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ON Σ-FINITE FAMILIES

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Let $\mathscr{U}$ be a family of subsets of a topological space X. We do not require $\mathscr{U}$ to be a covering of X, nor do we assume that the members of $\mathscr{U}$ are necessarily open. In this paper we shall assume that $\mathscr{U}$ is of a special sort, which we call Σ-finite. We show that a Σ-finite family is both locally finite and star-finite, and in particular that an open covering $\mathscr{U}$ of X is Σ-finite if and only if it is star-finite. We then prove that every Σ-finite family $\mathscr{U}$ is σ-discrete, so that in particular, every star-finite open covering of X is σ-discrete. There seems to be some applications of this fact.

We begin with some familiar definitions. A family $\mathscr{U}$ in a topological space X is called *locally finite* (*discrete*) if and only if every point of X has a neighborhood which intersects at most finitely many (one) of the members of $\mathscr{U}$. $\mathscr{U}$ is called *star-finite* if and only if each member of $\mathscr{U}$ intersects only finitely many other members of $\mathscr{U}$. $\mathscr{U}$ is called *σ-locally finite* (*σ-discrete*) if and only if it is the union of at most countably many locally finite (discrete) subfamilies. Similarly, $\mathscr{U}$ is called *σ-star-finite* if and only if $\mathscr{U}$ is the union of at most countably many star-finite subfamilies.

Before giving the definition of a Σ-finite family, we need some notation. Given a point x in the space X, and a neighborhood V_x of x, denote by $\mathscr{U}_x$ the (possibly empty) family consisting of all members of $\mathscr{U}$ which intersect V_x. (More precisely, we should denote this family by $\mathscr{U}_{V_x}$, since it depends, in general, upon the neighborhood V_x. However, in the interests of simpler notation we prefer to use simply $\mathscr{U}_x$, since this should lead to no confusion once this is understood.)

Definition 1. A family $\mathscr{U}$ of subsets of a topological space X is called *Σ-finite* (in X) if and only if the neighborhoods V_x can be chosen in such a way that for each $U \in \mathscr{U}$, the collection

$$\bigcup \{\mathscr{U}_x\colon U \in \mathscr{U}_x\}$$

is finite.

LEMMA 1. *Every Σ-finite family in a topological space is both locally finite and star-finite.*

Proof. Let $\mathscr{U}$ be a Σ-finite family in a topological space X.

Then $\mathscr{U}$ is locally finite. For the Σ-finiteness of $\mathscr{U}$ implies that for each $x \in X$, the family $\mathscr{U}_x$ is finite, which means that $\mathscr{U}$ is locally finite.

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Next, we show that $\mathscr{U}$ is star-finite. Suppose on the contrary that $\mathscr{U}$ is not star-finite. Then there exists a $U \in \mathscr{U}$ which intersects at least a countable number of distinct members of $\mathscr{U}$, say $\{U_i: i \in N\}$, where N denotes the set of all positive integers. Choose an $x_i \in U \cap U_i$ for each i. Then $U \in \mathscr{U}_{x_i}$ for each i and so

$$\{U_i: i \in N\} \subset \bigcup \{\mathscr{U}_{x_i}: i \in N\} \subset \bigcup \{\mathscr{U}_x: U \in \mathscr{U}_x\};$$

hence the latter family is infinite. Thus we conclude that $\mathscr{U}$ is not Σ-finite.

The following example clarifies the position of the Σ-finite families:

Example 1. Let X be the plane E^2 with the usual topology, let U be the x-axis with the points $(i, 0)$, $(i \in N)$ deleted, and for each $i \in N$ let U_i be the vertical line intersecting the x-axis in the point $(i, 0)$. Then the family $\mathscr{U}$ consisting of U and the lines U_i, $i \in N$, is locally finite and star-finite but not Σ-finite. For U belongs to each of the families $\mathscr{U}_{x_i}$, where x_i is the point $(i, 0)$ and hence

$$\bigcup \{\mathscr{U}_x: U \in \mathscr{U}_x\}$$

is infinite.

It follows that the collection of all Σ-finite families in a space X is properly contained in the collection of all locally finite, star-finite families. However, if we restrict ourselves to *open coverings* of X, we find that the Σ-finite open coverings of X coincide with the star-finite open coverings of X.

THEOREM 1. *Let $\mathscr{U}$ be an open covering of the space X. Then $\mathscr{U}$ is Σ-finite if and only if $\mathscr{U}$ is star-finite.*

Proof. By Lemma 1, every Σ-finite family is star-finite. Hence we need only to prove that if $\mathscr{U}$ is a star-finite open covering of X, then $\mathscr{U}$ is Σ-finite.

For each $x \in X$, define

$$V_x = \bigcap \{U: U \in \mathscr{U} \text{ and } x \in U\}.$$

Since $\mathscr{U}$ is a star-finite open covering of X, V_x is a neighborhood of x. With these neighborhoods V_x, $\mathscr{U}$ is Σ-finite. For consider any $U \in \mathscr{U}$. We will show that the family

$$\bigcup \{\mathscr{U}_x: U \in \mathscr{U}_x\}$$

is contained in the family

$$\{U' \in \mathscr{U}: \text{there exists } U'' \in \mathscr{U} \text{ with } U'' \cap U \neq \emptyset,\ U'' \cap U' \neq \emptyset\}.$$

This will complete the proof, since the latter collection is finite by the star-finiteness of $\mathscr{U}$.

Given any $U' \in \bigcup \{\mathscr{U}_x: U \in \mathscr{U}_x\}$, there exists an $x \in X$ such that U' and U are both members of $\mathscr{U}_x$, that is, $V_x \cap U \neq \emptyset$ and $V_x \cap U' \neq \emptyset$. By the definition of V_x, the required $U'' \in \mathscr{U}$ exists, and the proof is complete.

We now come to the main theorem of the paper.

THEOREM 2. *Every Σ-finite family in a topological space is σ-discrete.*

Proof. Let $\mathscr{U}$ be a Σ-finite family in a topological space X. By Lemma 1, $\mathscr{U}$ is both locally finite and star-finite. Hence, given any point $x \in X$, there exists a neighborhood V_x of x which intersects at most finitely many members of $\mathscr{U}$. As before, $\mathscr{U}_x$ denotes the collection of all members of $\mathscr{U}$ which intersect V_x:

$$\mathscr{U}_x = \{U_{x,\alpha} \colon \alpha \in F_x\}$$

where F_x is a finite set which indexes $\mathscr{U}_x$.

Roughly speaking, we wish to choose the sets F_x to be collections of positive integers in such a way that if a set $U_{x,\alpha}$ appears in more than one of the collections $\mathscr{U}_x$, then it has the same index in all of these collections, and if a member $U_{x,\alpha}$ of $\mathscr{U}_x$ intersects a member of a different collection $\mathscr{U}_y$, but does not equal this member, then these two members have different indices.

We shall use induction. Let $(X, <)$ be a well-ordering of X, and let x_0 be the first element of X with respect to this ordering. Consider the proposition

$P(x)$: F_x can be chosen to be a collection of distinct positive integers so as to satisfy the three conditions:
(1) If $U_{x,i} \in \mathscr{U}_x$ and $U_{x,i} = U_{y,j}$ for some $y < x$, then $i = j$.
(2) If $U_{x,i}, U'_{x,i} \in \mathscr{U}_x$, then $U_{x,i} = U'_{x,i}$.
(3) If for some $y < x$ we have $U_{x,i} \cap U_{y,j} \neq \emptyset$ and $U_{x,i} \neq U_{y,j}$, then $i \neq j$.

First, index the members of U_{x_0} with distinct positive integers. Then $P(x_0)$ is true because x_0 is the first element of X. Now suppose that $P(y)$ is true for all $y < x$. We shall show that $P(x)$ is true.

Consider the family

$$\mathscr{U}_x = \{U_{x,\alpha} \colon \alpha \in F_x\}.$$

Let us decompose $\mathscr{U}_x$ into three subfamilies:

$$\mathscr{U}_x^{(1)} = \{U_{x,\alpha} \colon U_{x,\alpha} \in \mathscr{U}_y \text{ for some } y < x\},$$
$$\mathscr{U}_x^{(2)} = \{U_{x,\alpha} \colon U_{x,\alpha} \notin \mathscr{U}_y \text{ for all } y < x \text{ but } U_{x,\alpha} \cap U_{y,j} \neq \emptyset \text{ for some } y < x \text{ and } j \in N\},$$
$$\mathscr{U}_x^{(3)} = \mathscr{U}_x - [\mathscr{U}_x^{(1)} \cup \mathscr{U}_x^{(2)}].$$

We first assign positive integral indices to the members of $\mathscr{U}_x^{(1)}$ as follows: if $U_{x,\alpha} \in \mathscr{U}_x^{(1)}$, then there exists a $y < x$ and a $j \in N$ such that $U_{x,\alpha} = U_{y,j}$. In this case, we replace α by j. Then the induction hypothesis insures that $U_{x,j} = U_{y,j}$ for any $y < x$ such that $U_{x,j} \in \mathscr{U}_y$. The remaining members of $\mathscr{U}_x^{(1)}$ are treated similarly.

Next, we assign indices to the members of $\mathscr{U}_x^{(2)}$. If $U_{x,\alpha} \in \mathscr{U}_x^{(2)}$, then this member of $\mathscr{U}$ appears for the first time, so that it has not been indexed pre-

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viously. However, there exists at least one $y < x$ and a $j \in N$ such that $U_{x,\alpha} \cap U_{y,j} \neq \emptyset$. Let $\{j_1, \ldots, j_n\}$ be the collection of all indices j_k such that there exists a $y_k < x$ for which $U_{x,\alpha} \cap U_{y_k,j_k} \neq \emptyset$ $(k = 1, \ldots, n)$. This collection is finite because the family $\mathscr{U}$ is star-finite.

It is at this stage that the Σ-finiteness of $\mathscr{U}$ is needed, for the following reason: Suppose at this point we assigned $U_{x,\alpha}$ some index r. Of course we would choose $r \neq j_k$ $(k = 1, \ldots, n)$, but this in general would not be enough to avoid later trouble. For, while indexing a collection $\mathscr{U}_y$, $y > x$, it could happen that $U_{x,r} = U_{y,\alpha} \in \mathscr{U}_y$, but that $\mathscr{U}_y$ has a member *already* having the index r by virtue of belonging to the collection $\mathscr{U}_y^{(1)}$. This would put us in an untenable position.

So instead of the just mentioned procedure, we proceed as follows: Since $\mathscr{U}$ is Σ-finite, the family

$$\bigcup \{\mathscr{U}_y \colon U_{x,\alpha} \in \mathscr{U}_y\}$$

is finite. Hence, the collection

$$\bigcup \{\mathscr{U}_y \colon y > x \text{ and } U_{x,\alpha} \in \mathscr{U}_y\} \cap \bigcup \{\mathscr{U}_y \colon y < x\}$$

is finite (or empty). These members of $\mathscr{U}$ have already been indexed by, say, $k_1, \ldots, k_m$. We now replace α by any positive integer i different from $j_1, \ldots j_n$, $k_1, \ldots, k_m$, and different from the indices already chosen for other members of $\mathscr{U}_x$. The remaining members of $\mathscr{U}_x^{(2)}$ are handled in the same way.

It follows that no member of $\mathscr{U}_x^{(2)}$ can have the same index as any member of a $\mathscr{U}_y$ $(y < x)$ which it intersects (but does not equal).

Finally, we assign indices to the members of $\mathscr{U}_x^{(3)}$. If $U_{x,\alpha} \in \mathscr{U}_x^{(3)}$, then $U_{x,\alpha}$ appears for the first time in $\mathscr{U}_x$, and $U_{x,\alpha}$ does not intersect any member of a previous family. For the same reasons as above, we again consider the collection

$$\bigcup \{\mathscr{U}_y \colon y > x \text{ and } U_{x,\alpha} \in \mathscr{U}_y\} \cap \bigcup \{\mathscr{U}_y \colon y < x\}$$

and replace α by any positive integer different from the indices occurring in this collection, and also different from any index already used for some other member of $\mathscr{U}_x$. The remaining members of $\mathscr{U}_x^{(3)}$ are handled in a similar manner.

It should be clear from the description of the indexing procedure that at most one member of the family $\mathscr{U}_x$ will be assigned a given index.

This procedure results in a new indexing set for $\mathscr{U}_x$, which we continue to call F_x. The new indexing set satisfies the proposition $P(x)$, so the induction is complete. Therefore, the proposition is true for every $x \in X$.

Now define

$$\mathscr{U}_n = \{U_{x,n} \colon x \in X_n\} \quad \text{for each } n \in N,$$

where

$$X_n = \{x \colon U_{x,n} \text{ exists}\}.$$

Then

$$\mathscr{U} = \bigcup \{\mathscr{U}_n : n \in N\}$$

and each $\mathscr{U}_n$ is discrete. For let $n \in N$ be fixed and consider any $x \in X$.

If $x \notin X_n$, then there is no set indexed $U_{x,n}$, so V_x intersects no member of $\mathscr{U}_n$. For, if $V_x \cap U_{y,n} \neq \emptyset$ for some $y \neq x$ then it follows that $U_{y,n} \in \mathscr{U}_x$, and therefore $U_{y,n} = U_{x,j}$ for some $j \neq n$. However, property (1) of our indexing procedure shows that this is impossible.

If $x \in X_n$, then V_x intersects precisely one member of $\mathscr{U}_n$, namely $U_{x,n}$. For, suppose $V_x \cap U_{y,n} \neq \emptyset$ where $U_{y,n} \neq U_{x,n}$. Then $y \neq x$ and $U_{y,n} = U_{x,i}$ for some $i \neq n$. But again property (1) of our indexing procedures gives us a contradiction. The proof of the theorem is now complete.

The following examples show that a family which is not Σ-finite need not be σ-discrete.

Example 2. Let X be the set of real numbers with the usual topology and let

$$\mathscr{U} = \{\{x\} : x \text{ is irrational}\}.$$

Then $\mathscr{U}$ is star-finite, but not locally finite. Hence, $\mathscr{U}$ is not Σ-finite. $\mathscr{U}$ cannot be σ-discrete, since every neighborhood of any point of X intersects uncountably many members of $\mathscr{U}$.

Example 3. Let I be the closed unit interval with the discrete topology, and let $X = I \times I$ with the product topology. Let $\mathscr{U}$ consist of all Γ-shaped figures in X with vertex on the diagonal. Each pair of members of $\mathscr{U}$ intersect in just one point, which is an open set of X, so $\mathscr{U}$ is locally finite. $\mathscr{U}$ is not star-finite because each member of $\mathscr{U}$ intersects infinitely many other members of $\mathscr{U}$. Hence, $\mathscr{U}$ is not Σ-finite. $\mathscr{U}$ is not σ-discrete because there are uncountably many members of $\mathscr{U}$, and each member intersects all the others.

Let us call a family $\mathscr{U}$ in a topological space X *σ-Σ-finite* if and only if it is the union of a countable number of subfamilies $\mathscr{U}_i$, each of which is Σ-finite in the space X, i.e.,

$$\mathscr{U} = \bigcup \{\mathscr{U}_i : i \in N\}$$

where each $\mathscr{U}_i$ is Σ-finite in X. Then we have

COROLLARY 1. *A family $\mathscr{U}$ in a topological space X is σ-Σ-finite if and only if it is σ-discrete.*

Proof. Suppose $\mathscr{U}$ is σ-Σ-finite. Then as above,

$$\mathscr{U} = \bigcup \{\mathscr{U}_i : i \in N\}$$

where each $\mathscr{U}_i$ is Σ-finite in X. By Theorem 2, each $\mathscr{U}_i$ is σ-discrete, so $\mathscr{U}$ is σ-discrete.

Conversely, a discrete family is Σ-finite, so a σ-discrete family is σ-Σ-finite.

By Theorem 1, an open covering of a space X is Σ-finite if and only if it is star-finite. Thus we have

COROLLARY 2. *Every star-finite open covering of a space X is σ-discrete.*

Nagata [**2**, p. 201] calls an open basis $\mathscr{B}$ for a topological space X a *σ-star-finite open basis* if and only if $\mathscr{B}$ is the union of a countable number of star-finite open *coverings* of X. Hence Corollary 2 gives us

COROLLARY 3. *Every σ-star-finite open basis for a topological space is σ-discrete.*

We next consider families of *closed* subsets of a topological space X. First, we show

THEOREM 3. *A family $\mathscr{U}$ of closed subsets of a topological space X is Σ-finite if and only if $\mathscr{U}$ is both locally finite and star-finite.*

Proof. By Lemma 1, every Σ-finite family is both locally finite and star-finite. Hence we need only prove that if $\mathscr{U}$ is locally finite and star-finite, then $\mathscr{U}$ is Σ-finite.

Given any $x \in X$, there is a neighborhood V_x' of x intersecting at most finitely many members of $\mathscr{U}$. By the local finiteness of $\mathscr{U}$,

$$X - \bigcup \{U: U \in \mathscr{U} \text{ and } x \notin U\}$$

is a neighborhood of x, so

$$\begin{aligned} V_x &= V_x' \cap [X - \bigcup \{U: U \in \mathscr{U} \text{ and } x \notin U\}] \\ &= V_x' - \bigcup \{U: U \in \mathscr{U} \text{ and } x \notin U\} \end{aligned}$$

is a neighborhood of x. Let $\mathscr{U}_x$ be the family of all members of $\mathscr{U}$ which V_x intersect. Then clearly, $\mathscr{U}_x$ is finite. Next, we note that for a given $U \in \mathscr{U}$,

$$U \in \mathscr{U}_x \quad \text{if and only if} \quad x \in U.$$

For if $x \in U$, then $V_x \cap U \neq \emptyset$, so $U \in \mathscr{U}_x$. Conversely, if $x \notin U$, then $V_x \cap U = \emptyset$, so $U \notin \mathscr{U}_x$. Hence a given $U \in \mathscr{U}$ can belong to only a finite number of distinct collections $\mathscr{U}_x$, because $\mathscr{U}$ is star-finite. Therefore

$$\bigcup \{\mathscr{U}_x: U \in \mathscr{U}_x\}$$

is finite, i.e., $\mathscr{U}$ is Σ-finite.

We immediately have, by Theorem 2,

COROLLARY 4. *Every star-finite, locally finite collection of closed subsets of a topological space is σ-discrete.*

Let us now consider some applications of some of the above results. We first recall some terminology. A space X is called *screenable* if and only if every open covering of X has a σ-disjoint open refinement, i.e., a refinement which is the union of a countable number of subfamilies, each consisting of pairwise disjoint open sets. X is said to be *strongly screenable* if and only if every open covering has a σ-discrete open refinement, and X is called *perfectly screenable* if and only if X has a σ-discrete open basis. Finally, X is called *strongly paracompact* if and only if every open covering has a star-finite open refinement.

Heath [**1**, p. 768] has proven that a space X is screenable if and only if every open covering has a σ-star-countable open refinement. By Corollary 2 we easily obtain the following special case of Heath's theorem.

COROLLARY 5. *A space X is screenable if and only if every open covering of X has a σ-star-finite open refinement.*

Proof. Let $\mathscr{U}$ be an open covering of X, and suppose that $\mathscr{U}$ has a σ-star-finite open refinement

$$\mathscr{V} = \bigcup \{\mathscr{V}_i: i \in N\}$$

where each $\mathscr{V}_i$ is star-finite. By Corollary 2, each $\mathscr{V}_i$ is σ-discrete (in $\bigcup \{V: V \in \mathscr{V}_i\}$), and so σ-disjoint. Therefore, $\mathscr{V}$ is σ-disjoint, and we conclude that X is screenable.

The converse is obvious.

Among *regular* spaces, the following characterization of paracompact spaces is well-known (see [**2**, p. 153]).

THEOREM 4. *A regular space X is paracompact if and only if X is strongly screenable.*

Again by Corollary 2, we are able to obtain a characterization of strongly paracompact spaces (regularity not assumed).

Let us call a covering of a space X *star-σ-discrete* if and only if it is σ-discrete, and a member of any of the discrete subfamilies intersects at most a finite number of the members of the remaining discrete subfamilies. Call a space X *star-strongly screenable* if and only if every open covering of X has a star σ-discrete open refinement.

THEOREM 5. *A space X is strongly paracompact if and only if X is star-strongly screenable.*

Proof. Let X be strongly paracompact, and let $\mathscr{U}$ be any open covering of X. By definition of strongly paracompact, $\mathscr{U}$ has a star-finite open refinement $\mathscr{V}$. By Corollary 2, $\mathscr{V}$ is σ-discrete, so $\mathscr{V}$ is star-σ-discrete.

The converse is clear.

Finally, let us note that a restatement of Corollary 3 is

THEOREM 6. *Every space with a σ-star-finite open basis is perfectly screenable.*

This is well-known in the case of *regular* spaces, since such a space is metrizable.

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