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ON HYPERSTABILITY OF GENERALISED LINEAR FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES

DONG ZHANG

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Abstract

We obtain some results on approximate solutions of the generalised linear functional equation $\sum_{i=1}^{m} L_i f(\sum_{j=1}^{n} a_{ij} x_j) = 0$ for functions mapping a normed space into a normed space. We show that, under suitable assumptions, the approximate solutions are in fact exact solutions. The theorems correspond to and complement recent results on the hyperstability of generalised linear functional equations.

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1. Introduction and main results

Studies of the stability of functional equations date back to Hyers [14] and Ulam [29] or, even earlier, to Pólya and Szegö [19, 20]. This is an active field with particular interest in the hyperstability of linear functional equations. In this paper, we concentrate on hyperstability of generalised linear functional equations in several variables of the form

$$\sum_{i=1}^{m} L_i f\Big(\sum_{j=1}^{n} a_{ij} x_j\Big) = 0.$$

Our main theorems correspond to and complement many results in the literature. We first recall some of these well-known results.

Proposition 1.1. *Let X, Y be two normed spaces with Y complete. Take $c \geq 0$ and let $p \neq 1$ be a fixed real number. Let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \backslash \{0\}.$$

Then there exists a unique solution $T : X \to Y$ of the functional equation $T(x+y) = T(x) + T(y)$ with $\|f(x) - T(x)\| \leq c\|x\|^p / |2^{p-1} - 1|$.

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This result is due to Hyers [14] ($p = 0$), Aoki [2] ($0 < p < 1$), Gajda [13] ($p > 1$) and Rassias [27] ($p < 0$). Moreover, Rassias [21, 22] also considered the case where $c(\|x\|^p + \|y\|^p)$ is replaced by $c\|x\|^p\|y\|^q$ with $p + q < 0$. The treatment in the proof of Proposition 1.1 can also be applied to other functional equations. Skof [28], Jun and Kim [15], Jung [16] and Fechner [12] treated the Hyers–Ulam stability of the quadratic equation as follows.

PROPOSITION 1.2. *Let X, Y be two normed spaces with Y complete. Take $c \geq 0$ and let $p \neq 2$ be a fixed real number. Let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X\backslash\{0\}.$$

Then there exists exactly one quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{2c\|x\|^p}{|4-2^p|} & p \neq 0, \\ c & p = 0. \end{cases}$$

The investigation of hyperstability is a new area of research; see, for example, [11]. Here we only list some typical recent results.

PROPOSITION 1.3 [18, Theorem 2]. *Let $\mathbb{F}, \mathbb{K}$ denote the fields of real or complex numbers. Let X be a normed space over the field $\mathbb{F}$, Y be a normed space over $\mathbb{K}$, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, $p < 0$ and let $f : X \to Y$ satisfy*

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X\backslash\{0\}.$$

Then f satisfies the equation

$$f(ax+by) - Af(x) - Bf(y) = 0, \quad x, y \in X\backslash\{0\}.$$

REMARK 1.4. Lemma 4.7 in [9] improves the statement of Proposition 1.3.

PROPOSITION 1.5 [5, Theorem 2]. *Let U be a nonempty subset of $X\backslash\{0\}$ such that there exists a positive integer n_0 with $nx \in U$ whenever $x \in U, n \in \mathbb{N}, n \geq n_0$. Let Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$, $p + q < 0$ and let $f : U \to Y$ satisfy*

$$\left\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\right\| \leq c\|x\|^p\|y\|^p, \quad x, y, \frac{x+y}{2} \in U.$$

Then f satisfies the Jensen functional equation $f(\frac{1}{2}(x+y)) = \frac{1}{2}(f(x)+f(y))$ on U.

Now we turn to our main results. We adopt the following basic assumptions throughout the paper.

(A) $\mathbb{F}, \mathbb{K}$ denote the fields of real or complex numbers and X, Y denote normed spaces over $\mathbb{F}, \mathbb{K}$, respectively.

(B) $n \geq 2$ and m are positive integers; $C \geq 0$, $a_{ij} \in \mathbb{F}$ and $L_i \in \mathbb{K}$ are given parameters for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

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(C) There exist $i_0 \in \{1, \dots, m\}$ and two different elements $j_1, j_2 \in \{1, 2, \dots, n\}$ such that $a_{i_0 j_1} \neq 0$, $a_{i_0 j_2} \neq 0$ and, for any $i \neq i_0$, $\gamma \neq 0$, there is $j \in \{1, \dots, n\}$ satisfying $a_{ij} \neq \gamma a_{i_0 j}$.

The last assumption concerns the 'nondegeneracy' of the matrix $(a_{ij})_{m\times n}$ and it can be expressed as follows: there is a row $A_{i_0} := (a_{i_0 j})_{1\times n}$ of the matrix $A := (a_{ij})_{m\times n}$ with at least two nonzero elements and such that no other row is a multiple of A_{i_0}.

Theorem 1.6. *Assume that all the parameters satisfy the basic assumptions. If there exists $p < 0$ such that*

$$\Big\|\sum_{i=1}^{m} L_i f\Big(\sum_{j=1}^{n} a_{ij}x_j\Big)\Big\| \leq C \sum_{j=1}^{n} \|x_j\|^p \tag{1.1}$$

for any $x_1, x_2, \dots, x_n \in X\backslash\{0\}$, then

$$\sum_{i=1}^{m} L_i f\Big(\sum_{j=1}^{n} a_{ij}x_j\Big) = 0 \tag{1.2}$$

holds on $X\backslash\{0\}$.

Theorem 1.7. *Assume that all the parameters satisfy the basic assumptions. If there exist real numbers $p_1, p_2, \dots, p_n$ such that $p_1 + p_2 + \cdots + p_n < 0$ and*

$$\Big\|\sum_{i=1}^{m} L_i f\Big(\sum_{j=1}^{n} a_{ij}x_j\Big)\Big\| \leq C \prod_{j=1}^{n} \|x_j\|^{p_j} \tag{1.3}$$

for any $x_1, x_2, \dots, x_n \in X\backslash\{0\}$, then (1.2) holds on $X\backslash\{0\}$.

Remark 1.8. Theorem 1.6 is closely related to [4, Theorem 2.1] and Theorem 1.7 corresponds to [8, Theorem 1.3].

We call the condition (1.1) in Theorem 1.6 the Aoki–Hyers type, and (1.3) in Theorem 1.7 the Rassias type. As far as we know, almost all the results on the hyperstability of generalised linear functional equations in the literature can be obtained immediately from our main theorems. This includes hyperstability results for the Cauchy equation, Fréchet equation, Jordan–von Neumann functional equation and the additive, quadratic, cubic, quartic and monomial functional equations. As examples, we present Proposition 1.3 above and the following results.

Proposition 1.9 [1, main result of Section 2]. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and $f : X \to Y$ satisfy*

$$\Big\|\sum_{i=0}^{n} (-1)^{n-i} C_n^i f(ix + y) - n!f(x)\Big\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X\backslash\{0\}.$$

Then f satisfies the equation

$$\sum_{i=0}^{n} (-1)^{n-i} C_n^i f(ix + y) - n!f(x) = 0, \quad x, y \in X\backslash\{0\}.$$

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PROPOSITION 1.10. *Let the fields* $\mathbb{F}, \mathbb{K}$ *denote the real or complex numbers. Let* $(X, \|\cdot\|_X)$ *be a normed space over* $\mathbb{F}$, $(Y, \|\cdot\|_Y)$ *be a Banach space over* $\mathbb{K}$, $c \geq 0$, $p < 0$ *and let the mapping* $f : X \to Y$ *satisfy*

$$\|f(x+y) + f(y+z) + f(x+z) - f(x) - f(y) - f(z) - f(x+y+z)\|_Y \\ \leq c(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p)$$

for all $x, y, z \in X\backslash\{0\}$. *Then* f *satisfies the equation*

$$f(x+y) + f(y+z) + f(x+z) = f(x) + f(y) + f(z) + f(x+y+z), \quad x, y, z \in X.$$

REMARK 1.11. Proposition 1.9 is related to [17, Theorems 2, 3 and 5] and Proposition 1.10 is a particular case of [3, Theorem 2.1].

Next, we will use Theorem 1.6 to prove Proposition 1.3 (for details we refer, for example, to [18]) and Proposition 1.9 (see also [1]). In fact, those results can be obtained immediately.

PROOF OF PROPOSITION 1.3. By the conditions, the two components of $A_1 := (a, b)$ are nonzero. For any $i \in \{2, 3\}$ and for all $\gamma \in \mathbb{F}$, $A_i \neq \gamma A_1$, where $A_2 = (1, 0)$, $A_3 = (0, 1)$. So, Proposition 1.3 follows from Theorem 1.6. □

PROOF OF PROPOSITION 1.9. Let $A_1 := (1, 1)$. For any $i \in \{2, 3, \ldots, n+1\}$ and for all $\gamma \in \mathbb{F}$, $A_i \neq \gamma A_1$, where $A_i = (i, 1)$, $i = 2, 3, \ldots, n$ and $A_{n+1} = (1, 0)$. So, by Theorem 1.6, we have Proposition 1.9. □

Furthermore, by Theorems 1.6 and 1.7, a cubic equation introduced in [24, 26],

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y),$$

a cubic equation considered in [15],

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

and a quartic equation given in [23, 25],

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y),$$

have hyperstability in the sense of Aoki–Hyers or Rassias when $p < 0$ (or $p_1 + p_2 < 0$).

2. Proof of theorems

In this section, we will prove the hyperstability of the generalised linear functional equation of Theorems 1.6 and 1.7. In essence, the method of proof was introduced in [6] and [7] and next used in [3, 5, 8, 18]. To prove our theorems, we need a useful result from [10], which we reproduce here for the reader's convenience. We introduce the following hypotheses.

(H1) X is a normed space, Y is a Banach space, $l_1, \ldots, l_m : X\backslash\{0\} \to X\backslash\{0\}$ and $L_1, \ldots, L_m : X \to \mathbb{R}_+^{X\backslash\{0\}}$.

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(H2) $T : Y^{X\backslash\{0\}} \to Y^{X\backslash\{0\}}$ satisfies

$$\|T\xi(x) - T\eta(x)\| \leq \sum_{i=1}^{m} L_i(x)\|\xi(l_i(x)) - \eta(l_i(x))\|, \quad \xi, \eta \in Y^{X\backslash\{0\}}, x \in X\backslash\{0\}.$$

(H3) $\Lambda : \mathbb{R}_+^{X\backslash\{0\}} \to \mathbb{R}_+^{X\backslash\{0\}}$ is given by

$$\Lambda\delta(x) = \sum_{i=1}^{m} L_i(x)\delta(l_i(x)), \quad \delta \in \mathbb{R}_+^{X\backslash\{0\}}, x \in X\backslash\{0\}.$$

Lemma 2.1 [10, Theorem 1]. *Let (H1)–(H3) hold and let* $\varepsilon : X\backslash\{0\} \to [0, +\infty)$, $f : X\backslash\{0\} \to Y$ *satisfy the conditions*

$$\|Tf(x) - f(x)\| \leq \varepsilon(x) \quad \textit{and} \quad \varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < +\infty, \quad x \in X\backslash\{0\}.$$

Then there exists a unique fixed point g of T with

$$\|f(x) - g(x)\| \leq \varepsilon^*(x), \quad x \in X\backslash\{0\}.$$

Moreover, g is given by $g(x) = \lim_{n\to+\infty} T^n f(x)$ for $x \in X\backslash\{0\}$.

We now give a complete proof of Theorem 1.6 and a brief proof of Theorem 1.7.

Proof of Theorem 1.6. We can assume that Y is complete, because otherwise we can replace Y by its completion $\overline{Y}$.

Without loss of generality, we may assume that $(a_{1j})_{1\times n}$ is the row satisfying condition (C). For $i = 1, \ldots, m$, let π_i denote the hyperplane $\sum_{j=1}^{n} a_{ij}t_j = 0$ in $\mathbb{F}^n$ and, for $k = 1, \ldots, n$, let $\pi_{c,k}$ be the coordinate plane $t_k = 0$. By the hypothesis on $(a_{1j})_{1\times n}$, it follows that π_1 is different from π_i $(i = 2, \ldots, m)$ and $\pi_{c,k}$ $(k = 1, \ldots, n)$ and so the set

$$\pi_1 \Big\backslash \bigcup_{k=1}^{n} \pi_{c,k} \Big\backslash \bigcup_{i=2}^{m} \pi_i$$

is not empty. Choose an element $(k_1, \ldots, k_n)$ from this set. Obviously, $(k_1, \ldots, k_n)$ satisfies

$$\begin{cases} \sum_{j=1}^{n} a_{1j}k_j = 0, & \\ k_j \neq 0, & j = 1, 2, \ldots, n, \\ \sum_{j=1}^{n} a_{ij}k_j \neq 0, & i = 2, \ldots, m. \end{cases}$$

Keeping the hypothesis on $(a_{1j})_{1\times n}$ in mind, it is easy to see that there exist $b_1, \ldots, b_n \in \mathbb{F}$ such that $\sum_{j=1}^{n} a_{1j}b_j = 1$. For a given large positive integer t and a nonzero number x, we set $x_j = (k_jt + b_j)x$, $j = 1, 2, \ldots, n$, and write $s_i(t) = \sum_{j=1}^{n} a_{ij}(k_jt + b_j)$, $i = 1, 2, \ldots, m$. Then

$$\left\| \sum_{i=1}^{m} L_i f(s_i(t)x) \right\| \leq C \sum_{j=1}^{n} |k_jt + b_j|^p \|x\|^p \tag{2.1}$$

and $s_1(t) \equiv s_1 = 1$.

Due to the homogeneity of degree one of both sides of the inequality (1.1), we can assume that $L_1 = -1$. Since $k_1, \ldots, k_n$ are all nonzero, we have $\lim_{t\to+\infty} |k_jt + b_j| = +\infty$, $i = 1, \ldots, n$. Define

$$\alpha_t := C \sum_{j=1}^{n} |k_jt + b_j|^p < 1,$$

so that $\lim_{t\to+\infty} \alpha_t = 0$. Therefore, we can suppose that t is sufficiently large so that $0 \leq \alpha_t < 1$.

Define operators T_t and Λ_t by

$$T_t\xi(x) = \sum_{i=2}^{m} L_i\xi(s_i(t)x),$$

$$\Lambda_t\delta(x) = \sum_{i=2}^{m} |L_i|\delta(s_i(t)x),$$

respectively. We can easily check that

$$\begin{aligned} \|T_t\xi(x) - T_t\eta(x)\| &= \left\| \sum_{i=2} L_i(\xi(s_i(t)x) - \eta(s_i(t)x)) \right\| \\ &\leq \sum_{i=2} |L_i| \, \|\xi(s_i(t)x) - \eta(s_i(t)x)\| = \Lambda_t(\|\xi(x) - \eta(x)\|). \end{aligned}$$

The inequality (2.1) can be written as

$$\|T_t f(x) - f(x)\| \leq \alpha_t \|x\|^p := \varepsilon_t(x).$$

It follows from the linearity of Λ_t that

$$\Lambda_t\varepsilon_t(x) = \Lambda_t(\alpha_t\|x\|^p) = \sum_{i=2}^{m} |L_i| \cdot \alpha_t\|s_i(t)x\|^p = \sum_{i=2}^{m} |L_i| \cdot |s_i(t)|^p\alpha_t\|x\|^p = \beta_t\alpha_t\|x\|^p,$$

where $\beta_t = \sum_{i=2}^{m} |L_i| \cdot |s_i(t)|^p$. Since

$$s_i(t) = \sum_{j=1}^{n} a_{ij}(k_jt + b_j) = \sum_{j=1}^{n} a_{ij}k_jt + \sum_{j=1}^{n} a_{ij}b_j$$

and $\sum_{j=1}^{n} a_{ij}k_j \neq 0$, $i = 2, \ldots, m$,

$$\lim_{t\to+\infty} |s_i(t)| = \lim_{t\to+\infty} \left| \sum_{j=1}^{n} a_{ij}k_jt + \sum_{j=1}^{n} a_{ij}b_j \right| = +\infty, \quad i = 2, \ldots, m.$$

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So, we get $\lim_{t\to+\infty}\beta_t = 0$ and therefore we can assume that $0 \le \beta_t < 1$.

Analogously, $\Lambda_t^n \varepsilon_t(x) = \Lambda_t^n(\alpha_t\|x\|^p) = \beta_t^n \alpha_t \|x\|^p$. Therefore,

$$\varepsilon_t^*(x) = \sum_{n=0}^{+\infty} \beta_t^n \alpha_t \|x\|^p = \frac{\alpha_t}{1-\beta_t}\|x\|^p$$

for sufficiently large t. According to Lemma 2.1, there exists a unique solution f_t of $T_t f_t(x) = f_t(x)$ satisfying $\|f_t(x) - f(x)\| \le \varepsilon_t^*(x)$ and $f_t(x) = \lim_{n\to+\infty} T_t^n f(x)$ for $x \ne 0$.

Next, we will prove that f_t satisfies the equation $\sum_{i=1}^m L_i f_t(\sum_{j=1}^n a_{ij}x_j) = 0$. To this end, we show by induction on r that

$$\Big\|\sum_{i=1}^m L_i T_t^r f\Big(\sum_{j=1}^n a_{ij}x_j\Big)\Big\| \le C\beta_t^r \sum_{j=1}^n \|x_j\|^p. \tag{2.2}$$

One can observe that the case $r = 0$ is indeed the inequality (1.1). We assume that the inequality (2.2) holds for $r := r$; then, for $r := r+1$,

$$\begin{aligned}
\Big\|\sum_{i=1}^m L_i T_t^{r+1} f\Big(\sum_{j=1}^n a_{ij}x_j\Big)\Big\| &= \Big\|\sum_{i=1}^m L_i \sum_{k=2}^m L_k T_t^r f\Big(s_k(t)\sum_{j=1}^n a_{ij}x_j\Big)\Big\| \\
&= \Big\|\sum_{k=2}^m L_k \sum_{i=1}^m L_i T_t^r f\Big(\sum_{j=1}^n a_{ij}s_k(t)x_j\Big)\Big\| \\
&\le \sum_{k=2}^m |L_k| \Big\|\sum_{i=1}^m L_i T_t^r f\Big(\sum_{j=1}^n a_{ij}s_k(t)x_j\Big)\Big\| \\
&\le \sum_{k=2}^m |L_k| C\beta_t^r \sum_{j=1}^n \|s_k(t)x_j\|^p \\
&= C\beta_t^r \sum_{k=2}^m |L_k|\cdot|s_k(t)|^p \sum_{j=1}^n \|x_j\|^p \\
&= C\beta_t^{r+1} \sum_{j=1}^n \|x_j\|^p.
\end{aligned}$$

Since $\lim_{r\to+\infty} C\beta_t^{r+1}\sum_{j=1}^n \|x_j\|^p = 0$ and $f_t(x) = \lim_{r\to+\infty} T_t^{r+1} f(x)$,

$$\sum_{i=1}^m L_i f_t\Big(\sum_{j=1}^n a_{ij}x_j\Big) = \lim_{r\to+\infty}\sum_{i=1}^m L_i T_t^{r+1} f\Big(\sum_{j=1}^n a_{ij}x_j\Big) = 0.$$

Finally, recall that

$$\|f_t(x) - f(x)\| \le \frac{\alpha_t}{1-\beta_t}\|x\|^p, \qquad \lim_{t\to+\infty}\frac{\alpha_t}{1-\beta_t} = 0$$

and that f_t satisfies (1.2) for sufficiently large t. Thus, by letting $t \to +\infty$, we see that f also satisfies (1.2). □

Proof of Theorem 1.7. Set $\alpha_t = \prod_{j=1}^{n} |k_j t + b_j|^{p_j}$. Then $\alpha_k \to 0$ as $t \to +\infty$ since $\sum_{j=1}^{n} p_j < 0$. The proof of Theorem 1.7 is now the same as the proof of Theorem 1.6. □

Remark 2.2. In Theorems 1.6 and 1.7, we can replace $X\backslash\{0\}$ by a subset $X' \subset X\backslash\{0\}$ with the property that $x \in X'$ implies $(k_j t + b_j)x \in X'$ for sufficiently large $t \in \mathbb{N}$, $j = 1, 2, \ldots, n$. Under the basic assumptions, Theorems 1.6 and 1.7 can be stated as follows.

If (1.1) or (1.3) holds for $x_1, \ldots, x_n \in X'$ with $\sum_{j=1}^{n} a_{ij} x_j \in X'$, $i = 1, 2, \ldots, m$, then (1.2) holds for $x_1, \ldots, x_n \in X'$ with $\sum_{j=1}^{n} a_{ij} x_j \in X'$, $i = 1, 2, \ldots, m$.

Remark 2.3. Theorem 2.2 in [9] allows us to generalise Theorem 1.6 to the following inhomogeneous version of (1.2):

$$\sum_{i=1}^{m} L_i f\Big(\sum_{j=1}^{n} a_{ij} x_j\Big) = F(x_1, \ldots, x_n),$$

where $F : X^n \to Y$ is a given function such that the equation has at least one solution $f_0 : X\backslash\{0\} \to Y$.

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DONG ZHANG, LMAM and School of Mathematical Sciences,
Peking University, Beijing 100871, PR China
e-mail: dongzhang@pku.edu.cn