilde{f}}(\operatorname{Im} \epsilon) = \bar{\lambda}\bar{\imath}(\operatorname{Im} \epsilon) = 0$, because $\operatorname{Im} \epsilon = \operatorname{Ker} \bar{\imath}$. Therefore $\bar{j} = 0$, so $\bar{\rho}$: $N/\underline{cN} \to N_1$ is an isomorphism. If we consider N as a factor of $N_2 \times P$, then $\bar{\rho}((0, p)) = \alpha(p)$, for $p \in P$. Using then that α is an epimorphism, and that $\delta\epsilon$ is the identity, so $f\delta\epsilon = f$, it is not hard to check that the diagram

$$\begin{tikzcd} c \otimes N_1 \arrow[r, "1 \otimes \bar{\rho}^{-1}"] \arrow[d, "f"'] & c \otimes N/cN \arrow[d, "m"] \\ N_2 \arrow[r, "j"] & cN \end{tikzcd}$$

is commutative. Then $(\bar{\rho}^{-1}, j)$: $(N_1, N_2, f) \to F(N) = (N/cN, cN, m)$ is an isomorphism; so F is dense.

This proposition applies to the special case when Λ satisfies 1) and $c = a$. We find that F: mod $\Lambda \to \mathscr{C}_1$ is dense. It is not true under this hypothesis that F is full. However, if we also assume that Λ satisfies 2), that is, that Λ is stably equivalent to an hereditary algebra, then F is full. The next pages are devoted to proving this. To prove that F is full when Λ satisfies 2) we use the following result of [**4**, Lemma 5.4].

LEMMA 1.8. *Let Λ be an artin ring satisfying* 2). *If Q and P are projective Λ-modules and if Q is indecomposable and contained in rP, then Q/rQ is torsion or projective.*

The next lemma describes the image of the map ϵ defined above when Λ satisfies 1).

LEMMA 1.9. *Let Λ be an artin algebra satisfying* 1), *let N be a Λ-module, and let $P \to N$ be a projective cover for N. Consider the exact sequence*

$$0 \to V \amalg Q \xrightarrow{i} P \to N/aN \to 0,$$

with Q projective and $V \subseteq aP$. Then the image of the map ϵ: $a \otimes N/aN \to (V \amalg Q) \otimes \Lambda/a$ is $V \otimes \Lambda/a \simeq V$.

Proof. We know by Lemma 1.2 that the sequence

$$0 \longrightarrow Q/aQ \xrightarrow{\mathrm{id}_{\Lambda/a} \otimes i} P/aP \longrightarrow N/aN \longrightarrow 0$$

is exact. Since $V \subseteq a\,P$, then $V \subseteq \operatorname{Ker}\,(\mathrm{id}_{\Lambda/a} \otimes i)$. The lemma is a consequence of these two facts and of the exactness of the sequence

$$0 \longrightarrow a \otimes N/aN \xrightarrow{\epsilon} V \amalg Q/aQ \xrightarrow{\mathrm{id}_{\Lambda/a} \otimes i} P/aP \longrightarrow N/aN \longrightarrow 0$$

(see Lemma 1.6).

We can prove now the following result.

PROPOSITION 1.10. *If Λ is stably equivalent to an hereditary ring then F is full.*

Proof. Let N be a Λ-module and let $\alpha\colon P \to N$ be a projective cover. Since a is contained in the radical of Λ, then the composition $P \to N \to N/a\,N$ is a projective cover of N/aN. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & aN \arrow[r, "i"] & N \arrow[r, "\pi"] & N/aN \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "\bar{\imath}"'] \arrow[u, "\alpha_2"'] & P \arrow[r, "\pi\alpha"'] \arrow[u, "\alpha"'] & N/aN \arrow[r] \arrow[u, equal] & 0,
\end{tikzcd}$$

where $\alpha_2 = \alpha|K$ and $i\colon aN \to N$, $\bar{\imath}\colon K \to P$ are, respectively, the kernels of π and $\pi\alpha$.

Let $\rho\colon K \to K/aK$ denote the canonical epimorphism and let $f\colon a \otimes N/aN \to aN/a^2N = aN$ be the map induced by multiplication, that is, $f(a \otimes \pi(n)) = a.n$. Then $F(N) = (N/aN, aN, f)$.

Let now N, N' be Λ-modules and assume that $(g_1, g_2)\colon F(N) = (N/aN, aN, f) \to F(N') = (N'/aN', aN', f')$ is a morphism. We want to prove that $(g_1, g_2) = F(g)$, for some $g\colon N \to N'$. Let $\alpha\colon P \to N$, $\alpha'\colon P' \to N'$ be projective covers of N and N', respectively. We consider the commutative diagram

$$\text{(I)}\qquad\begin{tikzcd}
0 \arrow[r] & aN \arrow[r, "i"] & N \arrow[r, "\pi"] & N/aN \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "\bar{\imath}"] \arrow[u, "\alpha_2"'] \arrow[d, "\theta_2"'] & P \arrow[r, "\bar{\alpha}"] \arrow[u, "\alpha"'] \arrow[d, "\theta"'] & N/aN \arrow[r] \arrow[u, equal] \arrow[d, "g_1"'] & 0 \\
0 \arrow[r] & K' \arrow[r, "\bar{\imath}'"] \arrow[d, "\alpha_2'"'] & P' \arrow[r, "\bar{\alpha}'"] \arrow[d, "\alpha'"'] & N'/aN' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & aN' \arrow[r, "i'"'] & N' \arrow[r, "\pi'"'] & N'/aN' \arrow[r] & 0,
\end{tikzcd}$$

where $\theta\colon P \to P'$ is a map such that $g_1\bar{\alpha} = \bar{\alpha}'\theta$; $V \subseteq aP$, $V' \subseteq aP'$, P and P' are projective modules, and $\theta_2 = \theta|K$. To define $g\colon N \to N'$ such that $F(g) = (g_1, g_2)$ we will prove that $\alpha'\theta|\operatorname{Ker}\alpha = 0$. Then g will be the map $P/\operatorname{Ker}\alpha \to N'$ induced by $\alpha'\theta$.

We also have the map $g_2\colon aN \to aN'$. We will prove first that $g_2\alpha_2 = \alpha_2'\theta_2$. We write $K = V \amalg Q$, $K' = V' \amalg Q'$, with $V \subseteq aP$, $V' \subseteq aP'$, and Q, Q' projective (Lemma 1.2). Since Λ satisfies 2) and $Q \subseteq rP$, then we know by Lemma 1.8 that the simples in Q/rQ are torsion or projective. Then $\operatorname{Hom}_\Lambda(Q, aN') = 0$ because aN' is annihilated by b and is, therefore, a sum of torsionless nonprojective simple modules. This proves that $g_2\alpha|Q = \alpha_2'\theta_2|Q = 0$.

We will prove now that $g_2\alpha_2|V = \alpha_2'\theta_2|V$. The maps α_2 and α_2' can be factored as

$$K \xrightarrow{\rho} K/aK \xrightarrow{\bar{\alpha}_2} aN \quad \text{and} \quad K' \xrightarrow{\rho'} K'/aK' \xrightarrow{\bar{\alpha}_2'} aN',$$

respectively. Let $\bar{\theta}_2$: $K/aK \to K'/aK'$ denote the map induced by θ_2: $K \to K'$. Then to prove that $g_2\alpha_2|V = \alpha_2'\theta_2|V$ we only need to prove that $g_2\bar{\alpha}_2|\rho(V) = \bar{\alpha}_2'\bar{\theta}_2|\rho(V)$; that is, that $g_2\bar{\alpha}_2\epsilon = \bar{\alpha}_2'\bar{\theta}_2\epsilon$, since $\operatorname{Im} \epsilon = \rho(V)$. (Lemma 1.9). We have the diagram

$$\begin{tikzcd}
K/aK \arrow[dr, "\bar{\alpha}_2"] \arrow[ddd, bend right=60, "\bar{\theta}_2"'] & \\
a \otimes N/aN \arrow[u, "\epsilon"] \arrow[ur, phantom, "1"] \arrow[r, "f"] \arrow[d, "1 \otimes g_1"'] \arrow[dr, phantom, "2"] & aN \arrow[d, "g_2"] \\
a \otimes N'/aN' \arrow[r, "f'"] \arrow[d, "\epsilon'"'] & aN' \\
K'/aK' \arrow[ur, "\bar{\alpha}_2'"'] \arrow[ur, phantom, "3"] &
\end{tikzcd}$$

where the subdiagrams 1, 2, and 3 commute, and where also $\bar{\theta}_2\epsilon = \epsilon'1 \otimes g_1$, (as one can check using the definitions of the maps).

As a consequence of the commutativity of this diagram it follows that $g_2\bar{\alpha}_2\epsilon = \bar{\alpha}_2'\bar{\theta}_2\epsilon$. This proves that $g_2\alpha_2 = \alpha_2'\theta_2$. Using this equality and the commutativity of diagram (I), one can check that $\alpha'\theta|\operatorname{Ker} \alpha = 0$. Then $\alpha'\theta$ induces a map g: $N = P/\operatorname{Ker} \alpha \to N'$, and this map verifies that $F(g) = (g_1, g_2)$. This proves that F is full and completes the proof of the proposition.

2. We assume for the rest of this paper that the artin algebra Λ is stably equivalent to an hereditary artin algebra, that is, that Λ satisfies 1) and 2). We have proven then that the ring $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ is hereditary and that the functor F: $\operatorname{mod} \Lambda \to \mathscr{C}_1$ is full and dense. Moreover, M in $\operatorname{mod} \Lambda$ is such that $F(M)$ is projective if and only if M is projective. If we identify the category $\mathscr{C}$ of triples with $\operatorname{mod} \Gamma$, then F induces a functor F: $\operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$. The Γ-modules that are not in the image of this functor correspond to the triples in $\mathscr{C}$ that are not in $\mathscr{C}_1$, that is, those of the form (O, B, O), where B is a Λ/a-module. These are projective modules in $\operatorname{mod} \Gamma$. So the projective Γ-modules are all the modules that are not in the image of F and those of the form $F(P)$, for some projective Λ-module P. Let $\operatorname{mod}_P \Lambda$ and $\operatorname{mod}_P \Gamma$ denote, respectively, the full subcategories of $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$ whose objects are the modules with no nonzero projective summands. Then F induces a full dense functor, that we will also denote by F:

$$F\colon \operatorname{mod}_P \Lambda \to \operatorname{mod}_P \Gamma.$$

Let M, N be in $\operatorname{mod}_P \Lambda$. Then it is easily seen that $\operatorname{Hom}_\Gamma(F(M), F(N)) = \operatorname{Hom}_\Lambda(M, N)/S(M, N)$, where $S(M, N)$ is the subgroup of $\operatorname{Hom}_\Lambda(M, N)$ consisting of the morphisms $f\colon M \to N$ such that $\operatorname{Im} f \subseteq aN$. The following lemma will prove that F induces an equivalence $\underline{F}\colon \underline{\operatorname{mod}}\ \Lambda \to \operatorname{mod}_P \Gamma$.

LEMMA 2.1. *If M, N are in $\operatorname{mod}_P \Lambda$, then $S(M, N)$ is equal to $P(M, N) = \{f \in \operatorname{Hom}_\Lambda(M, N)\colon f$ factors through a projective module$\}$.*

Proof. Let $f \in S(M, N)$. Then $\operatorname{Im}(f) \subseteq aN$, so f factors as

$$M \xrightarrow{\pi} M/aM \xrightarrow{\bar{f}} aN \subset N,$$

where π is the canonical epimorphism. Let $\alpha\colon P \to N$ be the projective cover of N. Then $\alpha|aP\colon aP \to aN$ is an epimorphism of Λ/b-modules. Since Λ/b is a semisimple ring, $\alpha|aP$ splits. Let $\phi\colon aN \to aP$ be such that $\alpha|aP.\ \phi = \mathrm{id}_{aN}$. Since ϕ is a Λ/b-homomorphism, then ϕ is also a Λ-homomorphism. We have the diagram

$$\begin{tikzcd}
M \arrow[r, "\pi"] & M/aM \arrow[r, "\bar{f}"] & aN \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, shift right, "\phi"'] & N \\
& & aP \arrow[r, phantom, "\underset{i}{\subset}"] \arrow[u, shift right, "\alpha|aP"'] & P \arrow[u, "\alpha"']
\end{tikzcd}$$

Then $f = i\bar{f}\pi = i(\alpha|aP \cdot \phi) \cdot \bar{f}\pi = \alpha(i\phi\bar{f}\pi)$, so f factors through P. This proves that $S(M, N) \subseteq P(M, N)$.

Let now $f \in P(M, N)$. Then f factors through a projective P, i.e., there is a commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & N. \\
& P \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Since M is in $\operatorname{mod}_P \Lambda$, the image of α does not contain projective summands. Then $\operatorname{Im}(\alpha)$ is a sum of simple torsionless nonprojective Λ-modules, because Λ satisfies 1). So $\operatorname{Im} \alpha \subseteq \tau_a(P) = aP$ (see Lemma 1.1). Thus $\operatorname{Im} f = \operatorname{Im} \beta\alpha = \beta(\operatorname{Im} \alpha) \subseteq \beta(aP) \subseteq aN$, that is, $f \in S(M, N)$, and this proves that $P(M, N) \subseteq S(M,N)$.

Since Γ is an hereditary ring, a map $f\colon M \to N$ between M and N in $\operatorname{mod}_P \Gamma$ factors through a projective if and only if $f = 0$. So $\operatorname{mod}_P \Gamma = \underline{\operatorname{mod}}\ \Gamma$. We obtain then the main result of this paper.

THEOREM 2.2. *Let Λ be an artin algebra stably equivalent to an hereditary algebra. Let $F\colon \operatorname{mod}\ \Lambda \to \operatorname{mod} \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ be the functor defined above. Then F induces an equivalence of categories $\underline{F}\colon \underline{\operatorname{mod}}\ (\Lambda) \to \underline{\operatorname{mod}}\ (\Gamma)$.*

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We give now an example describing the ring Γ for some ring Λ which is stably equivalent to a hereditary ring, but which is not hereditary and not of radical square zero.

Example 2.3. Let K be a field and let Λ be the subring of $M_{3\times 3}(K)$ defined by

$$\Lambda = \left\{\begin{bmatrix} \alpha & 0 & 0 \\ a_1 & a_2 & 0 \\ a_3 & a_4 & \alpha \end{bmatrix} : \alpha, a_i \in K\right\}.$$

(See [**1**, Example 3.1] or [**4**, Example 5.19].)

The indecomposable projective Λ-modules are $P_1 = \Lambda.e$, $P_2 = \Lambda.(1 - e)$, where

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $S_1 = P_1/rP_1$, $S_2 = P_2/rP_2$. The only proper submodules of P_1 are P_2 and S_1, and the only proper submodule of P_2 is S_1. So Λ is not hereditary, and $r^2 \neq 0$. However, Λ satisfies 1) and 2). The two sided ideal a is

$$a = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & a_4 & 0 \end{bmatrix}, \quad a_3, a_4 \in K\right\},$$

so

$$\Lambda/a \simeq \left\{\begin{bmatrix} a_1 & 0 \\ a_2 & a_3 \end{bmatrix}, \quad a_1, a_2, a_3 \in K\right\}, \quad \Lambda/b \simeq k \quad \text{and}$$

$$\Gamma \simeq \left\{\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix}, \quad a_i \in K, i = 1, \ldots, 6\right\}.$$

3. We devote this section to proving that if Λ has no semisimple summands, then $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ also has no semisimple summands and is therefore, the only hereditary artin algebra with no semisimple summands stably equivalent to Λ. (See [**1**, ch. III, Th. 2.1].) Since an artin algebra Λ has no semisimple summands when there are no projective injective simple Λ-modules, we will first describe the projective and the injective Λ-modules. Then it will follow easily that the simple projective modules are not injective.

The following general fact will be helpful to describe the injective and projective Γ-modules.

LEMMA 3.1. *Let F: $\mathscr{C} \to \mathscr{D}$ be a full dense functor between two categories $\mathscr{C}$ and $\mathscr{D}$.*

a) *Let F satisfy: If* $f: A \to B$ *is a morphism in* $\mathscr{C}$ *such that* $F(f)$ *is a monomorphism, then* f *is a monomorphism. Then, for every injective object in* $\mathscr{C}$, $F(I)$ *is injective in* $\mathscr{D}$.

b) *Let F satisfy: If* $f: A \to B$ *is a morphism in* $\mathscr{C}$ *such that* $F(f)$ *is an epimorphism, then* f *is an epimorphism. Then* $F(P)$ *is projective in* $\mathscr{D}$, *for every projective object* P *in* $\mathscr{C}$.

Proof. The conclusions follow in a straight forward manner from the definitions.

The functor F: mod $\Lambda \to$ mod Γ that we are considering satisfies the property a) of the lemma. Therefore if $S_1, \ldots, S_n$ is a complete set of nonisomorphic simple Λ-modules and $I_0(S_i)$ denotes the injective envelope of S_i, then $F(I_0(S_i)) = (I_0(S_i)/aI_0(S_i), aI_0(S_i), m_i)$, $i = 1, \ldots, n$, are injective Γ-modules. To finish the description of the injective Γ-modules, we need the following result of [**4**, Lemma 5.5].

LEMMA 3.2. *Assume that* Λ *satisfies* 1), *and let* $I_0(S)$ *be the injective envelope of the simple* Λ-*module* S. *Then* $aI_0(S) = S$ *if* S *is nonprojective torsionless, and* $aI_0(S) = 0$ *otherwise.*

For a Λ/a-module M, we denote by $I_{\Lambda/a}(M)$ the injective envelope of M. Then one can prove the following lemma whose proof we omit.

LEMMA 3.3. *Assume that* Λ *satisfies* 1) *and* 2) *and that* S *is a simple* Λ-*module. Then* $(I_{\Lambda/a}(S), 0, 0)$ *is injective in* mod Γ.

When S is torsion or projective then $a \cdot I_0(S) = 0$ (Lemma 3.2), and $I_{\Lambda/a}(S) = \{x \in I_0(S): ax = 0\} = I_0(S)$, so $F(I_0(S)) = (I_{\Lambda/a}(S), 0, 0)$. Combining the preceding results, we obtain the following description of the injective modules.

PROPOSITION 3.4. *Assume that* Λ *is stably equivalent to an hereditary ring. Then the indecomposable injective* Γ-*modules are the modules of the forms* $(I_{\Lambda/a}(S), 0, 0)$, *where* S *is a simple* Λ-*module, and* $(I_0(S)/S, S, m)$, *where* m *is the multiplication map and* S *is a torsionless nonprojective simple* Λ-*module.*

We can prove now the minimality of Γ.

THEOREM 3.5. *Let* Λ *be an artin algebra with no semisimple summands stably equivalent to an hereditary ring. Then* $\Gamma = \begin{bmatrix} \Lambda/a & 0 \\ a & \Lambda/b \end{bmatrix}$ *has no semisimple summands and is, therefore, the only hereditary artin algebra stably equivalent to* Λ *with no semisimple summands.*

Proof. Let A be a simple Γ-module. If $A = (O, S, O)$, where S is simple and torsionless nonprojective, then $I_0(A) = (I_0(S)/S, S, m)$. Since $aI_0(S) = S$ and $aS = 0$ (Lemma 3.2), then $I_0(S)/S \neq 0$. Therefore A is not injective.

Assume now that $A = (S, 0, 0)$, where S is simple. The projective cover of $(S, 0, 0)$ is $F(P_0(S)) = (P_0(S)/aP_0(S), aP_0(S), m)$, where $P_0(S)$ denotes the projective cover of S and m the map induced by multiplication. If $A = (S, 0, 0)$ is projective, then $aP_0(S) = 0$ and $P_0(S) = S$, and so S is projective. We know then by Proposition 3.4 that $I_0(A) = (I_{\Lambda/a}(S), 0, 0)$. If A is also injective, then $I_{\Lambda/a}(S) = S$, and so S is projective injective, which contradicts the hypotheses that Λ has no semisimple summands.

This proves that Γ has no simple injective projective modules and completes the proof of the theorem.

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