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k-11-REPRESENTATIONS OF GRAPHS, REVISITED

ZION HEFTY[1], PAUL HORN[1,2] AND COLBY MUIR[3]

[1] *Department of Mathematics, University of Denver, Denver, Colorado, USA* (zion.hefty@du.edu; paul.horn@du.edu)

[2] *Department of Mathematicsand Applied Mathematics, University of Johannesburg, Johannesburg, South Africa*

[3] *Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia, USA* (cmuir@gsu.edu)

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Abstract In this paper, we study the existence of k-11-representations of graphs. Inspired by work on permutation patterns, these representations are ways of representing graphs by words where adjacencies between vertices are captured by patterns in the corresponding letters. Our main result is that all graphs are 1-11-representable, answering a question originally raised by Cheon et al. in 2018 and repeated in several follow-up papers – including a very recent paper, where it was shown that all graphs on at most 8 vertices are 1-11-representable. Moreover, we prove that all graphs are permutationally 1-11-representable – that is representable as the concatenation of permutations of the vertices – answering the existence question in extremely strong fashion. Our construction leads to nearly optimal bounds on the length of the words, as well. It can, moreover, be adapted to represent all acyclic orientations of graphs; this generalizes the fact that word-representations capture semi-transitive orientations of graphs. Our construction also adapts easily to other $k \geq 2$ as well, giving representations using a linear number of permutations when the best known previous bounds used a quadratic number. Finally, we also consider the (non-)existence of 'even–odd'-representations of graphs. This answers a question raised by Wanless after a conference talk in 2018.

Keywords: Word-representations; 1-11-representations; permutations

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1. Introduction

An important topic, on the boundary of graph theory and computer science, is finding efficient ways of encoding graphs. One possible method is to encode graphs as strings. Of course there are many potential methods to do this, but a natural way is to consider strings whose alphabet is the vertex set and the (non-)adjacencies are encoded based on properties of the strings.

One fairly natural way to do this, first formally studied by Kitaev and Pyatkin [8] and motivated by earlier work of Kitaev and Seif from [9], is to encode edges between two vertices x and y by (strict) alternation of the characters x and y in the word. This gives rise to the *word-representation* of a graph. Unfortunately, not all graphs can be represented in this way, but the class of word-representable graphs has been well-studied in the literature.

Motivated by work on permutation patterns, a more general class of representations are the k-11-representations. These representation were first suggested by Remmel, but first studied in detail by Cheon et al. in [2] with follow-up work in several papers, including [4] and the very recent [1]. A more formal definition is given below, but here adjacency is again encoded by alternation, but up to k violations of being strict are allowed. The 0-11-representable graphs, then, are word-representable graphs and it was shown in [2] that all graphs are 2-11-representable. The question of whether all graphs are 1-11-representable was left as an open question. Indeed in the recent [1] this problem was settled for graphs on at most 8 vertices only after some effort.

In this paper, we settle the question affirmatively, for all n, in a strong fashion. Our main results are as follows:

- In Theorem 2.2, we prove that every graph is *permutationally* 1-11-representable, that is, it can be 1-11-represented by a word which is the concatenation of permutations of vertices. Our proof, which is constructive, also gives a variety of extensions to more restrictive representations and also to orientations of graphs. Not only does this answer the open question in [2] of the 1-11-representability of all graphs, but the (stronger) question asked in [2] about the existence of permutational representations.
- In Theorem 2.7, we prove the impossibility of even–odd-representations for almost all graphs – that is, representations by words where the *parity* and not the number of occurences of the 11 pattern determine adjacency. This answers a question raised by Ian Wanless in a 2018 conference[1] after a talk on the existence of 2-11-representations. We completely characterize the graphs that are permutationally representable in such a way (see Theorem 2.8.)
- We further study the length of these representations. We prove as Theorem 3.6 that every graph has a 1-11-representation consisting of the concatenation of $O(n)$ permutations, while proving that some graphs require $\Omega(n/\log n)$ permutations to permutationally represent, so our bounds are tight within a factor of $\log(n)$.
- We, in fact, present a slightly stronger lower bound (Theorem 3.1) showing that there are graphs that require $\Omega(n^2/\log n)$ characters in any 1-11-representation – whether permutational or not.
- Related to this lower bound: it is natural to suspect that graphs which are hard to word-represent are hard to 1-11-represent. We show this is not always the case. We

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show (see Theorem 3.4) that the crown graph $H_{n,n}$ – obtained by deleting a perfect matching from $K_{n,n}$ – is permutationally 1-11-representable by 4 permutations, independent of n, despite the fact that it requires $\lceil n/2 \rceil$ copies of each letter in any word-representant.
- Our construction (i.e. Theorem 3.6) easily adapts to give permutational k-11-representations for any $k \geq 1$ using $O(n)$ permutations. The previous construction from [2] building 2-11-representations for arbitrary graphs uses $\Omega(n^2)$ permutations, so our construction yields a significant improvement. The lower bound we present (Theorem 3.1) also adapts, so this is also tight within a logarithmic factor.

As a reminder of asymptotic notation, for functions $f(n)$ and $g(n)$, we say $f(n) = O(g(n))$ if there exists a constant $C > 0$ so that $|f(n)| \leq C|g(n)|$ for n sufficiently large. Meanwhile, $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$ as $n \to \infty$; in particular if $f(n) = o(1)$, then $f(n) \to 0$. A function $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$. All logarithms in the paper are taken to be the base-2 logarithm, unless otherwise specified. (In this paper, the base of the logarithm seldom matters as the logarithms are within $O(\cdot)$ notation, and changing the base only changes the implied constant.)

The remainder of the paper is organized as follows: in § 2, we give a simple construction which can be used to construct a 1-11-representation of all graphs, and discuss some extensions to more general representations and also to representing acyclic orientations of graphs. We further consider the existence of even–odd representations of graphs in this section. Then in § 3, we study the lengths of representations, both presenting our lower bounds and giving the optimizations to the main construction that lead to improved upper bounds.

2. 1-11-representations and even–odd-representations of graphs

Let $k \geq 0$. A graph $G = (V, E)$ is k-11-*representable* if there is a word w on alphabet V such that $u \sim v$ if and only if the number of occurrences of factors of the restriction $w|_{uv}$ that are uu or vv is less than or equal to k. We call the existence of such a factor a *defect*. A graph is *permutationally* k-11-*representable* if it is k-11-representable by a word that is a concatenation of permutations of the vertex set.

Before we begin with the thrust of the section – showing that every graph is permutationally 1-11-representable – we quickly present the following helpful observation.

Lemma 2.1. *Let w be a word 1-11-representing a graph $G = (V, E)$ and let π be the permutation of V consisting of the final occurrences of each letter in w. Then $w\pi$ 1-11-represents G.*

Proof. Let $u, v \in V$. If $u \sim v$ then u and v have at most one defect in w, and appending π adds alternation rather than a defect. If $u \not\sim v$ then in w, u and v have more than one defect in w and thus in $w\pi$ as well. Thus appending π preserves both edges and non-edges. □

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2.1. Constructing 1-11-representations

We begin with the main construction showing all graphs have 1-11-representations. Though there are optimizations that can be done, we present it here in such a way to easily admit generalizations that we then consider.

Theorem 2.2 *Let $G = (V, E)$ be a graph. Then there is a word w over alphabet V permutationally* 1-11*-representing G.*

Proof. Order the vertices of V arbitrarily as $v_1, \ldots, v_n$. We prove that the induced subgraph on $v_1, \ldots, v_t$ is permutationally 1-11-representable by induction on t. For the base case $t = 1$, the word v_1 is a permutational 1-11-representation of the induced subgraph on v_1.

Now suppose $t \geq 2$.

Let $v = v_t$. By the inductive hypothesis there exists a word w' over alphabet $\{v_1, \ldots, v_{t-1}\}$ permutationally 1-11-representing the induced subgraph on $\{v_1, \ldots, v_{t-1}\}$: $w' = \pi_1\pi_2 \cdots \pi_k$. Write the order of the vertices in the final permutation as $\pi_k = x_1x_2 \cdots x_{t-1}$; note that $x_1, x_2, \ldots, x_{t-1}$ is a permutation of $v_1, \ldots, v_{t-1}$.

For $1 \leq i \leq t-1$ we build a concatenation of 1 or 3 permutations based on x_i. Let

$$\sigma_i = \begin{cases} x_1x_2 \cdots x_{i-1}vx_i \cdots x_{t-1} & \text{if } x_i \sim v \\ (x_1x_2 \cdots x_{i-1}vx_i \cdots x_{t-1})(x_1x_2 \cdots x_ivx_{i+1} \cdots x_{t-1}) & \\ \quad \times (x_1x_2 \cdots x_{i-1}vx_i \cdots x_{t-1}) & \text{if } x_i \not\sim v \end{cases}.$$

Now consider the word

$$w = \pi_1v\pi_2v \cdots \pi_kv\sigma_{t-1}\sigma_{t-2} \cdots \sigma_1.$$

By the construction of w and Lemma 2.1, the subword of w induced by $X = \{x_i : 1 \leq i \leq t-1\}$ 1-11-represents $G[X]$. Thus we need only to check edges and non-edges on v.

- If $x_i \sim v$, then in $\pi_1v\pi_2v \cdots \pi_kv\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{i+1}$ the pattern of x_i and v is $x_ivx_iv \cdots x_iv$ and in $\sigma_i\sigma_{i-1} \cdots \sigma_1$ the pattern is $vx_ivx_i \cdots vx_i$. Together the pattern of x_i and v is only 1 defect away from alternation.
- If $x_i \not\sim v$, then in w the pattern of x_i and v is $x_ivx_i \ldots vx_ix_ivvx_i \ldots$, which actually has 3 defects from being alternating.

Therefore w truly does 1-11-represent G. □

This construction has a number of optimizations and generalizations. For instance, in the definition of σ_i when $x_i \not\sim v$, one only really needs the first two permutations. Making this change will shorten the resulting representation, though the number of defects for non-adjacent vertices will no longer always be exactly three – the last non-adjacency will only have 2. One can also process various bunches of vertices in bulk, depending on the adjacencies. These types of modifications, however, will still generate representations that are the concatenation of a quadratic number of permutations. In the next section we describe, in Theorem 3.6, a rather more complicated optimization which concatenates

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only a linear number of permutations to realize G. For that reason, we choose to keep this simple and easy to verify version here. In terms of generalizations, we mention a few.

One construction, extremely useful in the theory of word-representability, is that of an orientation from a word. Given a word representing a graph, one can build an acyclic orientation by ordering vertices according to their first appearance in the word; that is, an edge between x and y is oriented as $x \to y$ if the first occurrence of x occurs before the first occurrence of y in the word. In the case of standard word-representations (i.e. 0-11-representations), these orientations are 'semi-transitive', and indeed having a word-representation is *equivalent* to admitting a semi-transitive orientation as shown in [6].

In the case of 1-11-representations, this orientation still makes sense. That is, a word 1-11-representing a graph can be thought of as representing an acyclically oriented graph. One can recover almost immediately from our construction presented above the following strengthening.

Theorem 2.3 *Let $G = (V, E)$ be an acyclically oriented graph. Then there is a word w over alphabet V permutationally* 1-11-*representing G as an oriented graph.*

Proof. The only modification necessary is to carefully choose the ordering $v_1, \ldots, v_n$. One chooses v_n to be a sink, which exists because the orientation is acyclic. Then one inductively chooses v_{n-i} to be a sink in the graph induced on $V \setminus \{v_n, v_{n-1}, \ldots, v_{n-i+1}\}$ for $i = 1, \ldots, n$ – which again exists by the acyclicity of the orientation on the induced subgraph.

It is then easy to see that the construction described above will begin with the permutation $v_1 v_2 \ldots v_n$ which forces the orientation to coincide with that on the graph. □

Another generalization is to differ the pattern of defects occurring in words. Given integers (k, ℓ) (with $k \neq \ell$) we say that G is (k, ℓ)-11-representable if there is a word w so that:

- Whenever $x \sim y$, the pattern xx or yy appears in the subword $w|_{xy}$ *exactly* k times.
- Whenever $x \not\sim y$, the pattern xx or yy appears in the subword $w|_{xy}$ *exactly* ℓ times.

As observed in the proof, the 1-11 construction given in Theorem 2.2 actually yields a $(1, 3)$-11-representation of G. This is easily generalized to other values of (k, ℓ).

Theorem 2.4 *Suppose k and ℓ are distinct positive integers with the same parity. Let $G = (V, E)$ be a graph. Then there is a word w over V permutationally* (k, ℓ)-11-*representing G.*

Proof. As noted above, the proof of Theorem 2.2 above gives a $(1, 3)$-11-representation. If k and ℓ are both odd, then the modification of the proof is very straightforward: in the definition of the σ_i, instead of adding either 1 or 3 permutations, one adds k or ℓ permutations – moving vertex v back and forth past x_i. Each of

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these introduces exactly one x_ix_i or one vv. Since k and ℓ are odd, the last of these will always have v to the left of x_i, which is what is necessary to continue the construction without creating any more defects when considering x and v_i.

In the case where k and ℓ are even (and hence, $k, \ell \geq 2$), one can easily create a $(k-1, \ell-1)$-11-representation w as already described. This can easily be turned into a (k, ℓ)-11-representation by taking the permutation π consisting of the final occurrences of each letter in order, reversing π to get a permutation σ, and then taking $w' = w\sigma$. This reversal introduces exactly one new defect into each letter pair. □

2.2. Even–odd-representations of graphs

The restriction in Theorem 2.4 that k and ℓ have the same parity is perhaps curious. A related question, raised in 2018 by Ian Wanless after a talk on 2-11-representations of graphs, is the following: is it possible to represent every graph by a word so that for every edge the parity of the number of 11 patterns is (say) even, and for every non-edge the number of 11 patterns is odd? It turns out that the answer to this question is rather emphatically no: a *very* small fraction of graphs can be represented in this form, whether permutationally or not. This also shows that the restriction that k and ℓ must have the same parity, above, is unavoidable.

To make this precise, we say that G is *even-odd-representable* if there is a word w whose alphabet is the vertex set so that:

- Whenever $x \sim y$, the pattern xx or yy appears in the subword $w|_{xy}$ an even number of times.
- Whenever $x \not\sim y$, the pattern xx or yy appears in the subword an odd number of times.

We begin by making an observation regarding the parity of occurrences of 11-patterns in words.

Lemma 2.5. *Suppose w is a word in alphabet $\{x, y\}$. Let n_x and n_y be the number of occurrences of x and y in w respectively, and let $n = n_x + n_y$ be the length of w. Further, let*

$$e(w) = \begin{cases} 1 & \text{if the first and last character of } w \text{ are the same (or } n=1) \\ 0 & \text{else.} \end{cases}$$

Let $N(w)$ denote the number of occurrences of the pattern 11 in w. Then

$$N(w) \equiv n + e(w) \equiv n_x + n_y + e(w) \pmod 2.$$

Proof. Removing v_n decreases n by 1. If $v_n = v_{n-1}$ then $e(w)$ remains the same, while $N(w)$ decreases by 1. Otherwise, $e(w)$ changes parity, while $N(w)$ remains the same. In either case, the parity of $n - e(w) - N(w)$ does not change. □

The following observation then easily follows.

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Lemma 2.6. *Suppose G is even–odd-representable. Then there is a word w' representing G where each vertex appears at most 3 times.*

Proof. Suppose G is even–odd-representable and fix a word w even–odd-representing G. Suppose some vertex x appears at least 4 times in w. Let w' be the word obtained by removing two of the middle appearances of x from w, leaving the rest unchanged. We claim w' also represents G. Indeed, removing occurrences of x can only affect incidences between x and other vertices and, when considering the existence of an edge between x and v we observe that if $w|_{xv}$ is the string restricted to just x and v

- The parity of the length of $w|_{xv}$ and $w'|_{xv}$ is the same, as $w'|_{xv}$ is two characters shorter than $w|_{xv}$.
- The initial and final characters of $w|_{xv}$ and $w'|_{xv}$ are the same, as only the middle instances of x were changed.

But then, per Lemma 2.5, the parity of the number of occurrences of a 11-pattern in $w|_{xv}$ and $w'|_{xv}$ are the same – so w and w' represent the same graph. The shortest word representing G thus has the desired property. □

Finally we are ready to prove

Theorem 2.7 *Almost every graph is not even–odd-representable; that is, if G is an n vertex graph chosen uniformly at random, then with probability $1 - o(1)$, G is not even–odd-representable.*

Proof. Per Lemma 2.6, every even–odd-representable graph has order a string representing it of length at most $3n$. Quite naively, there are at most $\sum_{t=1}^{3n} n^t = (1 + o(1))n^{3n} = 2^{(1+o(1))3n\log(n)}$ such strings representing labelled graphs of order n, and hence at most that many even–odd-representable graphs. On the other hand, there are $2^{\binom{n}{2}} = 2^{(1+o(1))n^2/2}$ labelled graphs on n vertices. Clearly

$$\frac{2^{(1+o(1))3n\log(n)}}{2^{(1+o(1))n^2/2}} \to 0,$$

and thus almost every graph is not even–odd-representable. □

A simple extension to Lemma 2.6 allows us to completely characterize permutationally even–odd-representable graphs. To set it up, we quickly state the relevant definitions.

We recall that a (strict) partial order $(P, \prec)$ is a relation $\prec$ on P that is irreflexive, anti-symmetric and transitive. A linear extension of $(P, \prec)$ is a total ordering consistent with the partial ordering. A realizer of $(P, \prec)$ is a collection of linear extensions $\pi_1, \ldots, \pi_t$ so that $x \prec y$ in $(P, \prec)$ if and only if $x \prec y$ in all π_i. The dimension of a poset, then, is the minimum cardinality of a realizer. Partially ordered sets, their realizers, and their dimension are closely related to the word-representation of graphs; see eg. [7]. The compatibility graph of a poset $(P, \prec)$ is a graph so on vertex set P, so that $a \sim b$ if and only if either $a \prec b$ or $b \prec a$.

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Theorem 2.8 *G is permutationally even–odd-representable if and only if it is the comparability graph of a poset of dimension at most two.*

Proof. Suppose G is permutationally even–odd-representable; let $w = \pi_1 \dots \pi_t$ be the word. We begin by noting that one may assume $t \leq 2$. Otherwise, consider w' obtained by removing one of the middle permutations. Then for each pair u, v over vertices, considering the restriction $w|_{uv}$ of w to those vertices, we note:

- The parity of the length of $w|_{uv}$ and $w'|_{uv}$ is the same, as the number of us and vs were both reduced by 1.
- The initial and final characters of $w|_{uv}$ and $w'|_{uv}$ is the same, as only the middle instances of u and v were changed.

Thus w and w' even–odd-represent the same graph.

Taking w to be the shortest word, we see that either $w = \pi_1$, in which case G is complete (and the comparability graph of a poset of dimension 1), or $w = \pi_1\pi_2$. In the latter case, it is easy to see that even–odd-representability is the same as word-representability and the graph obtained is the comparability graph of the poset with realizer π_1, π_2.

Conversely, if G is the comparability graph of a poset of dimension at most two, the concatenation of a realizer viewed as a word gives a word-representation of its comparability graph – but word-representation is easily seen to correspond to even–odd-representation in this case. □

As mentioned in the beginning of the section, this immediately implies the following corollary, as otherwise the words generated would contradict Theorem 2.8.

Corollary 2.9. *In Theorem 2.4, the condition that k and ℓ have the same parity is necessary.*

3. The length of 1-11-representations

In this section, we study the length of 1-11-representations of graphs. In Theorem 2.2, we showed that all graphs admit a permutational 1-11-representations. The simple construction given above uses $\Omega(n^2)$ permutations to represent a graph. In this section, we prove lower bounds on the length of a 1-11-representation, and then show that the construction of § 2 can be optimized to use far fewer permutations (albeit at the cost of some complexity.)

For this section, we restrict ourselves to dealing with 1-11-representations, although simple modifications allow one to recover essentially the same asymptotic bounds for k-11-representations for any $k \geq 1$.

To this end, we let

- $\mathcal{R}_\pi(G)$ denote the *permutational* 1-11-*representation number of* G – the minimum number of permutations in any 1-11 permutational representation of G. We further let
- $\mathcal{R}(G)$ denote the 1-11-*representation number of* G, the minimum number of *characters* in any 1-11-representation of G.

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Note that $\mathcal{R}(G) \leq n\mathcal{R}_\pi(G)$ as the concatenation of $\mathcal{R}_\pi(G)$ permutations of length n is a word-representing G on $n\mathcal{R}_\pi(G)$ characters.

3.1. Lower bounding representation length

We begin by offering lower bounds for the length of any 1-11-representation (or even k-11- representation) of a graph. These follow by simple counting arguments; unfortunately it seems somewhat difficult to prove that any particular graph has a large representation number.

Theorem 3.1 *There exist n vertex graphs with*

$$\mathcal{R}(G) \geq (1 + o(1))\frac{n^2}{2\log(n)}.$$

Proof. Suppose t is such that every labelled n vertex graph has $\mathcal{R}(G) \leq t$. On one hand, there are at most $\sum_{k=1}^{t} n^k = O(n^t) = 2^{(1+o(1))t\log(n)}$ such words. On the other hand, there are $2^{\binom{n}{2}} = 2^{(1+o(1))n^2/2}$ labelled graphs on n vertices. As each word represents a single graph, we must have

$$2^{(1+o(1))t\log(n)} \geq 2^{\binom{n}{2}},$$

and hence $t \geq (1 + o(1))\frac{n^2}{2\log n}$. □

An immediate corollary is that

Corollary 3.2. *There exist n vertex graphs with*

$$\mathcal{R}_\pi(G) \geq (1 + o(1))\frac{n}{2\log(n)}.$$

There are $2^{(1+o(1))n^2/4}$ labelled bipartite graphs on partite sets (X, Y) with $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$, so the same argument gives

Corollary 3.3. *There exist n-vertex bipartite graphs with*

$$\mathcal{R}(G) \geq (1 + o(1))\frac{n^2}{4\log n},$$

and

$$\mathcal{R}_\pi(G) \geq (1 + o(1))\frac{n}{4\log n}.$$

Although these counting arguments are quite naive – and indeed imply that *almost every n* vertex graph satisfy the existential lower bound of Theorem 3.1 – it seems difficult to give a non-trivial lower bound on $\mathcal{R}(G)$ for any particular graph. One potential idea

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is to look at graphs which are hard to word-represent efficiently. In [2], the authors showed that a word-representation of a graph can be turned into a 1-11-representation by appending at most n characters.

The crown graph, $H_{n,n}$, is a bipartite graph on $2n$ vertices, where each part has size n. It consists of a complete bipartite graph $K_{n,n}$ with a matching removed. It is known, see [5], that this has a large word-representation number and is perhaps a natural candidate for a graph with a large 1-11-representation number. Unfortunately, this turns out not to be the case.

Theorem 3.4 *The crown graph $H_{n,n}$ satisfies*

$$\mathcal{R}_\pi(H_{n,n}) \leq 4.$$

Proof. Denote the partite sets of $H_{n,n}$ by $A = \{a_i : 1 \leq i \leq n\}$ and $B = \{b_i : 1 \leq i \leq n\}$ where $a_i \sim b_j$ if $i \neq j$. The word

$$(a_1 a_2 \cdots a_n b_1 b_2 \cdots b_n)(a_n a_{n-1} \cdots a_1 b_n b_{n-1} \cdots b_1)(b_1 a_1 b_2 a_2 \cdots b_n a_n)(a_1 b_1 a_2 b_2 \cdots a_n b_n),$$

1-11-represents $H_{n,n}$. □

We can, however, prove one general lower bound which can be applied to specific graphs inspired by a previous result of the authors and Owens from [7] on word-representations. To this end, we make a definition. Let $N(x)$ denote the neighbourhood of a vertex x in a graph. For a graph G and $X \subseteq V(G)$, we define

$$\mathcal{N}(X) = \{N(z) \cap X : z \in (V(G) \setminus X)\}.$$

$|\mathcal{N}(X)|$ then is the number of distinct neighbourhoods that other vertices have within X. Then

Theorem 3.5 *Suppose G is a graph, and $X \subseteq V(G)$. Then*

$$\mathcal{R}_\pi(G) \geq \frac{\log|\mathcal{N}(X)|}{\log(|X|+1)}.$$

Proof. Fix a permutational representation of G with $t = \mathcal{R}_\pi(G)$ permutations. Fix a vertex z. Within each of these t permutations, z sits in one of $|X| + 1$ locations with respect to the vertices of X – before the first, between the first and second, …, or after the last. These locations, in the t permutations, completely determine the adjacencies from z to the vertices in X. But then

$$(|X| + 1)^t \geq |\mathcal{N}(X)|.$$

Rearranging, we obtain the result. □

Unfortunately, while this can yield non-trivial bounds, they are quite a bit weaker than those coming from Theorem 3.1, as $|N(X)| \leq n$. On the other hand, $|N(X)|$ can

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be as large as $2^{|X|}$. One particular example – where Theorem 3.5 can be seen to give a fairly tight bound is the following: consider a bipartite graph G on vertex set (X, Y) where $|X| = t$, $|Y| = 2^t$, and the vertices in Y are adjacent to the 2^t possible distinct neighbourhoods in X. Then Theorem 3.5 implies that this graph has $\mathcal{R}_\pi(G) \geq \frac{t}{\log(t+1)}$. Theorem 3.8 below implies that $\mathcal{R}_\pi(G) = O(t)$, so at least for this graph Theorem 3.5 gives a reasonably tight bound.

3.2. Upper bounding representation length

The main purpose in this section is to describe some optimizations to the construction above that prove that the lower bound arising from Theorem 3.1 is nearly tight.

In particular we prove:

Theorem 3.6 *Suppose G is a graph on n vertices. Then*

$$\mathcal{R}_\pi(G) \leq 4n + O(\log n).$$

The key observation is that one can implement a 'divide-and-conquer' approach by realizing two halves of a graph simultaneously, and then 'shuffle' one half through the other – essentially in parallel – to realize the edges between the halves. It is this divide and conquer paradigm that allows us to decrease the number of permutations in a realization from quadratic to linear. This is realized through the following lemma.

Lemma 3.7. *Suppose G is an n vertex graph and let X and Y partition the vertex set of G. Let G_1 and G_2 be the induced graphs on X and Y respectively. Then*

$$\mathcal{R}_\pi(G) \leq \max\{\mathcal{R}_\pi(G_1), \mathcal{R}_\pi(G_2)\} + 2n.$$

Let us derive Theorem 3.6 from Lemma 3.7, then we will return to the proof of the Lemma.

Proof of Theorem 3.6. Let

$$f(n) = \max_{G:|V(G)|=n} \mathcal{R}_\pi(G).$$

Note that $f(n)$ is an increasing function of n as every $n-1$ vertex graph is an induced subgraph of an n vertex graph, and a 1-11-representation of a graph G contains a 1-11-representation of its $n-1$ vertex subgraphs. We prove that

$$f(n) \leq 4n + 2\log_{3/2}(n),$$

by induction on n. This is easy to verify for $n \leq 2$, so suppose $n \geq 3$. Let G_n be a graph on n vertices maximizing $\mathcal{R}_\pi(G)$. Partition $V(G_n)$ into parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and let H_1 and H_2 be the induced subgraphs on these parts. Then, By Lemma 3.7,

$$\begin{aligned} f(n) = \mathcal{R}_\pi(G_n) &\le \max\{\mathcal{R}_\pi(H_1), \mathcal{R}_\pi(H_2)\} + 2n \\ &\le \max\{f(\lfloor n/2 \rfloor), f(\lceil n/2 \rceil)\} + 2n \\ &= f(\lceil n/2 \rceil) + 2n \\ &\le 4(\lceil n/2 \rceil) + 2\log_{3/2}(\lceil n/2 \rceil) + 2n \\ &\le 4(n+1)/2 + 2\log_{3/2}(2n/3) + 2n \\ &= 4n + 2\log_{3/2}(n). \end{aligned}$$

Note that here we used the fact that $\lceil n/2 \rceil \le 2n/3$ for $n \ge 3$, and this completes the proof. □

We now turn to the proof of Lemma 3.7, which follows by modifying the construction of Theorem 2.2.

Proof of Lemma 3.7. Suppose X and Y partition the vertex set $V(G)$, and G_1 and G_2 be the graphs induced on X and Y respectively. Let $n_1 = |X|$ and $n_2 = |Y|$, so that $n_1 + n_2 = n$. Let $s = \mathcal{R}_\pi(G_1)$ and $t = \mathcal{R}_\pi(G_2)$, and assume that without loss of generality that $s \ge t$. Then there exists a permutational representation $\pi_1\pi_2\ldots\pi_s$ of G_1, and a permutational representation $\pi'_1\pi'_2\ldots\pi'_t$ of G_2. By Lemma 2.1, this representation of G_2 can be extended to a representation of length s by taking $\pi'_{t+1} = \pi'_{t+2} = \cdots = \pi'_s = \pi'_t$. One immediately sees then that

$$w_1 = \pi_1\pi'_1\pi_2\pi'_2\ldots\pi_s\pi'_s,$$

is a word representing G_1 and G_2 with a complete bipartite graph between them. We now proceed to augment this word with additional permutations to 'fix' the edges between them. The procedure is similar to that illustrated in Theorem 2.2, but designed to take care of adjacencies of multiple vertices simultaneously.

Suppose $\pi_s = v_1v_2\ldots v_{n_1}$ and $\pi'_s = x_1x_2\ldots x_{n_2}$. Then for $1 \le i \le n_1 + n_2 = n$ we define permutations σ_i and σ'_i as follows:

- σ_i consists of the first i elements of π'_s being shuffled into the last i vertices of π_s. For $1 \le j \le i$, this means that the element x_j will be between $v_{n_1-i+j-1}$ and v_{n_1-i+j}, where $n_i - i + j \le 1$ means x_j precedes all of π_s. For any x_j that precede all of π_s, we keep them in their same order as in π'_s – that is x_{j-1} precedes x_j. For $j \ge i$, the x_j come after all elements of π_s, again keeping their order.
- σ'_i is the same as σ_i, except for if x_j is not adjacent to v_{n_1-i+j}, it is moved to be immediately after v_{n_1-i+j}, while keeping it preceding x_{j+1}.

Thus these two permutations 'fix' the edges between the x_j and v_{n_1-i+j} – they add a single instance of a 11 pattern if x_j is adjacent to v_{n_1-i+j} and two instances if not. We note that when σ_{i+1} is added, another 11 pattern is added when x_j is not adjacent to v_{n_1-i+j} but not when x_j is adjacent to v_{n_1-i+j}.

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Through this procedure we have $\sigma_n = \pi'_s \pi_s$ – that is to say, that by σ'_n all edges have been fixed, and hence

$$w = \pi_1 \pi'_1 \pi_2 \pi'_2 \dots \pi_s \pi'_s \sigma_1 \sigma'_1 \sigma_2 \sigma'_2 \dots \sigma_n \sigma'_n,$$

1-11-represents G as desired. □

We remark that, if one desires different k-11-representations, the modifications to Theorem 2.2 can also be adapted to Lemma 3.7 and Theorem 3.6. This will change the constants involved, but the construction will still involve a linear number of permutations (with the constant depending on k).

The construction in the proof of Lemma 3.7 can also be improved somewhat in the case where one of the graphs is an independent set: in this case after the initial word (w_1 in the proof) is constructed, all edges within the independent set have already been destroyed and we do not need to respect the final permutation of vertices of that graph, when shuffling through the other. A particular application of this idea is the following.

Theorem 3.8 *Suppose G is a bipartite graph with bipartition (X, Y) where $|X| < |Y|$. Then*

$$\mathcal{R}_\pi(G) \leq 2|X| + 3.$$

Proof. Essentially we follow the proof of Lemma 3.7 with minor modifications. Let G_1 be the independent set on X and G_2 be an independent set on Y. Let π_1 (respectively π_2) be an arbitrary permutation of vertices in X (resp. Y). Let π'_1 and π'_2 be the reverses of π_1 and π_2, respectively. Then

$$w_1 = \pi_1 \pi_2 \pi'_1 \pi'_2 \pi_1 \pi_2,$$

is easily seen to 1-11-represent the complete bipartite graph on (X, Y). Supposing $|X| = t$, we now create $\sigma_1, \sigma_2, \dots, \sigma_t$ and $\sigma'_1, \sigma'_2, \dots, \sigma'_t$ to append the word, to fix the adjacencies between the two partite sets.

If $\pi_1 = v_1 v_2 \dots v_t$, then for $1 \leq i \leq t$ we define

- $\sigma_i = v_1 v_2 \dots v_{t-i} \pi_2 v_{t-i+1} \dots v_t$ to be the permutation obtained by moving all of π_2 between v_{t-i} and v_{t-i+1}.
- σ'_i is obtained from σ_i by moving all vertices in Y not adjacent to v_{t-i+1} to the right of v_{t-i+1}.

This construction will introduce exactly one 11 pattern between an adjacent pair $v_j \in X$ and $y \in Y$, and at least two instances if not – two when the $\sigma_i \sigma'_i$ are added, and one additional when σ_{i+1} is added. It may add additional 11 patterns between vertices within Y, but these already are non-edges so no additional edges are deleted. Then the final word is

$$w_1 \sigma_1 \sigma'_1 \sigma_2 \sigma'_2 \dots \sigma_t \sigma'_t.$$

□

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As noted above, Theorem 3.8 shows the near-sharpness of the example after Theorem 3.5.

4. Conclusion and open problems

In some sense, Theorem 2.2 settles the main question about the class of 1-11-representable graphs. It is, indeed, all graphs. There are still some open questions about 1-11-representations, and related questions, that would be interesting to study.

- Theorem 3.1 gives a lower bound on the number of permutations needed in a permutational representation of some graphs that is close to optimal, but it is ineffective. Is there an easy structural property that implies that $\mathcal{R}(G)$ or $\mathcal{R}_\pi(G)$ is large? Can one prove, for an explicitly chosen graph, that $\mathcal{R}_\pi(G)$ is large?

  A natural place to look would be random-like graphs, as random graphs have large $\mathcal{R}_\pi(G)$ with high probability. Perhaps one can show that the Paley graphs have large $\mathcal{R}_\pi(G)$ as this family is known to be quasi-random in the sense of Chung, Graham, and Wilson [3].
- While Theorem 3.6 cannot be significantly improved – it is tight within a factor of $O(\log n)$ – perhaps it can be improved in some cases.

  For many families of graphs (e.g. bipartite graphs, graphs with chromatic number exactly k, split graphs, ...) the number of graphs in the family of order n grows like 2^{cn^2} and in these cases, our counting bound adapts to give lower bounds a similar $O(\log n)$ factor away from our upper bound.

  The family of planar graphs seems particularly interesting. There are only exponentially many planar graphs and counting essentially gives no non-trivial lower bound. Can one either find planar graphs with high $R_\pi(G)$ or prove an $o(n)$ or even $O(n^{1-\epsilon})$ upper bound for $\mathcal{R}_\pi(G)$ for graphs in this class?
- The example of $H_{n,n}$ shows that sometimes the word-representation number can be significantly larger than $R_\pi(G)$ of the same graph. This is despite the fact that, in a sense, one needs more effort to encode non-edges. It would be interesting to study when this happens: what causes a graph to be easier to k-11-represent than it is to $(k-1)$-11-represent?

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References

1. M. Alshammari, S. Kitaev, C. Tang, T. Tao and J. Zhang, On 1-11-representability and multi-1-11-representability of graphs, *Util. Math.* **122** (2025), 29–40.
2. G. -S. Cheon, J. Kim, M. Kim, S. Kitaev and A. Pyatkin, On k-11-representable graphs, *J. Comb.* **10**(3) (2019), 491–513.

https://doi.org/10.1017/S0013091525101211 Published online by Cambridge University Press

3. F. R. K. Chung, R. L. Graham and R. M. Wilson, Quasi-random graphs, *Combinatorica* **9**(4) (1989), 345–362.
4. M. Futorny, S. Kitaev and A. Pyatkin, New tools to study 1-11-representation of graphs, *Graphs Combin.* **40**(5) (2024), 13.
5. M. Glen, S. Kitaev and A. Pyatkin, On the representation number of a crown graph, *Discrete Appl. Math.* **244** (2018), 89–93.
6. M. M. Halldórsson, S. Kitaev and A. Pyatkin, Semi-transitive orientations and word-representable graphs, *Discrete Appl. Math.* **201** (2016), 164–171.
7. Z. Hefty, P. Horn, C. Muir and A. Owens, Word-representable graphs: orientations, posets, and bounds, *Electron. J. Combin.* **31**(4) (2024), 26.
8. S. Kitaev and A. Pyatkin, On representable graphs, *J. Autom. Lang. Comb.* **13**(1) (2008), 45–54.
9. S. Kitaev and S. Seif, Word problem of the Perkins semigroup via directed acyclic graphs, *Order* **25**(3) (2008), 177–194.