Adv. Appl. Probab. 1–42 (2025)
*doi:*10.1017/apr.2025.10023

THE DISTANCE ON THE SLIGHTLY SUPERCRITICAL RANDOM SERIES–PARALLEL GRAPH

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Abstract

We consider the random series–parallel graph introduced by Hambly and Jordan (2004 *Adv. Appl. Probab.* **36**, 824–838), which is a hierarchical graph with a parameter $p \in [0, 1]$. The graph is built recursively: at each step, every edge in the graph is either replaced with probability p by a series of two edges, or with probability $1-p$ by two parallel edges, and the replacements are independent of each other and of everything up to then. At the nth step of the recursive procedure, the distance between the extremal points on the graph is denoted by $D_n(p)$. It is known that $D_n(p)$ possesses a phase transition at $p = p_c := \frac{1}{2}$; more precisely, $\frac{1}{n} \log \mathbb{E}[D_n(p)] \to \alpha(p)$ when $n \to \infty$, with $\alpha(p) > 0$ for $p > p_c$ and $\alpha(p) = 0$ for $p \le p_c$. We study the exponent $\alpha(p)$ in the slightly supercritical regime $p = p_c + \varepsilon$. Our main result says that as $\varepsilon \to 0^+$, $\alpha(p_c + \varepsilon)$ behaves like $\sqrt{\zeta(2)\,\varepsilon}$, where $\zeta(2) := \frac{\pi^2}{6}$.

Keywords: Random graph; critical behaviour

2010 Mathematics Subject Classification: Primary 60J80; 05C80
Secondary 82B20; 82B27

1. Introduction

1.1. The model and the main result

Hierarchical lattices were studied in the physics literature [5, 7, 13, 24] as lattices tailored for real-space renormalization techniques to become exact; in the absence of disorder, they allow one to calculate exactly critical points and exponents, by analysing the neighbourhood of

Received 26 June 2025; accepted 26 June 2025.
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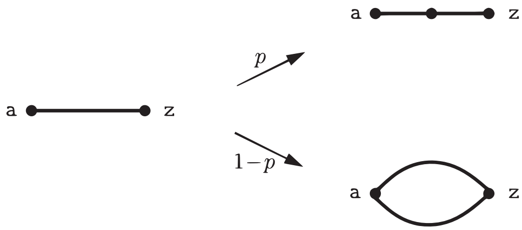

FIGURE 1. At each step of the construction of the hierarchical lattice, each edge of the lattice is replaced by two edges in series with probability p or by two edges in parallel with probability $1-p$. The left shows graph $\texttt{Graph}_0(p)$ and the right shows the two possibilities for the graph $\texttt{Graph}_1(p)$.

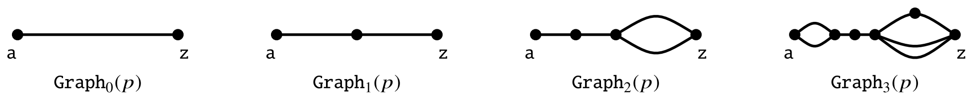

FIGURE 2. An example of the first four graphs in the sequence $(\texttt{Graph}_n(p))_{n\geq 0}$.

fixed points of some dynamical systems. In the disordered case, like in the problem of directed polymers or interfaces in a random medium [6, 8, 11] as well as their simplified versions [9, 16], disordered spin models [22], or percolation clusters [15], the hierarchical structure allows one to formulate the problem as a recursion for a probability distribution. Despite this simple and elegant formulation, these recursions are often hard to analyse, especially near transition points. This is why most models with disorder on hierarchical lattices remain unsolved.

The series–parallel random graph, investigated by Hambly and Jordan [14], is an example of a hierarchical lattice with disorder. The original motivation was to view this graph as a random environment which can be studied as a dynamical system on probability measures; see the introduction in [14] for more details.

To define the series–parallel random graph, we fix a parameter $p \in [0,\ 1]$, and consider two vertices denoted by $\texttt{a}$ and $\texttt{z}$, respectively. The graph $\texttt{Graph}_0(p)$ is simply composed of vertices $\texttt{a}$ and $\texttt{z}$, as well as a (non-oriented) edge connecting $\texttt{a}$ and $\texttt{z}$. We now define $\texttt{Graph}_n(p)$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$, recursively as follows: $\texttt{Graph}_{n+1}(p)$ is obtained by replacing *independently* each edge of $\texttt{Graph}_n(p)$, either by two edges in series with probability p, or by two parallel edges with probability $1-p$. See Figure 1.

The graph $\texttt{Graph}_n(p)$ is called the *series–parallel random graph of order n*. See Figure 2 for an example.

Let $D_n(p)$ denote the graph distance between vertices $\texttt{a}$ and $\texttt{z}$ on $\texttt{Graph}_n(p)$, i.e. the minimal number of edges necessary to connect $\texttt{a}$ and $\texttt{z}$ on $\texttt{Graph}_n(p)$. Since $n \mapsto D_n(p)$ is non-decreasing, we have $D_n(p) \uparrow D_\infty(p) := \sup_{k\geq 1} D_k(p)$ (which may be infinite). It is known [14] that $D_\infty(p) < \infty$ almost surely (a.s.) for $p \in [0,\ \frac{1}{2})$, and $D_\infty(p) = \infty$ a.s. for $p \in [\frac{1}{2},\ 1]$. As such, there is a phase transition for $D_n(p)$ at $p = p_c := \frac{1}{2}$. In this article we focus on the slightly supercritical case: $p = p_c + \varepsilon$ when $\varepsilon > 0$ is sufficiently small.

Let us fix $p \in [0,\ 1]$ for the time being. There is a simple recursive formula for the law of $D_n(p)$. In fact, $D_0(p) = 1$; for $n \in \mathbb{N}$, by considering the two edges of $\texttt{Graph}_1(p)$, we have

$$D_{n+1}(p) \overset{\text{law}}{=} (D_n(p) + \widehat{D}_n(p))\mathscr{E}_n + \min\,(D_n(p), \widehat{D}_n(p))(1-\mathscr{E}_n), \tag{1.1}$$

https://doi.org/10.1017/apr.2025.10023 Published online by Cambridge University Press

where "$\overset{\text{law}}{=}$" denotes identity is distribution, $\widehat{D}_n(p)$ is an independent copy of $D_n(p)$, $\mathscr{E}_n$ is a Bernoulli(p) random variable $\mathbb{P}(\mathscr{E}_n = 1) = p = 1 - \mathbb{P}(\mathscr{E}_n = 0)$, and $D_n(p)$, $\widehat{D}_n(p)$, and $\mathscr{E}_n$ are assumed to be independent. Equation (1.1) defines a random iterative system. There is an important literature on random iteration functions, see for example [12, 17, 19, 23, 25]. Equation (1.1) is also an interesting addition to the list of recursive distributional equations analyzed in the seminal paper [3].

Let us briefly explain that the limit $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[D_n(p)] =: \alpha(p)$ exists. To this end, we fix $m, n \in \mathbb{N}$. By definition, at step n, there exists a certain path Γ_n on $\texttt{Graph}_n(p)$ of length $D_n(p)$ (i.e. number of edges) connecting vertices $\texttt{a}$ and $\texttt{z}$. After m additional steps, edges in Γ_n are transformed into independent copies of $\texttt{Graph}_m(p)$. Accordingly,

$$D_{n+m}(p) \le \sum_{i=1}^{D_n} \Delta_i,$$

where, given $D_n(p)$, the Δ_i are (conditionally) independent having the same law as $D_m(p)$. Taking expectation, we get that $\mathbb{E}[D_{n+m}(p)] \le \mathbb{E}[D_n(p)]\,\mathbb{E}[D_m(p)]$. Since $D_n(p) \le 2^n$ and by the Fekete lemma on subadditive sequences, we obtain that

$$\alpha(p) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[D_n(p)] = \inf_{k\ge 1} \frac{1}{k} \log \mathbb{E}[D_k(p)] \in [0,\ \log 2], \tag{1.2}$$

exists. An easy coupling also implies that $p \mapsto \alpha(p)$ is non-decreasing on $[0,\ 1]$. The main result of the paper concerns the behaviour of the exponent $\alpha(p_c + \varepsilon)$, when $\varepsilon > 0$ in the neighbourhood of 0. Let us mention here that the results stated in this article concern the law of the $D_n(p)$ only and that they hold true for any sequence of random variables satisfying the distributional equation (1.1).

Theorem 1. *Let $(D_n(p))_{n\in\mathbb{N}}$ satisfy (1.1) with $D_0(p) = 1$. Let $\alpha(\cdot)$ be as in (1.2). Then*

$$\lim_{\varepsilon\to 0^+} \frac{\alpha\left(\frac{1}{2} + \varepsilon\right)}{\sqrt{\varepsilon}} = \frac{\pi}{\sqrt{6}}. \tag{1.3}$$

Proof. See Sections 3 and 4. □

We have not been able to prove a convergence of $\frac{1}{n} \log D_n$ when $p \in \left(\frac{1}{2}, 1\right)$, However, we prove the following theorem which provides a partial answer as $p \downarrow p_c = \frac{1}{2}$.

Theorem 2. *Let $(D_n(p))_{n\in\mathbb{N}}$ satisfy (1.1) with $D_0(p) = 1$. Then there exists a function $\widetilde{\alpha} : [\frac{1}{2}, 1] \to [0, \log 2]$ such that $\lim_{\varepsilon\to 0^+} \widetilde{\alpha}(p_c + \varepsilon)/\sqrt{\varepsilon} = \pi/\sqrt{6}$ and $\mathbb{P}$-a.s.*

$$\widetilde{\alpha}(p_c + \varepsilon) \le \liminf_{n\to\infty} \frac{1}{n} \log D_n(p_c + \varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log D_n(p_c + \varepsilon) \le \alpha(p_c + \varepsilon). \tag{1.4}$$

Thus, $\mathbb{P}$-a.s.

$$\lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\varepsilon}} \liminf_{n\to\infty} \frac{1}{n} \log D_n(p) = \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\varepsilon}} \limsup_{n\to\infty} \frac{1}{n} \log D_n(p) = \frac{\pi}{\sqrt{6}}. \tag{1.5}$$

Proof. See Section 5. □

1.2. Existing results and other problems

Considering $\texttt{Graph}_n(p)$ as an electric network (as in [10] or [20]) and assigning a unit resistance to each edge, the effective resistance $R_n(p)$ of $\texttt{Graph}_n(p)$ from $\texttt{a}$ to $\texttt{z}$ is such that $R_0(p) = 1$ and that for each $n \geq 0$,

$$\mathbb{P}(R_{n+1}(p) = R_n(p) + \widehat{R}_n(p)) = p = 1 - \mathbb{P}\left(\frac{1}{R_{n+1}(p)} = \frac{1}{R_n(p)} + \frac{1}{\widehat{R}_n(p)}\right),$$

where $\widehat{R}_n(p)$ denotes an independent copy of $R_n(p)$. It is known [14] that $\mathbb{E}[R_n(p)] \to \infty$ for $p > \frac{1}{2}$ and $\mathbb{E}[R_n(p)] \to 0$ for $p < \frac{1}{2}$. As such, the effective resistance has a phase transition at $p = \frac{1}{2}$, as for the graph distance (Hambly and Jordan [14] proved that $p = \frac{1}{2}$ is also critical for the Cheeger constant of $\texttt{Graph}_n(p)$, though we do not study the resistance nor the Cheeger constant in this article). On the other hand, it is a straightforward observation that at $p = p_c$, $R_n(p_c)$ has the same distribution as $1/R_n(p_c)$. Hambly and Jordan [14] predicted that for all $y \geq x > 0$,

$$\mathbb{P}\big(R_n(p_c) \in [x,\ y]\big) \to 0, \quad n \to \infty. \tag{1.6}$$

Even though (1.6) looks quite plausible, no rigorous proof has been made available yet: it remains an open problem. Let us also mention that a more quantitative prediction has been made by Addario-Berry et al. [2] (see also [1]): at $p = p_c$, $\frac{|\log R_n(p_c)|}{c\, n^{1/3}}$ would converge weakly to a (shifted) beta distribution, with an appropriate but unknown constant $c \in (0,\ \infty)$.

For all $p \in [0,\ 1]$, Hambly and Jordan [14] also conjectured the existence of a constant $\theta(p) \in \mathbb{R}$ such that $\frac{1}{n} \log R_n(p) \to \theta(p)$ in probability.

Hambly and Jordan [14] also studied the first passage percolation problem on $\texttt{Graph}_n(p)$, which amounts to study the same recursion (1.1) as for the graph distance $D_n(p)$, but with a more general initial distribution instead of $D_0(p) = 1$. As far as the distance $D_n(p)$ is concerned, we mention that weak convergence for $D_n(p)$ at criticality $p = p_c := \frac{1}{2}$ has been investigated by Auffinger and Cable [4]. Let us also mention that the hierarchical structure in $\texttt{Graph}_n(p)$ turned out to be very convenient for study of the 'ant problem' for reinforcement [18].

The rest of the article is structured as follows. In Section 2, we present some heuristics leading to Theorem 1, and give an outline of the proof. The lower and upper bounds in Theorem 1 are proved in Sections 3 and 4, respectively.

2. Heuristics and Description of the Approach

This aim of this short section is two-fold: the first part describes some heuristics about what led us to the conclusion in Theorem 1 and the second part provides an outline of the proof of Theorem 1.

2.1. Heuristics for Theorem 1

Let $p \in (0,\ 1)$. Write

$$a_n(k) := \mathbb{P}(D_n = k), \quad n \geq 0,\ k \geq 1.$$

By (1.1), we get, for $k \geq 1$,

$$a_{n+1}(k) = p \sum_{1 \leq i < k} a_n(i) a_n(k-i) + (1-p)\left(2a_n(k)\left(1 - \sum_{1 \leq i < k} a_n(i)\right) - a_n(k)^2\right). \tag{2.1}$$

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Let $p = p_c + \varepsilon = \frac{1}{2} + \varepsilon$ and assume that for large n and for k such that $\log k = \mathcal{O}(\sqrt{n})$ that $a_n(k)$ takes the following scaling form:

$$a_n(k) = \frac{1}{k\sqrt{n}} f\left(n, \ \frac{\log k}{\sqrt{n}}\right), \tag{2.2}$$

with an appropriate bivariate function $(t, \ x) \mapsto f(t, \ x)$. This scaling was already used by Auffinger and Cable [4] in the case $p = p_c$. Using the fact that

$$\frac{1}{i(k-i)} = \frac{1}{k}\left(\frac{1}{i} + \frac{1}{k-i}\right),$$

we show that (2.1) implies heuristically the following equation (the details of the derivation of (2.3) from (2.1) are given in Appendix A)

$$t\frac{\partial f}{\partial t} = \frac{x}{2}\frac{\partial f}{\partial x} + \frac{f}{2} - Kf\frac{\partial f}{\partial x} + \varepsilon t\left(-2f + 4f\int_0^x f(t, \ y)\,\mathrm{d}y\right), \tag{2.3}$$

where

$$K := \int_0^1 \frac{\log\left(\frac{1}{1-u}\right)}{u}\,\mathrm{d}u = \frac{\pi^2}{6}.$$

For $\varepsilon = 0$ one can solve (2.3) exactly for an arbitrary initial condition, $F_0(x) \equiv f(t_0, x)$ at time $t_0 > 0$. It is in fact easy to write the solution in a parametric form at arbitrary time $t \geq t_0$:

$$f(t, \ x) = \sqrt{\frac{t}{t_0}}F_0(y); \quad x = K\left(\sqrt{\frac{t}{t_0}} - \sqrt{\frac{t_0}{t}}\right)F_0(y) + y\sqrt{\frac{t_0}{t}}. \tag{2.4}$$

If one writes

$$\mathrm{d}x = \left[K\left(\sqrt{\frac{t}{t_0}} - \sqrt{\frac{t_0}{t}}\right)F_0'(y) + \sqrt{\frac{t_0}{t}}\right]\mathrm{d}y, \tag{2.5}$$

and if $F_0(y)$ vanishes at the boundaries, then one can see that

$$\int f(t, x)\,\mathrm{d}x = \int F_0(y)\,\mathrm{d}y, \tag{2.6}$$

so that the normalization of F_0 is conserved. However, after a finite time $t^* > t_0$, given by the first time for which $\mathrm{d}x/\mathrm{d}y$ as in (2.5) vanishes for some y i.e.

$$t^* = t_0\left[1 - \frac{1}{K}\max_y \frac{1}{F_0'(y)}\right],$$

the solution F of (2.4) becomes multi-valued (see Figure 3 below). Then the solution is still given by (2.4) for some ranges of x with one of several shocks as on the right of Figure 3. In the example of Figure 3, if $f_a(t, x) > f_b(t, x) > f_c(t, x)$ are the three expressions of $f(t, x)$ given by the parametric form (2.4) in the region $(x_1(t), \ x_2(t))$ where $f(t, \ x)$ is multi-valued, the true solution is

$$f(t, x) = \begin{cases} f_a(t, x) & \text{for } x_1 < x < x_*, \\ f_c(t, x) & \text{for } x_* < x < x_2, \end{cases}$$

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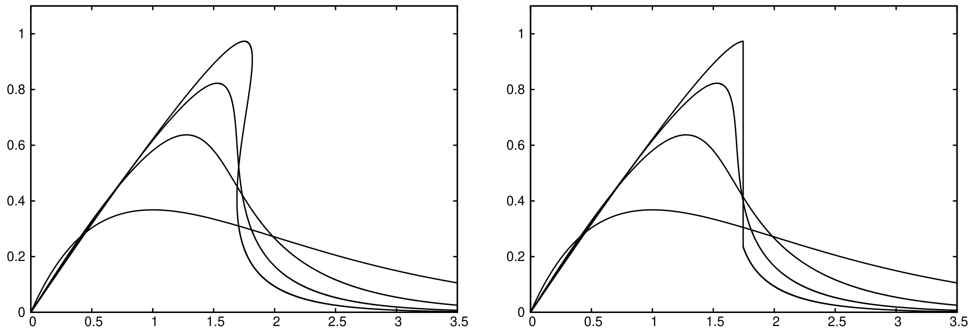

FIGURE 3. On the left, solution $F(t,\ x)$ using the parametric form (2.4) at times t_0, $3t_0$, $5t_0$, and $7t_0$ for $F_0(z) = z\,\mathrm{e}^{-z}$. The same with the shock on the right.

with $x_*(t)$ determined by

$$\int_{x_1(t)}^{x_*(t)} [f_a(t,x) - f_b(t,x)]\,\mathrm{d}x = \int_{x_*(t)}^{x_2(t)} [f_b(t,x) - f_c(t,x)]\,\mathrm{d}x.$$

With this choice of $x_*(t)$ the normalization (2.6) of f is preserved.

In the long time limit, in that way from (2.4) one obtains that

$$f(t,x) = \begin{cases} \frac{x}{K} & \text{for } 0 < x < x_*(\infty), \\ 0 & \text{for } x \geq x_*(\infty), \end{cases} \qquad \text{with } x_*(\infty) := \sqrt{2K}. \tag{2.7}$$

For $\varepsilon = 0$, this is in agreement with the description in [4].

For $\varepsilon \neq 0$, we did not succeed in solving (2.3) for an arbitrary initial condition. However, if one starts at $t = 0$ with the $\varepsilon = 0$ solution (2.7), one can solve (2.3) explicitly:

$$f(t,\ x) = \frac{1}{\mathrm{e}^{2\varepsilon t} - 1}\sqrt{\frac{\varepsilon t}{K}}\,\sinh\left(2x\sqrt{\frac{\varepsilon t}{K}}\right). \tag{2.8}$$

Since $\sum_{k=1}^{\infty} a_n(k) = 1$ for all n, the function $x \mapsto f(t,\ x)$ is a probability density function for all t. By analogy with the case $\varepsilon = 0$, we assume that (2.8) is valid only for $x \in [0,\ x_*(t)]$, where $x_*(t)$ is such that

$$\int_0^{x_*(t)} \frac{1}{\mathrm{e}^{2\varepsilon t} - 1}\sqrt{\frac{\varepsilon t}{K}}\,\sinh\left(2x\sqrt{\frac{\varepsilon t}{K}}\right)\mathrm{d}x = 1$$

and we take $f(t,\ x) = 0$ for $x > x_*(t)$. When $t \to \infty$, we have

$$x_*(t) \approx \sqrt{K\varepsilon t}.$$

This implies that for $n \to \infty$,

$$\log \mathbb{E}[D_n(p_c + \varepsilon)] \approx \sqrt{K\varepsilon}\,n.$$

Consequently,

$$\alpha(p_c + \varepsilon) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[D_n(p_c + \varepsilon)] \approx \sqrt{K\varepsilon}, \qquad \varepsilon \to 0^+,$$

which leads to the statement of Theorem 1.

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2.2. Outline of the proof of Theorem 1

It does not seem obvious how to turn the heuristics described in Section 2.1 directly into a rigorous argument. Our proof goes along different lines, even though the ideas are guided by the heuristics.

To prove Theorem 1, we study the following dynamical system on the set $\mathcal{M}_1$ of all probability measures on $\mathbb{R}_+^* := (0, \infty)$: we use the notation $a \wedge b := \min\{a, b\}$. Let $p \in [0, 1]$; for all $\mu \in \mathcal{M}_1$, we define the probability measure $\Psi_p(\mu)$ as follows:

$$\Psi_p(\mu) \text{ is the law of } (X + \widehat{X})\mathscr{E} + (X \wedge \widehat{X})(1 - \mathscr{E}), \tag{2.9}$$

where X, $\widehat{X}$, and $\mathscr{E}$ are independent, X and $\widehat{X}$ have law μ, and $\mathscr{E}$ is a Bernoulli random variable with parameter p (all the random variables mentioned in this article are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that is assumed to be sufficiently rich to carry as many independent random variables as needed). We observe that if μ_n stands for the law of $D_n(p)$, then $\mu_{n+1} = \Psi_p(\mu_n)$ and $\mu_0 = \delta_1$.

Let us briefly mention two basic properties of Ψ_p. First, Ψ_p is *homogeneous*; namely,

$$\text{for all } \theta \in \mathbb{R}_+^*, \text{ for all } \mu \in \mathcal{M}_1, \quad \Psi_p(M_\theta(\mu)) = M_\theta(\Psi_p(\mu)), \tag{2.10}$$

where, for all $\theta \in \mathbb{R}_+^*$ and for any random variable X with law $\mu \in \mathcal{M}_1$, $M_\theta(\mu)$ denotes the law of θX.

We next observe that Ψ_p *preserves the stochastic order* $\overset{\text{st}}{\leq}$ on $\mathcal{M}_1$ that is defined as follows: $\mu \overset{\text{st}}{\leq} \nu$ if $\mu((x, \infty)) \leq \nu((x, \infty))$ for all $x \in \mathbb{R}_+^*$ (which is equivalent to the existence of two random variables X and Y whose laws are respectively μ and ν and such that $X \leq Y$, $\mathbb{P}$-a.s.). Indeed, for all $\mu, \nu \in \mathcal{M}_1$,

$$\big(\mu \overset{\text{st}}{\leq} \nu\big) \Longrightarrow \big(\Psi_p(\mu) \overset{\text{st}}{\leq} \Psi_p(\nu)\big). \tag{2.11}$$

In the rest of the article, it will sometimes be convenient to abuse slightly the notation by writing $X \overset{\text{st}}{\leq} Y$ for any pair of random variables X and Y to mean that the law of X is stochastically less than that of Y.

2.2.1. *Strategy for the lower bound in Theorem 1.* We fix $p = \frac{1}{2} + \varepsilon$ and denote by μ_n the law of $D_n(p)$ as defined in (1.1). As already mentioned, $\mu_{n+1} = \Psi_p(\mu_n)$ for all integers $n \geq 0$ and $\mu_0 = \delta_1$. Let $\theta, \beta \in \mathbb{R}_+^*$, and $n_0 \in \mathbb{N}^*$. Suppose that we are able to find an $\mathcal{M}_1$-valued sequence $(\nu_n)_{n \geq n_0}$ such that for $n \geq n_0$,

$$M_\theta(\nu_{n_0}) \overset{\text{st}}{\leq} \mu_{n_0} \quad \text{and} \quad M_\beta(\nu_{n+1}) \overset{\text{st}}{\leq} \Psi_p(\nu_n). \tag{2.12}$$

We call the ν_n the *lower bound laws*. For all integers $n \geq n_0$, let Y_n and U_n be random variables with respective laws ν_n and $\Psi_p(\nu_n)$. Then (2.12) can be rewritten as $\theta Y_{n_0} \overset{\text{st}}{\leq} D_{n_0}(p)$ and $\beta Y_{n+1} \overset{\text{st}}{\leq} U_n$. By (2.10) and (2.11), for all $n \geq n_0$,

$$M_{\theta\beta^{n-n_0}}(\nu_n) \overset{\text{st}}{\leq} \mu_n;$$

in other words, $\theta\beta^{n-n_0} Y_n \overset{\text{st}}{\leq} D_n(p)$, which, in turn, implies that

$$\mathbb{E}[D_n(p)] \geq \theta\beta^{n-n_0}\mathbb{E}[Y_n], \quad n \geq n_0. \tag{2.13}$$

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Here, β and $\mathbb{E}[Y_n]$ turn out to be sufficiently explicit in terms of n and ε to provide a good lower bound for $\mathbb{E}[D_n(p)]$ and, thus, for $\alpha(\frac{1}{2}+\varepsilon) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[D_n(p)]$. To find appropriate lower bound laws ν_n (i.e. laws such that (2.12) holds), we are guided by the heuristics described in Section 2.1.

Remark 1. In the proof of the lower bound in Theorem 1 in Section 3, we need to fix several parameters in (the law of) the random variable Y_n introduced in the strategy in the previous paragraph. Even though we are guided by the heuristics in Section 2.1, the choice of these parameters is not obvious, and is in fact rather delicate, resulting from a painful procedure of adjustment. A similar remark applies to the proof of the upper bound in Theorem 1.

2.2.2. *Strategy for the upper bound in Theorem 1*. The strategy is identical, except that the values of the parameters n_0, β and θ, are different: suppose we are able to find an $\mathcal{M}_1$-valued sequence $(\nu'_n)_{n\geq n_0}$ such that for $n \geq n_0$,

$$\mu_{n_0} \overset{\text{st}}{\leq} M_\theta(\nu'_{n_0}) \quad \text{and} \quad \Psi_p(\nu'_n) \overset{\text{st}}{\leq} M_\beta(\nu'_{n+1}); \tag{2.14}$$

we call ν'_n the *upper bound laws*. For all integer $n \geq n_0$, let Z_n and V_n be random variables with respective laws ν'_n and $\Psi_p(\nu'_n)$. By (2.10) and (2.11), for all $n \geq n_0$,

$$\mu_n \overset{\text{st}}{\leq} M_{\theta\beta^{n-n_0}}(\nu'_n); \tag{2.15}$$

i.e. $D_n(p) \overset{\text{st}}{\leq} \theta\beta^{n-n_0} Z_n$, which, in turn, implies that

$$\mathbb{E}[D_n(p)] \leq \theta\beta^{n-n_0}\mathbb{E}[Z_n].$$

As in the lower bound, β and $\mathbb{E}[Z_n]$ will be sufficiently explicit in terms of n and ε to provide an upper bound for $\mathbb{E}[D_n(p)]$ and, thus, for $\alpha(\frac{1}{2}+\varepsilon)$. To find appropriate upper bound laws ν'_n (i.e. laws such that (2.14) holds), again we follow the heuristics described in Section 2.1.

3. Proof of Theorem 1: The Lower Bound

3.1 Definition of the lower bound laws

In this section we construct *lower bound laws*, i.e. an $\mathcal{M}_1$-valued sequence $(\nu_n)_{n\geq 1}$ satisfying (2.12).

First note that the polynomial function $P(\eta) = (1-\eta)(1+\eta)^2$ is such that $P(0) = 1$ and that $P'(0) = 1 > 0$. Thus, there exists $\eta_0 \in (0, 1)$ such that

$$\forall \eta \in (0, \eta_0), \quad (1-\eta)(1+\eta)^2 > 1, \quad \widetilde{\eta} := \frac{12(1+\eta)^2\eta}{\pi^2} < 1. \tag{3.1}$$

We recall that

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Let $\varepsilon \in (0, \frac{1}{2})$. We set, for $n \in \mathbb{N}$,

$$\delta = \frac{2}{\sqrt{\zeta(2)}}(1+\eta)\sqrt{\varepsilon} \quad \text{and} \quad a_n = \frac{1}{4}\big(1-2\varepsilon+\eta\delta^2\big)^n = \frac{1}{4}\big(1-2(1-\widetilde{\eta})\varepsilon\big)^n. \tag{3.2}$$

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The following increasing function $\varphi_\delta : [1, \infty) \to \mathbb{R}_+$ plays a key role in the rest of the paper:

$$\varphi_\delta(x) := \cosh(\delta \log x) - 1 = \frac{1}{2}\big(x^\delta + x^{-\delta}\big) - 1, \quad x \in [1, \infty). \tag{3.3}$$

In the following lemma, we introduce the laws ν_n and provide basic properties of these laws. They are shown to satisfy (2.12) in Lemma 2, the proof of which is the main technical step of the proof of the lower bound in Theorem 1. Let us mention that the notation $\mathcal{O}_{\eta,\varepsilon}(a_n)$ denotes an expression which, for fixed (η, ε) is $\mathcal{O}(a_n)$. The remark also applies to forthcoming expressions such as $\mathcal{O}_\eta(\cdot)$ or $\mathcal{O}_\delta(\cdot)$.

Lemma 1. *Let $\eta \in (0, \eta_0)$ and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Let δ and $(a_n)_{n\in\mathbb{N}}$ be defined in (3.2). Recall the function φ_δ from (3.3). For all $n \in \mathbb{N}$, let $\lambda_n \in (1, \infty)$ be the unique real number such that $2a_n\varphi_\delta(\lambda_n) = 1$ and let $\nu_n = \nu_n(\eta, \varepsilon)$ be the unique element of $\mathcal{M}_1$ such that*

$$\nu_n((0, 1]) = 0 \quad \text{and} \quad \nu_n((0, x]) = 2a_n\varphi_\delta(x \wedge \lambda_n), \quad x \in [1, \infty). \tag{3.4}$$

Then the following hold.

(i) *We have $\lambda_n = a_n^{-1/\delta}\big(1 + \mathcal{O}_{\eta,\varepsilon}(a_n)\big)$ as $n \to \infty$.*

(ii) *We have $\lim_{n\to\infty} 2a_n\varphi_\delta(\eta^{1/\delta}\lambda_n) = \eta$.*

(iii) *There exists $\epsilon_1(\eta) \in \left(0, \frac{1}{2}\right)$ such that for all $\varepsilon \in (0, \epsilon_1(\eta))$ there is $n_1(\eta, \varepsilon) \in \mathbb{N}$ satisfying $\lambda_{n+1} - \lambda_n + 1 \le \delta\lambda_n$ for all $n \ge n_1(\eta, \varepsilon)$.*

(iv) *Let Y_n be a random variable with law ν_n. Then*

$$\mathbb{E}[Y_n] = \delta a_n \left(\frac{\lambda_n^{1+\delta} - 1}{1+\delta} - \frac{\lambda_n^{1-\delta} - 1}{1-\delta}\right). \tag{3.5}$$

Furthermore,

$$\frac{1}{\sqrt{\varepsilon}} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[Y_n] = \frac{-\log(1 - 2(1-\widetilde{\eta})\varepsilon)}{\delta\sqrt{\varepsilon}} \xrightarrow[\varepsilon\to 0^+]{} \sqrt{\zeta(2)}\,\frac{1-\widetilde{\eta}}{1+\eta} \xrightarrow[\eta\to 0^+]{} \sqrt{\zeta(2)}. \tag{3.6}$$

Proof. Recall that the inverse on $\mathbb{R}_+$ of cosh is the function $\operatorname{arcosh}(y) = \log(y + \sqrt{y^2 - 1})$, $y \in [1, \infty)$ and that $\operatorname{arcosh}(\frac{y}{2}) = \log y - y^{-2} + \mathcal{O}(y^{-4})$ as $y \to \infty$. Thus, we get

$$\begin{aligned}\delta \log \lambda_n &= \operatorname{arcosh}\left(1 + \frac{1}{2a_n}\right) = \log\left(\frac{1+2a_n}{a_n}\right) - \frac{a_n^2}{(1+2a_n)^2} + \mathcal{O}\left(\frac{a_n^4}{(1+2a_n)^4}\right)\\ &= \log\frac{1}{a_n} + \mathcal{O}_{\eta,\varepsilon}(a_n),\end{aligned}$$

which immediately implies (i). Since $\varphi_\delta(x) \sim_{x\to\infty} \frac{1}{2}x^\delta$, we get $2a_n\varphi_\delta(\eta^{1/\delta}\lambda_n) \sim_{n\to\infty} \eta a_n\lambda_n^\delta$ which tends to η as $n \to \infty$. This proves (ii).

Let us prove (iii). Recall here that $\eta \in (0, \eta_0)$ is fixed. To simply notation, we set $\rho := (1 - 2(1-\widetilde{\eta})\varepsilon)^{-1} > 1$, so that $a_n = \frac{1}{4}\rho^{-n}$. It follows from (i) that

$$\lambda_{n+1} - \lambda_n = (\rho^{1/\delta} - 1)\lambda_n(1 + \mathcal{O}_{\eta,\varepsilon}(a_n)), \quad n \to \infty. \tag{3.7}$$

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We also set $c=2\zeta(2)^{-1/2}$; thus, $\delta=c(1+\eta)\sqrt{\varepsilon}$. We observe that

$$\frac{\log\rho}{\delta}=\frac{2(1-\widetilde{\eta})\varepsilon+\mathcal{O}_\eta(\varepsilon^2)}{c(1+\eta)\sqrt{\varepsilon}}=\frac{2(1-\widetilde{\eta})}{c^2(1+\eta)^2}\delta+\mathcal{O}_\eta(\varepsilon^{3/2}),\quad \varepsilon\to 0^+. \tag{3.8}$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\delta^{-1}(\rho^{1/\delta}-1)=\frac{2(1-\widetilde{\eta})}{c^2(1+\eta)^2}<\frac{2}{c^2}=\frac{\pi^2}{12}<1$$

and there exists $\epsilon_1(\eta)\in(0,\,1/2)$ such that for all $\varepsilon\in(0,\,\epsilon_1(\eta)(\eta))$, $\rho^{1/\delta}-1<\delta$, which, combined with (3.7), yields (iii).

Let us prove (iv). Observe that

$$\begin{aligned}\mathbb{E}[Y_n]&=\int_1^{\lambda_n}2a_n\varphi_\delta'(x)x\,\mathrm{d}x=2a_n\delta\int_1^{\lambda_n}\sinh(\delta\log x)\,\mathrm{d}x\\&=2a_n\delta\int_0^{\log\lambda_n}\sinh(\delta u)\mathrm{e}^u\,\mathrm{d}u=a_n\delta\int_0^{\log\lambda_n}\left(\mathrm{e}^{u(1+\delta)}-\mathrm{e}^{u(1-\delta)}\right)\mathrm{d}u,\end{aligned}$$

which implies (3.5). By (i), we thus get $\mathbb{E}[Y_n]\sim_{n\to\infty}(\delta/1+\delta)a_n^{-1/\delta}$. Since $a_n=\frac{1}{4}\rho^{-n}$ with $\rho=(1-2(1-\widetilde{\eta})\varepsilon)^{-1}$, this implies that $\lim_{n\to\infty}n^{-1}\log\mathbb{E}[Y_n]=\delta^{-1}\log\rho$, which readily yields (3.6) by means of (3.8).

The following lemma asserts that the laws ν_n defined in Lemma 1 satisfy the right-hand side of (2.12) with $\beta=1$.

Lemma 2. *Let $\eta\in(0,\,\eta_0)$ and let $\varepsilon\in(0,\,\frac{1}{2})$. Let δ and $(a_n)_{n\in\mathbb{N}}$ be as in (3.2) and let ν_n be defined by (3.4). For all $n\in\mathbb{N}$, we denote by Y_n and $\widehat{Y}_n$ two independent random variables with common law ν_n. Then there exists $\varepsilon_2(\eta)\in(0,\,\frac{1}{2})$ such that for all $\varepsilon\in(0,\,\varepsilon_2(\eta))$, there is $n_2(\eta,\,\varepsilon)\in\mathbb{N}$ such that for all integers $n\geq n_2(\eta,\,\varepsilon)$ and all $y\in\mathbb{R}_+^*$,*

$$\left(\frac{1}{2}+\varepsilon\right)\mathbb{P}\Big(Y_n+\widehat{Y}_n>y\Big)+\left(\frac{1}{2}-\varepsilon\right)\mathbb{P}\Big(Y_n>y\Big)^2-1+\mathbb{P}\Big(Y_{n+1}\leq y\Big)\geq 0. \tag{3.9}$$

Proof. See Section 3.2.

Proof of the lower bound in Theorem 1. Let us admit Lemma 2 for the time being and prove that it implies the lower bound in Theorem 1.

Recall that $p=\frac{1}{2}+\varepsilon$, so (3.9) implies for all $n\geq n_2(\eta,\varepsilon)$ that $\nu_{n+1}((y,\infty))\leq\Psi_p(\nu_n)((y,\infty))$ for all $y\in\mathbb{R}_+^*$, i.e. $\nu_{n+1}\overset{\text{st}}{\leq}\Psi_p(\nu_n)$. On the other hand, we note that a.s. $Y_{n_2(\eta,\varepsilon)}\leq\lambda_{n_2(\eta,\varepsilon)}$ and $1\leq D_{n_2(\eta,\varepsilon)}(p)$. Thus, a.s. $Y_{n_2(\eta,\varepsilon)}/\lambda_{n_2(\eta,\varepsilon)}\leq D_{n_2(\eta,\varepsilon)}(p)$, which implies

$$M_\theta(\nu_{n_2(\eta,\varepsilon)})\overset{\text{st}}{\leq}\mu_{n_2(\eta,\varepsilon)},$$

where $\theta:=1/\lambda_{n_2(\eta,\varepsilon)}$ (as before, μ_n stands for the law of $D_n(p)$). As such, the laws ν_n satisfy (2.12) with $n_0=n_2(\eta,\,\varepsilon)$, $\theta=1/\lambda_{n_2(\eta,\varepsilon)}$ and $\beta=1$. It follows from (2.13) that for all $n\geq n_2(\eta,\varepsilon)$, $\mathbb{E}[D_n(p)]\geq\theta\,\mathbb{E}[Y_n]$, thus $\alpha\left(\frac{1}{2}+\varepsilon\right)=\lim_{n\to\infty}(1/n)\log\mathbb{E}[D_n(p)]\geq\lim_{n\to\infty}(1/n)\log\mathbb{E}[Y_n]$ and we get $\liminf_{\varepsilon\to 0^+}\alpha\left(\frac{1}{2}+\varepsilon\right)/\sqrt{\varepsilon}\geq\sqrt{\zeta(2)}$ by (3.6) in Lemma 1. □

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3.2. Proof of Lemma 2

We fix $\eta \in (0,\ \eta_0)$ and $\varepsilon \in (0,\ \frac{1}{2})$. Writing $\mathrm{LHS}_{(3.9)}$ for the expression on the left-hand side of (3.9), we need to check that $\mathrm{LHS}_{(3.9)} \geq 0$ for all large n, uniformly in $y \in [1,\ \lambda_{n+1}]$ because this inequality is obvious if $y \in [0,\, 1)$ and $y \in (\lambda_{n+1},\, \infty)$. Let $n_3(\eta,\, \varepsilon)$ be such that $\eta^{1/\delta}\lambda_n \geq 2$ for all $n \geq n_3(\eta,\, \varepsilon)$. Thus, to prove (3.9), we suppose that $n \geq n_3(\eta,\, \varepsilon)$ and we consider three situations:

Case 1: $y \in [1,\ \eta^{1/\delta}\lambda_n]$, **Case 2:** $y \in (\eta^{1/\delta}\lambda_n,\ \lambda_n]$, and **Case 3:** $y \in (\lambda_n,\ \lambda_{n+1}]$.

For the sake of clarity, these situations are dealt with in three distinct parts.

Proof of Lemma 2: first case $y \in [1,\ \eta^{1/\delta}\lambda_n]$. In this case, we first note that

$$\mathbb{P}\big(Y_n + \widehat{Y}_n > y\big) \geq 1 - \mathbb{P}\big(Y_n \leq y;\, \widehat{Y}_n \leq y\big) = 1 - \mathbb{P}\big(Y_n \leq y\big)^2.$$

Therefore, writing $F_n(y) := \mathbb{P}(Y_n \leq y)$, we obtain

$$\begin{aligned}\mathrm{LHS}_{(3.9)} &\geq \left(\frac{1}{2} + \varepsilon\right)\left(1 - F_n(y)^2\right) + \left(\frac{1}{2} - \varepsilon\right)\big(1 - F_n(y)\big)^2 - 1 + F_{n+1}(y)\\ &= -(1-2\varepsilon)F_n(y) - 2\varepsilon F_n(y)^2 + F_{n+1}(y).\end{aligned}$$

For $y \in [1,\ \eta^{1/\delta}\lambda_n]$, we have $y \leq \lambda_n < \lambda_{n+1}$, thus $F_n(y) = 2a_n\varphi_\delta(y \wedge \lambda_n) = 2a_n\varphi_\delta(y)$, and $F_{n+1}(y) = 2a_{n+1}\varphi_\delta(y) = (1 - 2\varepsilon + \eta\delta^2)2a_n\varphi_\delta(y)$. Accordingly,

$$\mathrm{LHS}_{(3.9)} \geq 2\eta\delta^2 a_n\varphi_\delta(y) - 2\varepsilon(2a_n\varphi_\delta(y))^2.$$

Observe that $\delta^2/(2\varepsilon) = 12(1+\eta)^2/\pi^2 > 1$, thus $\frac{\eta\delta^2}{2\varepsilon} > \eta$, which equals $\lim_{n\to\infty} 2a_n\varphi_\delta(\eta^{1/\delta}\lambda_n)$ by Lemma 1(ii). Thus, there exists $n_4(\eta,\, \varepsilon) \geq n_3(\eta,\, \varepsilon)$ such that for all $n \geq n_4(\eta,\, \varepsilon)$ and all $y \in [1,\, \eta^{1/\delta}\lambda_n]$, we have

$$2a_n\varphi_\delta(y) \leq \frac{\eta\delta^2}{2\varepsilon},$$

which implies that

$$\mathrm{LHS}_{(3.9)} \geq 2\eta\delta^2 a_n\varphi_\delta(y) - 2\varepsilon\frac{\eta\delta^2}{2\varepsilon}2a_n\varphi_\delta(y) = 0,$$

as desired. □

Proof of Lemma 2: second case $y \in [\eta^{1/\delta}\lambda_n,\ \lambda_n]$. In this case and in the next case, the convolution term $Y_n + \widehat{Y}_n$ matters more specifically; we use the following lemma to get estimates for the law of $Y_n + \widehat{Y}_n$. This is a key lemma where the constant $\zeta(2)$ appears.

Lemma 3. *We keep the previous notation and we recall, in particular, δ from (3.2). Let $r \in (0,\, 1]$. We define*

$$\mathtt{cvl}_{\delta,r}(x) := \int_1^{rx-1} \varphi'_\delta(t)\,\big(\varphi_\delta(x) - \varphi_\delta(x-t)\big)\,\mathrm{d}t, \quad x \in \left[\frac{2}{r},\ \infty\right). \tag{3.10}$$

For $r \in (0,\, 1]$, we also set $\kappa(r) := \sum_{k=1}^{\infty} \frac{r^k}{k^2}$ and for $x \in \left[\frac{2}{r},\, \infty\right)$,

$$\kappa_{\delta,r}(x) := \kappa\left(r - \frac{1}{x}\right) - \frac{2}{x} - c_0\,\delta\coth\left(\delta\log x\right),$$

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with

$$c_0 := \int_0^1 \left(\log \frac{1}{s(1-s)}\right)\left(\log \frac{1}{1-s}\right)\frac{ds}{s} \in (0,\ \infty).$$

Then, for $r \in (0,\,1]$ *and* $x \in [2/r,\ \infty)$*,*

$$\kappa_{\delta,r}(x)\big(\delta \sinh(\delta \log x)\big)^2 \le \mathtt{cvl}_{\delta,r}(x) \le \kappa(r)\,\big(\delta \sinh(\delta \log x)\big)^2. \tag{3.11}$$

Proof. We first prove the upper bound in (3.11). Let $\widetilde{\varphi}_\delta(u) := \varphi_\delta(e^u) = \cosh(\delta u) - 1$, for $u \in \mathbb{R}_+$. We fix $r \in (0,\,1]$ and $x \in [2/r,\,\infty)$. Since $\widetilde{\varphi}_\delta$ is convex on $\mathbb{R}_+$, we get $\widetilde{\varphi}_\delta(\log x) - \widetilde{\varphi}_\delta(\log(x-t)) \le (\log x - \log(x-t))\widetilde{\varphi}'_\delta(\log x)$ for $t \ge 0$ and $x \ge t+1$. Thus,

$$\mathtt{cvl}_{\delta,r}(x) = \int_1^{rx-1} \varphi'_\delta(t)\big(\widetilde{\varphi}_\delta(\log x) - \widetilde{\varphi}_\delta(\log(x-t))\big)\,dt \le \widetilde{\varphi}\,'_\delta(\log x)\int_1^{rx-1} \varphi'_\delta(t)\log\frac{x}{x-t}\,dt.$$

By definition, $\widetilde{\varphi}\,'_\delta(\log x) = \delta \sinh(\delta \log x)$, and $\varphi'_\delta(t) = \frac{\delta}{t}\sinh(\delta \log t) \le \frac{\delta}{t}\sinh(\delta \log x)$ if $t \le rx-1$. Thus,

$$\begin{aligned}\mathtt{cvl}_{\delta,r}(x) &\le \big(\delta \sinh(\delta \log x)\big)^2 \int_1^{rx-1} \frac{1}{t}\log\frac{x}{x-t}\,dt \le \big(\delta \sinh(\delta \log x)\big)^2 \int_0^{rx} \frac{1}{t}\log\frac{x}{x-t}\,dt \\ &= \big(\delta \sinh(\delta \log x)\big)^2 \int_0^{r} \frac{\log\frac{1}{1-u}}{u}\,du.\end{aligned}$$

By Fubini–Tonelli,

$$\int_0^r \frac{\log\frac{1}{1-u}}{u}\,du = \sum_{k\ge 1}\int_0^r \frac{1}{u}\frac{u^k}{k}\,du = \kappa(r), \tag{3.12}$$

which yields the upper bound in (3.11).

We turn to the proof of the lower bound in (3.11). Since on $\mathbb{R}_+$, all the derivatives of $\widetilde{\varphi}_\delta$ are positive, $\widetilde{\varphi}_\delta$ and $\widetilde{\varphi}'_\delta$ are convex which implies for all $b \ge a \ge 0$ that

$$\frac{\widetilde{\varphi}_\delta(b) - \widetilde{\varphi}_\delta(a)}{b-a} \ge \widetilde{\varphi}'_\delta(a) = \widetilde{\varphi}'_\delta(b) - (\widetilde{\varphi}'_\delta(b) - \widetilde{\varphi}'_\delta(a)) \ge \widetilde{\varphi}'_\delta(b) - (b-a)\widetilde{\varphi}''_\delta(b).$$

We suppose that $1 \le t \le rx-1$. Taking $b = \log x$ and $a = \log(x-t)$, we get that

$$\begin{aligned}\mathtt{cvl}_\delta(x) &\ge \int_1^{rx-1} \varphi'_\delta(t)\left(\log\frac{x}{x-t}\right)\left(\widetilde{\varphi}'_\delta(\log x) - \left(\log\frac{x}{x-t}\right)\widetilde{\varphi}''_\delta(\log x)\right)dt \\ &= \mathrm{RHS}^{(1)}_{(3.13)} - \mathrm{RHS}^{(2)}_{(3.13)},\end{aligned} \tag{3.13}$$

where we have set

$$\mathrm{RHS}^{(1)}_{(3.13)} := \widetilde{\varphi}'_\delta(\log x)\int_1^{rx-1} \varphi'_\delta(t)\left(\log\frac{x}{x-t}\right)dt,$$

$$\mathrm{RHS}^{(2)}_{(3.13)} := \widetilde{\varphi}''_\delta(\log x)\int_1^{rx-1} \varphi'_\delta(t)\left(\log\frac{x}{x-t}\right)^2 dt.$$

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We first look at $\mathrm{RHS}^{(2)}_{(3.13)}$. Since $\varphi'_\delta(t) = \frac{\delta}{t}\sinh(\delta\log t) \le \frac{\delta}{t}\sinh(\delta\log x)$, we have

$$\int_1^{rx-1} \varphi'_\delta(t)\Big(\log\frac{x}{x-t}\Big)^2\,\mathrm{d}t \le \delta\sinh(\delta\log x)\int_1^{rx-1}\frac{1}{t}\Big(\log\frac{x}{x-t}\Big)^2\,\mathrm{d}t.$$

We observe that

$$\int_1^{rx-1}\frac{1}{t}\Big(\log\frac{x}{x-t}\Big)^2\,\mathrm{d}t \le \int_0^x \frac{1}{t}\Big(\log\frac{x}{x-t}\Big)^2\,\mathrm{d}t = \int_0^1\frac{1}{s}\Big(\log\frac{1}{1-s}\Big)^2\,\mathrm{d}s =: c_1.$$

Therefore,

$$\mathrm{RHS}^{(2)}_{(3.13)} \le c_1\delta\sinh(\delta\log x)\widetilde{\varphi}''_\delta(\log x) = c_1\delta^3\sinh(\delta\log x)\cosh(\delta\log x). \tag{3.14}$$

We now turn to $\mathrm{RHS}^{(1)}_{(3.13)}$ and look for a lower bound. We still assume that $1 \le t \le rx-1$. Since the function sinh is convex on $\mathbb{R}_+$, we have $\sinh(\delta\log t) \ge \sinh(\delta\log x) - \delta(\log x - \log t)\cosh(\delta\log x)$. Therefore,

$$\begin{aligned}
&\int_1^{rx-1}\delta\sinh(\delta\log t)\frac{1}{t}\big(\log\frac{x}{x-t}\big)\,\mathrm{d}t\\
&\ge \delta\int_1^{rx-1}\Big(\sinh(\delta\log x) - \delta\Big(\log\frac{x}{t}\Big)\cosh(\delta\log x)\Big)\frac{1}{t}\Big(\log\frac{x}{x-t}\Big)\,\mathrm{d}t\\
&= \delta\sinh(\delta\log x)\int_1^{rx-1}\frac{\log\frac{x}{x-t}}{t}\,\mathrm{d}t - \delta^2\cosh(\delta\log x)\int_1^{rx-1}\frac{\big(\log\frac{x}{t}\big)\log\frac{x}{x-t}}{t}\,\mathrm{d}t.
\end{aligned}$$

Let us look at the two integrals on the right-hand side. The second integral is easy to deal with:

$$\int_1^{rx-1}\frac{\big(\log\frac{x}{t}\big)\log\frac{x}{x-t}}{t}\,\mathrm{d}t \le \int_0^x\frac{\big(\log\frac{x}{t}\big)\log\frac{x}{x-t}}{t}\,\mathrm{d}t = \int_0^1\frac{\big(\log\frac{1}{s}\big)\log\frac{1}{1-s}}{s}\,\mathrm{d}s =: c_2 \in (0,\ \infty).$$

The first integral is handled as follows: let $r_1 := r - \frac{1}{x} \in (0,\ r)$. Then $rx - 1 = r_1 x$, so that

$$\int_1^{rx-1}\frac{\log\frac{x}{x-t}}{t}\,\mathrm{d}t = \int_0^{r_1x}\frac{\log\frac{x}{x-t}}{t}\,\mathrm{d}t - \int_0^1\frac{\log\frac{x}{x-t}}{t}\,\mathrm{d}t = \kappa(r_1) - \kappa\Big(\frac{1}{x}\Big),$$

since $\int_0^r \frac{1}{u}\log(1-u)^{-1}\,\mathrm{d}u = \kappa(r)$ by (3.12). Consequently,

$$\int_1^{rx-1}\delta\sinh(\delta\log t)\,\frac{1}{t}\,\Big(\log\frac{x}{x-t}\Big)\,\mathrm{d}t \ge \Big(\kappa(r_1) - \kappa\Big(\frac{1}{x}\Big)\Big)\delta\sinh(\delta\log x) - c_2\delta^2\cosh(\delta\log x).$$

This implies that

$$\begin{aligned}
\mathrm{RHS}^{(1)}_{(3.13)} &\ge \widetilde{\varphi}'_\delta(\log x)\Big(\Big(\kappa(r_1) - \kappa\Big(\frac{1}{x}\Big)\Big)\delta\sinh(\delta\log x) - c_2\delta^2\cosh(\delta\log x)\Big)\\
&= (\delta\sinh(\delta\log x))^2\Big(\kappa(r_1) - \kappa\Big(\frac{1}{x}\Big) - c_2\delta\coth(\delta\log x)\Big).
\end{aligned}$$

Now observe that $x\kappa\big(\frac{1}{x}\big) \le \zeta(2) \le 2$. Together with (3.13) and (3.14), this yields the lower bound in (3.11) with $c_0 = c_1 + c_2$, and completes the proof of Lemma 3.

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Let us now proceed to the proof of (3.9) (i.e. Lemma 2) in the second case $y \in (\eta^{1/\delta}\lambda_n,\ \lambda_n]$. We write

$$
\begin{aligned}
\mathrm{LHS}_{(3.9)} &= \left(\frac{1}{2}+\varepsilon\right)\left(1-\mathbb{P}\left(Y_n+\widehat{Y}_n \le y\right)\right)+\left(\frac{1}{2}-\varepsilon\right)\left(1-F_n(y)\right)^2-1+F_{n+1}(y)\\
&= -\left(\frac{1}{2}+\varepsilon\right)\mathbb{P}(Y_n+\widehat{Y}_n \le y)-(1-2\varepsilon)F_n(y)+\left(\frac{1}{2}-\varepsilon\right)F_n(y)^2+F_{n+1}(y)\\
&= \frac{1}{2}\,\mathrm{I}_n(y)+\mathrm{I\,I}_n(y)+\varepsilon\,\mathrm{I\,I\,I}_n(y),
\end{aligned}
\tag{3.15}
$$

where we have set

$$
\begin{aligned}
\mathrm{I}_n(y) &:= F_n(y)^2-\mathbb{P}(Y_n+\widehat{Y}_n \le y),\\
\mathrm{I\,I}_n(y) &:= F_{n+1}(y)-F_n(y)\ \text{(which is negative)},\\
\mathrm{I\,I\,I}_n(y) &:= 2F_n(y)-F_n(y)^2-\mathbb{P}(Y_n+\widehat{Y}_n \le y).
\end{aligned}
$$

We first look for a lower bound for $\mathrm{I}_n(y)$. Note that $F_n(y)^2 \ge F_n(y)\,F_n(y-1) = F_n(y)\ \int_1^{y-1} F_n'(t)\,\mathrm{d}t$ and that $\mathbb{P}(Y_n+\widehat{Y}_n \le y) = \int_1^{y-1} F_n'(t)F_n(y-t)\,\mathrm{d}t$. Therefore,

$$
\begin{aligned}
\mathrm{I}_n(y) &\ge \int_1^{y-1} F_n'(t)\big(F_n(y)-F_n(y-t)\big)\,\mathrm{d}t\\
&= 4a_n^2\int_1^{y-1}\varphi_\delta'(t)\big(\varphi_\delta(y)-\varphi_\delta(y-t)\big)\,\mathrm{d}t = 4a_n^2\,\mathtt{cvl}_{\delta,1}(y),
\end{aligned}
$$

where $\mathtt{cvl}_{\delta,1}$ is defined in Lemma 3. By the first inequality in (3.11) of Lemma 3, $\mathtt{cvl}_{\delta,1}(y) \ge \kappa_{\delta,1}(y)(\delta\sinh(\delta\log y))^2$ for $y \in [2,\infty)$. Since

$$
\text{for all } t \in \mathbb{R}_+,\qquad \cosh(t) \ge \sinh(t) \ge \cosh(t)-1 \ge 0, \tag{3.16}
$$

we have $\mathtt{cvl}_{\delta,1}(y) \ge \kappa_{\delta,1}(y)\delta^2\varphi_\delta(y)^2$. Thus, $\mathrm{I}_n(y) \ge \kappa_{\delta,1}(y)\delta^2F_n(y)^2$. Since the function coth decreases to 1 and since $\kappa_{\delta,1}$ is increasing on $[2,\ \infty)$, we get, for $y \in [\eta^{1/\delta}\lambda_n,\ \lambda_n]$,

$$
\begin{aligned}
\kappa_{\delta,1}(y) &\ge \kappa_{\delta,1}(\eta^{1/\delta}\lambda_n)\\
&= \kappa\left(1-\frac{1}{\eta^{1/\delta}\lambda_n}\right)-\frac{2}{\eta^{1/\delta}\lambda_n}-c_0\delta\coth(\delta\log(\eta^{1/\delta}\lambda_n)) \xrightarrow[n\to\infty]{} \zeta(2)-c_0\delta,
\end{aligned}
$$

because $\kappa(1)=\zeta(2)$. By definition, $\delta = 2\zeta(2)^{-1/2}(1+\eta)\sqrt{\varepsilon}$, thus

$$
\lim_{n\to\infty}\delta^2\kappa_{\delta,1}(\eta^{1/\delta}\lambda_n) = \delta^2\zeta(2)-c_0\delta^3 = 4(1+\eta)^2\varepsilon-8c_0\zeta(2)^{-3/2}(1+\eta)^3\varepsilon^{3/2}.
$$

Let $\varepsilon_3(\eta) \in (0,\ \frac{1}{2})$ be such that for $\varepsilon \in (0,\ \varepsilon_3(\eta))$,

$$
4(1+\eta)^2\varepsilon-8c_0\zeta(2)^{-3/2}(1+\eta)^3\varepsilon^{3/2} > 4\varepsilon.
$$

Therefore, there exists an integer $n_5(\eta,\varepsilon) \ge n_4(\eta,\varepsilon)$ such that

$$
\text{for all } n \ge n_5(\eta,\varepsilon),\ \text{ for all } y \in [\eta^{1/\delta}\lambda_n,\ \lambda_n],\qquad \mathrm{I}_n(y) \ge 4\varepsilon F_n(y)^2. \tag{3.17}
$$

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On the other hand, since $a_{n+1}/a_n = 1 - 2\varepsilon(1-\widetilde{\eta})$, we get

$$\mathrm{I\,I}_n(y) = 2(a_{n+1} - a_n)\varphi_\delta(y) = -2(1-\widetilde{\eta})\varepsilon F_n(y),$$
$$\mathrm{I\,I\,I}_n(y) = 2(F_n(y) - F_n(y)^2) + \mathrm{I}_n(y) \geq 2(F_n(y) - F_n(y)^2).$$

Thus, for all $\varepsilon \in (0, \varepsilon_3(\eta))$, all integers $n \geq n_5(\eta, \varepsilon)$ and all $y \in [\eta^{1/\delta}\lambda_n,\ \lambda_n]$,

$$\mathrm{LHS}_{(3.9)} \geq 2\varepsilon F_n(y)^2 - 2(1-\widetilde{\eta})\varepsilon F_n(y) + 2\varepsilon(F_n(y) - F_n(y)^2) = 2\widetilde{\eta}\varepsilon F_n(y) \geq 0.$$

This completes the proof of Lemma 2 in the second case $y \in [\eta^{1/\delta}\lambda_n,\ \lambda_n]$.

Proof of Lemma 2: third and last case $y \in (\lambda_n,\ \lambda_{n+1}]$. Here again the law of $Y_n + \widehat{Y}_n$ matters specifically and we use Lemma 3 to handle it. Let us fix $\varepsilon \in (0, \varepsilon_3(\eta))$ and $n \geq n_5(\eta, \varepsilon)$. We first observe that

$$\mathrm{LHS}_{(3.9)} \geq \frac{1}{2}\,\mathbb{P}\big(Y_n + \widehat{Y}_n > y\big) - 1 + F_{n+1}(y) \geq \frac{1}{2}\,\mathbb{P}\big(Y_n + \widehat{Y}_n > \lambda_{n+1}\big) - 1 + F_{n+1}(\lambda_n).$$

Since $F_{n+1}(\lambda_n) = \big(1 - 2(1-\widetilde{\eta})\varepsilon\big)F_n(\lambda_n) = 1 - 2(1-\widetilde{\eta})\varepsilon$, we get for all $y \in (\lambda_n,\ \lambda_{n+1}]$ that

$$\mathrm{LHS}_{(3.9)} \geq \frac{1}{2}\,\mathbb{P}\big(Y_n + \widehat{Y}_n > \lambda_{n+1}\big) - 2(1-\widetilde{\eta})\varepsilon. \tag{3.18}$$

We now look for a lower bound for $\mathbb{P}(Y_n + \widehat{Y}_n > \lambda_{n+1})$. By Lemma 1(iii), there is $\varepsilon_4(\eta) \in (0, \varepsilon_3(\eta))$ such that for $\varepsilon \in (0, \varepsilon_4(\eta))$, there exists an integer $n_6(\eta, \varepsilon) \geq n_5(\eta, \varepsilon)$ which satisfies $\lambda_{n+1} - \lambda_n + 1 \leq \delta\lambda_n$, for all $n \geq n_6(\eta, \varepsilon)$. We have

$$\begin{aligned}\mathbb{P}\big(Y_n + \widehat{Y}_n > \lambda_{n+1}\big) &\geq \mathbb{P}\big(\lambda_n - 1 > Y_n > \lambda_{n+1} - \lambda_n; Y_n + \widehat{Y}_n > \lambda_{n+1}\big)\\ &= \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} F_n'(t)\big(1 - F_n(\lambda_{n+1} - t)\big)\,\mathrm{d}t.\end{aligned}$$

Writing $1 - F_n(\lambda_{n+1} - t) = F_n(\lambda_n) - F_n(\lambda_{n+1} - t)$, this leads to

$$\begin{aligned}\mathbb{P}\big(Y_n + \widehat{Y}_n > \lambda_{n+1}\big) &\geq 4a_n^2 \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} \varphi_\delta'(t)\big(\varphi_\delta(\lambda_n) - \varphi_\delta(\lambda_{n+1} - t)\big)\,\mathrm{d}t\\ &= \mathrm{RHS}^{(1)}_{(3.19)} - \mathrm{RHS}^{(2)}_{(3.19)} - \mathrm{RHS}^{(3)}_{(3.19)},\end{aligned} \tag{3.19}$$

where:

- $\mathrm{RHS}^{(1)}_{(3.19)} := 4a_n^2 \int_1^{\lambda_n - 1} \varphi_\delta'(t)\big(\varphi_\delta(\lambda_n) - \varphi_\delta(\lambda_n - t)\big)\,\mathrm{d}t$;
- $\mathrm{RHS}^{(2)}_{(3.19)} := 4a_n^2 \int_1^{\lambda_{n+1}-\lambda_n} \varphi_\delta'(t)\big(\varphi_\delta(\lambda_n) - \varphi_\delta(\lambda_n - t)\big)\,\mathrm{d}t$;
- $\mathrm{RHS}^{(3)}_{(3.19)} := 4a_n^2 \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} \varphi_\delta'(t)\big(\varphi_\delta(\lambda_{n+1} - t) - \varphi_\delta(\lambda_n - t)\big)\,\mathrm{d}t$.

We apply Lemma 3 to $\mathrm{RHS}^{(1)}_{(3.19)}$ to get that

$$\begin{aligned}\mathrm{RHS}^{(1)}_{(3.19)} &= 4a_n^2 \mathtt{cvl}_{\delta,1}(\lambda_n) \geq 4a_n^2 \kappa_{\delta,1}(\lambda_n)\delta^2 \sinh(\delta\log\lambda_n)^2\\ &\overset{\text{by (3.16)}}{\geq} \kappa_{\delta,1}(\lambda_n)\delta^2(2a_n\varphi_\delta(\lambda_n))^2 = \kappa_{\delta,1}(\lambda_n)\delta^2.\end{aligned}$$

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Note that $\lim_{n\to\infty} \kappa_{\delta,1}(\lambda_n)\delta^2 = 4(1+\eta)^2\varepsilon - 8c_0\zeta(2)^{-3/2}(1+\eta)^2\varepsilon^{3/2}$. Therefore there exists $\varepsilon_5(\eta) \in (0, \varepsilon_4(\eta))$ such that for all $\varepsilon \in (0, \varepsilon_5(\eta))$ there is an integer $n_7(\eta, \varepsilon) \ge n_6(\eta, \varepsilon)$ satisfying

$$\text{for all } n \ge n_7(\eta, \varepsilon), \quad \mathrm{RHS}^{(1)}_{(3.19)} \ge \left(1 - \frac{1}{3}\eta\right)4(1+\eta)^2\varepsilon. \tag{3.20}$$

Let us next provide an upper bound for $\mathrm{RHS}^{(2)}_{(3.19)}$. For $n \ge n_7(\eta, \varepsilon)$, we have $\lambda_{n+1} - \lambda_n + 1 \le \delta\lambda_n$; thus

$$\begin{aligned}
\mathrm{RHS}^{(2)}_{(3.19)} &\le 4a_n^2 \int_1^{\delta\lambda_n - 1} \varphi'_\delta(t)\big(\varphi_\delta(\lambda_n) - \varphi_\delta(\lambda_n - t)\big)\,\mathrm{d}t = 4a_n^2 \mathtt{cvl}_{\delta,\delta}(\lambda_n) \\
&\overset{\text{by (3.11)}}{\le} 4a_n^2\kappa(\delta)\delta^2 \sinh(\delta \log \lambda_n)^2 \overset{\text{by (3.16)}}{\le} \kappa(\delta)\delta^2(2a_n\varphi_\delta(\lambda_n) + 2a_n)^2 \\
&= \kappa(\delta)\delta^2(1+2a_n)^2 \xrightarrow[n\to\infty]{} \kappa(\delta)\delta^2 = \zeta(2)^{-1}\kappa(\delta)4(1+\eta)^2\varepsilon = o_\eta(\varepsilon).
\end{aligned}$$

Therefore, there exists $\varepsilon_6(\eta) \in (0, \varepsilon_5(\eta))$ such that for all $\varepsilon \in (0, \varepsilon_6(\eta))$, there is an integer $n_8(\eta, \varepsilon) \ge n_7(\eta, \varepsilon)$ satisfying

$$\text{for all } n \ge n_8(\eta, \varepsilon), \quad \mathrm{RHS}^{(2)}_{(3.19)} \le \frac{1}{3}\eta\, 4(1+\eta)^2\varepsilon. \tag{3.21}$$

We finally look for an upper bound for $\mathrm{RHS}^{(3)}_{(3.19)}$. Since $\varphi'_\delta(t) = \frac{\delta}{t}\sinh(\delta \log t)$, we have

$$\begin{aligned}
\mathrm{RHS}^{(3)}_{(3.19)} &= 4a_n^2 \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} \mathrm{d}t\, \frac{\delta}{t}\sinh(\delta\log t) \int_{\lambda_n - t}^{\lambda_{n+1}-t} \mathrm{d}s\, \frac{\delta}{s}\sinh(\delta \log s) \\
&\le 4a_n^2\delta^2 \sinh(\delta\log\lambda_n)^2 \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} \frac{\mathrm{d}t}{t} \int_{\lambda_n - t}^{\lambda_{n+1}-t} \frac{\mathrm{d}s}{s} \\
&\le \delta^2(2a_n\varphi_\delta(\lambda_n) + 2a_n)^2 \int_{\lambda_{n+1}-\lambda_n}^{\lambda_n - 1} \frac{\log(\lambda_{n+1} - t) - \log(\lambda_n - t)}{t}\,\mathrm{d}t \\
&\le \delta^2(1+2a_n)^2 \int_0^{\lambda_n} \frac{\log(\lambda_{n+1} - t) - \log(\lambda_n - t)}{t}\,\mathrm{d}t.
\end{aligned}$$

Since $\lambda_{n+1} - \lambda_n + 1 \le \delta\lambda_n$, we get $\lambda_{n+1} \le (1+\delta)\lambda_n$, and thus

$$\begin{aligned}
\mathrm{RHS}^{(3)}_{(3.19)} &\le \delta^2(1+2a_n)^2 \int_0^{\lambda_n} \frac{\log((1+\delta)\lambda_n - t) - \log(\lambda_n - t)}{t}\,\mathrm{d}t \\
&\le \delta^2(1+2a_n)^2 \int_0^1 \frac{\log(1+\delta-s) - \log(1-s)}{s}\,\mathrm{d}s \\
&\xrightarrow[n\to\infty]{} \delta^2 \int_0^1 \frac{\log(1+\delta-s) - \log(1-s)}{s}\,\mathrm{d}s = o_\eta(\varepsilon).
\end{aligned}$$

Thus, there exists $\varepsilon_7(\eta) \in (0, \varepsilon_6(\eta))$ such that for all $\varepsilon \in (0, \varepsilon_7(\eta))$, there is an integer $n_9(\eta, \varepsilon) \ge n_8(\eta, \varepsilon)$ such that

$$\text{for all } n \ge n_9(\eta, \varepsilon), \quad \mathrm{RHS}^{(3)}_{(3.19)} \le \frac{1}{3}\eta\, 4(1+\eta)^2\varepsilon.$$

Combined with (3.20) and (3.21), it entails that

$$\mathrm{RHS}^{(1)}_{(3.19)} - \mathrm{RHS}^{(2)}_{(3.19)} - \mathrm{RHS}^{(3)}_{(3.19)} \geq \left(1 - \frac{1}{3}\eta\right)4(1+\eta)^2\varepsilon - \frac{1}{3}\eta\, 4(1+\eta)^2\varepsilon - \frac{1}{3}\eta\, 4(1+\eta)^2\varepsilon$$
$$= (1-\eta)(1+\eta)^2 4\varepsilon \overset{\text{by (3.1)}}{>} 4\varepsilon.$$

Returning to (3.18), we obtain $\mathrm{LHS}_{(3.9)} \geq 2\varepsilon - 2(1-\widetilde{\eta})\varepsilon = 2\widetilde{\eta}\varepsilon > 0$, which proves Lemma 2 in the third and last case $y \in (\lambda_n,\ \lambda_{n+1}]$.

4. Proof of Theorem 1: The Upper Bound

Compared with the previous section, in the following proof, the numbering for the constants $n_i(\eta,\ \varepsilon), n_{i+1}(\eta,\ \varepsilon), \ldots$ and $\varepsilon_i(\eta), \varepsilon_{i+1}(\eta), \ldots$ starts again from $i = 1$.

4.1. Definition of the upper bound laws

Recall from (3.3) that for all $q \in (0,\ 1)$ and all $x \in [1, \infty)$, we have set $\varphi_q(x) := \cosh(q \log x) - 1$. The following lemma gives a list of properties of the function φ_q that are used to define the forthcoming laws ν'_n. These laws are proved to satisfy (2.14); see Lemma 6 that is the key technical step in the proof of the upper bound in Theorem 1.

Lemma 4. *Let $q \in (0,\ 1)$.*

(i) *We have $\varphi'_q([1, \infty)) = [0,\ M_q]$ with $M_q := \sup_{y\in[1,\,\infty)} \varphi'_q(y)$, and*

$$x_q := \left(\frac{1+q}{1-q}\right)^{1/(2q)}$$

is the unique $x \in [1,\ \infty)$ such that $\varphi'_q(x) = M_q$. Moreover, $x_q \to \mathrm{e}$ and $M_q \sim \mathrm{e}^{-1}q^2$ as $q \to 0^+$.

(ii) *The function $\varphi'_q : [1, x_q] \to [0,\ M_q]$ is a C^1 increasing bijection whose inverse is denoted by $\ell_q : [0,\ M_q] \to [1,\ x_q]$ and $\varphi'_q : [x_q,\ \infty) \to (0,\ M_q]$ is a C^1 decreasing bijection whose inverse is denoted by $r_q : (0,\ M_q] \to [x_q,\ \infty)$. As $y \to 0^+$, we get*

$$\ell_q(y) = 1 + q^{-2}y(1 + \mathcal{O}_q(y)) \quad \text{and} \quad r_q(y) \sim (2y/q)^{-\frac{1}{1-q}}.$$

(iii) *For all $y \in (0, M_q]$, we set $\Phi_q(y) = \varphi_q(r_q(y)) - \varphi_q(\ell_q(y))$. Then $\Phi_q : (0,\ M_q] \to \mathbb{R}_+$ is a C^1 decreasing bijection whose inverse is denoted by $\Phi_q^{-1} : \mathbb{R}_+ \to (0,\ M_q]$. As $x \to \infty$, we get $\Phi_q^{-1}(x) \sim (q/2)(2x)^{-\frac{1-q}{q}}$,*

$$r_q\big(\Phi_q^{-1}(x)\big) \sim (2x)^{\frac{1}{q}} \quad \textit{and} \quad \ell_q\big(\Phi_q^{-1}(x)\big) = 1 + \frac{1}{2q}(2x)^{-\frac{1-q}{q}}\left(1 + \mathcal{O}_q\left(x^{-\frac{1-q}{q}}\right)\right).$$

(iv) *Let $a \in \mathbb{R}_+^*$ and for all $x \in [1,\ \infty)$ set $g(x) = \varphi_q(x+a) - \varphi_q(x)$. Then, $\lim_{x\to\infty} g(x) = 0$. Moreover, suppose that there is $x^* \in [1,\ \infty)$ such that $g'(x^*) = 0$. Then, g is strictly decreasing on $[x^*,\ \infty)$.*

Proof. See Appendix B. □

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We fix $\eta \in (0,\ 1)$ and $\varepsilon \in \left(0,\ \frac{1}{36}\right)$, and write

$$\gamma := \frac{2}{\sqrt{\zeta(2)}}(1-\eta)\sqrt{\varepsilon} < \frac{1}{3} \quad \text{and} \quad b_n := \frac{1}{4}\big(1-(2+\eta)\varepsilon\big)^n, \qquad n \in \mathbb{N}. \tag{4.1}$$

We keep the notation in Lemma 4, and set

$$\sigma_n := \ell_\gamma\left(\Phi_\gamma^{-1}\left(\frac{1}{2b_n}\right)\right) \quad \text{and} \quad \tau_n := r_\gamma\left(\Phi_\gamma^{-1}\left(\frac{1}{2b_n}\right)\right) - \ell_\gamma\left(\Phi_\gamma^{-1}\left(\frac{1}{2b_n}\right)\right). \tag{4.2}$$

Then

$$2b_n(\varphi_\gamma(\sigma_n+\tau_n)-\varphi_\gamma(\sigma_n)) = 1 \quad \text{and} \quad \varphi'_\gamma(\sigma_n+\tau_n) = \varphi'_\gamma(\sigma_n),$$

and these two equations characterize σ_n and τ_n. By Lemma 4(ii) and (iii), σ_n decreases to 1, τ_n increases to ∞, and

$$\sigma_n = 1 + \frac{1}{2\gamma} b_n^{\frac{1-\gamma}{\gamma}}\left(1+\mathcal{O}_{\eta,\varepsilon}\left(b_n^{\frac{1-\gamma}{\gamma}}\right)\right) \quad \text{and} \quad \tau_n \sim b_n^{-\frac{1}{\gamma}}, \qquad n \to \infty. \tag{4.3}$$

By Lemma 4(iv), $x \in [\sigma_n,\ \infty) \mapsto 2b_n(\varphi_\gamma(x+\tau_n)-\varphi_\gamma(x))$ decreases to 0. Therefore, there is a unique measure $\nu'_n \in \mathcal{M}_1$ such that

$$\nu'_n((0,\sigma_n]) = 0 \quad \text{and} \quad \nu'_n((x,\infty)) = 2b_n(\varphi_\gamma(x+\tau_n)-\varphi_\gamma(x)), \qquad x \in [\sigma_n,\infty). \tag{4.4}$$

In the rest of this section, Z_n denotes a random variable with law ν'_n, and

$$G_n(z) := \mathbb{P}(Z_n \le z) = \nu'_n((-\infty,\ z]), \qquad z \in \mathbb{R}.$$

Lemma 5. *We keep the previous notation. There exists* $\varepsilon_1(\eta) \in \left(0,\ \frac{1}{36}\right)$ *such that for all* $\varepsilon \in (0,\ \varepsilon_1(\eta))$*, there exists* $n_1(\eta,\ \varepsilon) \in \mathbb{N}$ *satisfying* $\mathbb{E}[Z_n \wedge \widehat{Z}_n] \le 2\tau_n$ *for all* $n \ge n_1(\eta,\varepsilon)$*, where* $\widehat{Z}_n$ *is an independent copy of* Z_n.

Proof. Observe that

$$\mathbb{E}\big[Z_n \wedge \widehat{Z}_n\big] = \int_0^\infty \mathbb{P}(Z_n > x)^2 \,\mathrm{d}x \le \tau_n + \int_{\tau_n}^\infty \mathbb{P}(Z_n > x)^2 \,\mathrm{d}x.$$

For all sufficiently large n, $\tau_n \ge \sigma_n$, so that for $x \in [\tau_n,\ \infty)$,

$$\begin{aligned}\mathbb{P}(Z_n > x) &= 2b_n\big(\varphi_\gamma(x+\tau_n)-\varphi_\gamma(x)\big) \le b_n\big((x+\tau_n)^\gamma - x^\gamma\big)\\ &= \gamma b_n \int_x^{x+\tau_n} y^{-(1-\gamma)}\,\mathrm{d}y \le \gamma b_n \tau_n x^{-(1-\gamma)}.\end{aligned}$$

Since $\gamma \to 0$ as $\varepsilon \to 0$, we choose $\varepsilon_1(\eta) \in \left(0,\ \frac{1}{36}\right)$ such that for all $\varepsilon \in (0, \varepsilon_1(\eta))$, we have $2(1-\gamma) > 1$ (i.e. $1-2\gamma > 0$) and that $\gamma^2/(1-2\gamma) < 1$. Therefore,

$$\int_{\tau_n}^\infty \mathbb{P}(Z_n > x)^2\,\mathrm{d}x \le (\gamma b_n \tau_n)^2 \int_{\tau_n}^\infty x^{-2(1-\gamma)}\,\mathrm{d}x = \frac{\gamma^2}{1-2\gamma}(b_n\tau_n^\gamma)^2\tau_n \sim_{n\to\infty} \frac{\gamma^2}{1-2\gamma}\tau_n,$$

by (4.3). Thus, for all $\varepsilon \in (0, \varepsilon_1(\eta))$, there is $n_1(\eta,\varepsilon) \in \mathbb{N}$ such that $\int_{\tau_n}^\infty \mathbb{P}(Z_n > x)^2\,\mathrm{d}x < \tau_n$ for all $n \ge n_1(\eta,\varepsilon)$, which entails the desired inequality. □

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The following lemma, which is the key point in the proof of the upper bound in Theorem 1, tells us that the laws ν'_n satisfy (2.14). Its proof is postponed to Section 4.2. The proof of the upper bound in Theorem 1 relies on this lemma, as well as on the arguments in Section 2.2 and on Lemma 5.

Lemma 6. *We keep the previous notation. There exists* $\varepsilon_2(\eta) \in (0,\ \varepsilon_1(\eta))$ *such that for all* $\varepsilon \in (0,\ \varepsilon_2(\eta))$*, there is an integer* $n_2(\eta, \varepsilon) \geq n_1(\eta, \varepsilon)$ *satisfying, for* $n \geq n_2(\eta, \varepsilon)$ *and* $z \in \big[(1+\eta\gamma)\sigma_{n+1},\ \infty\big)$*,*

$$\left(\frac{1}{2}+\varepsilon\right)\mathbb{P}\big(Z_n+\widehat{Z}_n>z\big)+\left(\frac{1}{2}-\varepsilon\right)\big(1-G_n(z)\big)^2-1+G_{n+1}\left(\frac{z}{1+\eta\gamma}\right)\leq 0, \tag{4.5}$$

where $\widehat{Z}_n$ *is an independent copy of* Z_n*.*

Proof. See Section 4.2.

Proof of the upper bound in Theorem 1. We recall that $p=\frac{1}{2}+\varepsilon$. We admit Lemma 6 and prove that it implies the upper bound in Theorem 1. We keep the previous notation. Fix $\eta \in (0,\ 1)$ and $\varepsilon \in (0,\ \varepsilon_2(\eta))$. Note that a.s. $D_{n_2(\eta,\varepsilon)}(p) \leq 2^{n_2(\eta,\varepsilon)} \leq 2^{n_2(\eta,\varepsilon)} Z_{n_2(\eta,\varepsilon)}$ because $Z_n \geq \sigma_n \geq 1$. Then $\mu_{n_0} \overset{\text{st}}{\leq} M_\theta(\nu'_{n_0})$ with $n_0 = n_2(\eta, \varepsilon)$ and $\theta = 2^{n_2(\eta,\varepsilon)}$.

For $n \geq n_2(\eta,\ \varepsilon)$ and $z \in [0,\ (1+\eta\gamma)\sigma_{n+1})$, we have

$$[\Psi_p(\nu'_n)]((z,\ \infty)) \leq 1 = \mathbb{P}\big((1+\eta\gamma)Z_{n+1}>z\big).$$

Combined with (4.5) that holds for $z \geq (1+\eta\gamma)\sigma_{n+1}$, this implies that $\Psi_p(\nu'_n) \overset{\text{st}}{\leq} M_{1+\eta\gamma}(\nu'_{n+1})$. Thus, the laws ν'_n satisfy (2.14) with $\beta = 1+\eta\gamma$. By (2.15),

$$D_n(p) \overset{\text{st}}{\leq} 2^{n_2(\eta,\varepsilon)}(1+\eta\gamma)^{n-n_2(\eta,\varepsilon)} Z_n, \quad n \geq n_2(\eta, \varepsilon). \tag{4.6}$$

We denote by $\widehat{D}_n(p)$ (respectively, $\widehat{Z}_n$) an independent copy of $D_n(p)$ (respectively, of Z_n). Then for $n \geq n_2(\eta, \varepsilon)$,

$$\begin{aligned}
\mathbb{E}[D_{n+1}(p)] &= \left(\frac{1}{2}+\varepsilon\right)\mathbb{E}\big[D_n(p)+\widehat{D}_n(p)\big]+\left(\frac{1}{2}-\varepsilon\right)\mathbb{E}\big[D_n(p)\wedge\widehat{D}_n(p)\big] \\
&\leq (1+2\varepsilon)\mathbb{E}[D_n(p)] + 2^{n_2(\eta,\varepsilon)}(1+\eta\gamma)^{n-n_2(\eta,\varepsilon)}\mathbb{E}\big[Z_n\wedge\widehat{Z}_n\big] \\
&\leq (1+2\varepsilon)\mathbb{E}[D_n(p)] + 2^{n_2(\eta,\varepsilon)+1}(1+\eta\gamma)^{n-n_2(\eta,\varepsilon)}\tau_n \quad \text{(by Lemma 5)}.
\end{aligned}$$

Iterating the inequality yields that for $n \geq n_0 := n_2(\eta,\ \varepsilon)$ and using the fact that $j \mapsto \tau_j$ is non-decreasing, we get

$$\begin{aligned}
\mathbb{E}[D_n(p)] &\leq (1+2\varepsilon)^{n-n_0}\mathbb{E}[D_{n_0}(p)] + \sum_{i=0}^{n-n_0-1}(1+2\varepsilon)^i 2^{n_0+1}(1+\eta\gamma)^{n-i-1-n_0}\tau_{n-i-1} \\
&\leq (1+2\varepsilon)^{n-n_0}\mathbb{E}[D_{n_0}(p)] + n(1+2\varepsilon)^n + 2^{n_0+1}(1+\eta\gamma)^n\tau_n.
\end{aligned}$$

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By (4.3), for $\varepsilon \in (0,\ \varepsilon_2(\eta))$,

$$\begin{aligned}\alpha\left(\frac{1}{2}+\varepsilon\right) &= \lim_{n\to\infty} n^{-1}\log \mathbb{E}[D_n(p)]\\ &\le \log(1+2\varepsilon)+\log(1+\eta\gamma)-\frac{1}{\gamma}\log\big(1-(2+\eta)\varepsilon\big)\\ &= \left(\frac{2\eta(1-\eta)}{\sqrt{\zeta(2)}}+\frac{1+\frac{1}{2}\eta}{1-\eta}\sqrt{\zeta(2)}\right)\sqrt{\varepsilon}+\mathcal{O}_\eta(\varepsilon),\end{aligned}$$

which implies the upper bound in Theorem 1, as $\eta > 0$ can be made as small as possible. Provided Lemma 6 holds true, it also completes the proof of Theorem 1. □

4.2. Proof of Lemma 6

This section is devoted to the proof of Lemma 6. We fix $\eta \in (0,\ 1)$ and $\varepsilon \in (0,\ \varepsilon_1(\eta))$. Recall from (4.1) the definition of γ and b_n and from (4.2) the definition of σ_n and τ_n. By (4.3), $\lim_{n\to\infty}\sigma_n = 1 < 1+\eta\gamma = \lim_{n\to\infty}(1+\eta\gamma)\sigma_{n+1} < \lim_{n\to\infty}(\eta\varepsilon)^{1/\gamma}\tau_n = \infty$. Therefore, there is an integer $n_3(\eta,\ \varepsilon) \ge n_1(\eta,\ \varepsilon)$ such that

$$2\sigma_n < 2(1+\eta\gamma)\sigma_{n+1} < (\eta\varepsilon)^{1/\gamma}\tau_n < \gamma^2\tau_n < \mathrm{e}^{1/\sqrt{\gamma}}\tau_n, \quad n \ge n_3(\eta, \varepsilon). \tag{4.7}$$

Writing $\mathrm{LHS}_{(4.5)}$ for the expression on the left-hand side of (4.5), we need to check that $\mathrm{LHS}_{(4.5)} \le 0$ for all small enough $\varepsilon > 0$, all sufficiently large integer n and all $z \ge (1+\eta\gamma)\sigma_{n+1}$. This is done by distinguishing four cases:

Case 1: $(1+\eta\gamma)\sigma_{n+1} \le z < (\eta\varepsilon)^{1/\gamma}\tau_n$, **Case 2:** $(\eta\varepsilon)^{1/\gamma}\tau_n \le z < \gamma^2\tau_n$,

Case 3: $\gamma^2\tau_n \le z < \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$, and **Case 4:** $z \ge \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$.

Proof of Lemma 6: first case $(1+\eta\gamma)\sigma_{n+1} \le z < (\eta\varepsilon)^{1/\gamma}\tau_n$. Here $\varepsilon \in (0, \varepsilon_1(\eta))$ and $n \ge n_3(\eta, \varepsilon)$. We use the trivial fact $\mathbb{P}\big(Z_n + \widehat{Z}_n > z\big) \le 1$, so that

$$\begin{aligned}\mathrm{LHS}_{(4.5)} &\le \frac{1}{2}+\varepsilon+\left(\frac{1}{2}-\varepsilon\right)\big(1-G_n(z)\big)^2-1+G_{n+1}\left(\frac{z}{1+\eta\gamma}\right)\\ &= G_{n+1}\left(\frac{z}{1+\eta\gamma}\right)+\left(\frac{1}{2}-\varepsilon\right)G_n(z)^2-(1-2\varepsilon)G_n(z)\\ &\le G_{n+1}\left(\frac{z}{1+\eta\gamma}\right)+\frac{1}{2}G_n(z)^2-(1-2\varepsilon)G_n(z).\end{aligned} \tag{4.8}$$

Recall the notation $G_n(z) := \mathbb{P}(Z_n \le z)$. Then

$$\begin{aligned}G_n(z) &\overset{\text{by (4.4)}}{=} 1-2b_n\big(\varphi_\gamma(z+\tau_n)-\varphi_\gamma(z)\big)\\ &\overset{\text{by (4.4)}}{=} 2b_n\big(\varphi_\gamma(z)-\big(\varphi_\gamma(z+\tau_n)-\varphi_\gamma(\sigma_n+\tau_n)\big)-\varphi_\gamma(\sigma_n)\big)\\ &\le 2b_n\varphi_\gamma(z).\end{aligned} \tag{4.9}$$

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We have used the fact (by (4.7)) that $z \geq \sigma_n$ in the last inequality. Since $\varphi_\gamma(z) \leq \cosh(\gamma \log z)$ and $z < (\eta\varepsilon)^{1/\gamma}\tau_n$, we get that

$$G_n(z) \leq 2b_n \cosh\big(\gamma \log((\eta\varepsilon)^{1/\gamma}\tau_n)\big) \sim_{n\to\infty} b_n\big((\eta\varepsilon)^{1/\gamma}\tau_n\big)^\gamma \xrightarrow[n\to\infty]{\text{by (4.3)}} \eta\varepsilon.$$

Thus, there exists $n_4(\eta,\ \varepsilon) \geq n_3(\eta,\ \varepsilon)$ such that $G_n(z) < 2\eta\varepsilon$ for all $n \geq n_4(\eta,\ \varepsilon)$ and for all $z \in [(1+\eta\gamma)\sigma_{n+1},\ (\eta\varepsilon)^{1/\gamma}\tau_n]$. Thus, we get

$$\frac{1}{2}G_n(z)^2 - (1-2\varepsilon)G_n(z) \leq -\big(1-(2+\eta)\varepsilon\big)G_n(z) = -2b_{n+1}\frac{G_n(z)}{2b_n}.$$

By (4.8), this leads to

$$\begin{aligned}
\mathrm{LHS}_{(4.5)} &\leq G_{n+1}\left(\frac{z}{1+\eta\gamma}\right) - 2b_{n+1}\big(\varphi_\gamma(z) - \big(\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n)\big) - \varphi_\gamma(\sigma_n)\big)\\
&\overset{\text{by (4.9)}}{\leq} 2b_{n+1}\varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) - 2b_{n+1}\big(\varphi_\gamma(z) - \varphi_\gamma(z+\tau_n) + \varphi_\gamma(\sigma_n+\tau_n) - \varphi_\gamma(\sigma_n)\big)\\
&= 2b_{n+1}\left(\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n) + \varphi_\gamma(\sigma_n) - \left(\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)\right)\right).
\end{aligned}$$

We claim for all sufficiently small ε and large n, and all $z \in [(1+\eta\gamma)\sigma_{n+1},\ (\eta\varepsilon)^{1/\gamma}\tau_n]$, that

$$\begin{aligned}
\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n) &\leq \frac{1}{2}\left(\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)\right) \quad \text{and}\\
\varphi_\gamma(\sigma_n) &\leq \frac{1}{2}\left(\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)\right),
\end{aligned} \tag{4.10}$$

which will readily imply Lemma 6 in the first case.

To check the second inequality in (4.10), we look for a suitable upper bound for $\varphi_\gamma(\sigma_n) := \cosh(\gamma \log \sigma_n) - 1$. We note that $\cosh(\lambda) - 1 \leq 2\lambda^2$ for $\lambda \in [0,\ 2]$ (indeed, since $\cosh(2) < 4$, the second derivative of $\lambda \in [0,\ 2] \mapsto \cosh(\lambda) - 1 - 2\lambda^2$ is negative, so is the derivative, and the function itself is decreasing). Since $\sigma_n \to 1^+$ as $n \to \infty$, there exists $n_5(\eta,\ \varepsilon) \geq n_4(\eta,\ \varepsilon)$ such that $\gamma \log \sigma_n \in [0,\ 2]$ for $n \geq n_5(\eta,\ \varepsilon)$; accordingly,

$$\varphi_\gamma(\sigma_n) \leq 2(\gamma \log \sigma_n)^2 \leq 2\gamma^2(\sigma_n - 1)^2, \tag{4.11}$$

since $\log x \leq x - 1$ for $x \geq 1$.

We now look for lower bounds for $\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)$. Using the formula $\cosh(a) - \cosh(b) = 2\sinh\left(\frac{a+b}{2}\right)\sinh\left(\frac{a-b}{2}\right)$, we get that

$$\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) = 2\sinh\left(\frac{\gamma}{2}\log(1+\eta\gamma)\right)\ \sinh\left(\gamma \log z - \frac{\gamma}{2}\log(1+\eta\gamma)\right).$$

Since $z \in [(1+\eta\gamma)\sigma_{n+1},\ (\eta\varepsilon)^{1/\gamma}\tau_n]$, observe that

$$\gamma \log z - \frac{\gamma}{2}\log(1+\eta\gamma) \geq \gamma \log(1+\eta\gamma) + \gamma \log \sigma_{n+1} - \frac{\gamma}{2}\log(1+\eta\gamma) \geq 0. \tag{4.12}$$

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Since $\sinh(x) \geq x$ (for $x \in \mathbb{R}_+$) and $\log(1+x) \geq \frac{1}{2}x$ (for $x \in [0,\ 1]$), we have

$$\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) \geq \gamma\,[\log(1+\eta\gamma)]\sinh\left(\gamma\log z - \frac{\gamma}{2}\log(1+\eta\gamma)\right)$$
$$\geq \frac{1}{2}\eta\gamma^2\sinh\left(\gamma\log z - \frac{\gamma}{2}\log(1+\eta\gamma)\right) \tag{4.13}$$

$$\overset{\text{by (4.12)}}{\geq} \frac{1}{2}\eta\gamma^2\left(\gamma\log z - \frac{\gamma}{2}\log(1+\eta\gamma)\right) \geq \frac{1}{4}\eta\gamma^3\log(1+\eta\gamma). \tag{4.14}$$

Since $2\gamma^2(\sigma_n - 1)^2 \to 0$ as $n \to \infty$, there exists $n_6(\eta,\ \varepsilon) \geq n_5(\eta,\ \varepsilon)$ such that $2\gamma^2(\sigma_n - 1)^2 < \frac{1}{8}\eta\gamma^3\log(1+\eta\gamma)$ for $n \geq n_6(\eta,\ \varepsilon)$; in view of (4.11), this implies the desired second inequality in (4.10):

$$\varphi_\gamma(\sigma_n) \leq \frac{1}{2}\left(\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)\right), \quad n \geq n_6(\eta, \varepsilon),\ z \in \big[(1+\eta\gamma)\sigma_{n+1}, (\eta\varepsilon)^{1/\gamma}\tau_n\big).$$

We now turn to the proof of the first inequality in (4.10). Observe that

$$\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n) = \gamma\int_{\sigma_n+\tau_n}^{z+\tau_n}\sinh(\gamma\log x)\frac{\mathrm{d}x}{x}$$
$$\leq \frac{\gamma}{2}\int_{\sigma_n+\tau_n}^{z+\tau_n}\frac{\mathrm{d}x}{x^{1-\gamma}} \leq \frac{\gamma(z-\sigma_n)}{2(\sigma_n+\tau_n)^{1-\gamma}} \leq \frac{\gamma(z-1)}{2\tau_n^{1-\gamma}}, \tag{4.15}$$

since $\sigma_n \geq 1$. Consequently, for $z \in \big[(1+\eta\gamma)\sigma_{n+1},\ \mathrm{e}^{2/\gamma}\big)$,

$$\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n) \leq \frac{\gamma(\mathrm{e}^{2/\gamma}-1)}{2\tau_n^{1-\gamma}} \xrightarrow[n\to\infty]{} 0 < \frac{1}{8}\eta\gamma^3\log(1+\eta\gamma).$$

Thus, by (4.14), there is $n_7(\eta,\ \varepsilon) \geq n_6(\eta,\ \varepsilon)$ such that for $n \geq n_7(\eta,\ \varepsilon)$ and $z \in \big[(1+\eta\gamma)\sigma_{n+1},\ \mathrm{e}^{2/\gamma}\big)$,

$$\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n) \leq \frac{1}{2}\left(\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right)\right).$$

To complete the proof of the first inequality in (4.10), we still need to consider $z \in [\mathrm{e}^{2/\gamma},\ (\eta\varepsilon)^{1/\gamma}\tau_n)$ (with $n \geq n_7(\eta, \varepsilon)$). Noting that $\frac{\gamma}{2}\log(1+\eta\gamma) < 1$, we have $\gamma\log z - \frac{\gamma}{2}\log(1+\eta\gamma) > \gamma\log z - 1 \geq 1$. By (4.13), and writing $c_0 := \inf_{x\in[2,\ \infty)}[\mathrm{e}^{-x}\sinh(x-1)]$,

$$\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) \geq \frac{1}{2}\eta\gamma^2\sinh(\gamma\log z - 1) \geq \frac{1}{2}c_0\eta\gamma^2\mathrm{e}^{\gamma\log z}.$$

Since $\mathrm{e}^{\gamma\log z} = \frac{z}{z^{1-\gamma}} \geq \frac{z-1}{((\eta\varepsilon)^{1/\gamma}\tau_n)^{1-\gamma}} = \frac{1}{\gamma(\eta\varepsilon)^{(1-\gamma)/\gamma}}\,\frac{\gamma(z-1)}{\tau_n^{1-\gamma}}$, this implies that

$$\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) \geq \frac{1}{2}c_0\eta\gamma^2\frac{z-1}{((\eta\varepsilon)^{1/\gamma}\tau_n)^{1-\gamma}} = \frac{c_1(1-\eta)}{\eta^{\frac{1}{\gamma}-2}\,\varepsilon^{\frac{1}{\gamma}-\frac{3}{2}}}\,\frac{\gamma(z-1)}{2\tau_n^{1-\gamma}},$$

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with $c_1 := \frac{2}{\sqrt{\zeta(2)}} c_0$. Since $\frac{c_1(1-\eta)}{\eta^{\frac{1}{\gamma}-2}\varepsilon^{\frac{1}{\gamma}-\frac{3}{2}}} \to \infty$ as $\varepsilon \to 0^+$, there exists $\varepsilon_3(\eta) \in (0, \varepsilon_1(\eta))$ such that $\frac{c_1(1-\eta)}{\eta^{(1/\gamma)-2}\varepsilon^{1/\gamma-3/2}} > 2$ for $\varepsilon \in (0, \varepsilon_3(\eta))$ and there exists $n_8(\eta, \varepsilon) \ge n_7(\eta, \varepsilon)$ such that for $n \ge n_8(\eta, \varepsilon)$ and $z \in [\mathrm{e}^{2/\gamma}, (\eta\varepsilon)^{1/\gamma}\tau_n)$,

$$\varphi_\gamma(z) - \varphi_\gamma\left(\frac{z}{1+\eta\gamma}\right) > 2 \times \frac{\gamma(z-1)}{2\tau_n^{1-\gamma}} \overset{\text{by (4.15)}}{\ge} 2\big(\varphi_\gamma(z+\tau_n) - \varphi_\gamma(\sigma_n+\tau_n)\big).$$

This yields (4.10) and, thus, implies Lemma 6 in the first case $(1+\eta\gamma)\sigma_{n+1} \le z < (\eta\varepsilon)^{1/\gamma}\tau_n$.

Proof of Lemma 6: second case $(\eta\varepsilon)^{1/\gamma}\tau_n \le z < \gamma^2\tau_n$. We first prove the following lemma.

Lemma 7. *We keep the previous notation. For* $0 < a < b < \infty$, *we have*

$$\limsup_{n\to\infty} \sup_{\theta\in[a,\, b]} |G_n(\theta\tau_n)\theta^{-\gamma} - 1| = \frac{(1+b)^\gamma - 1}{b^\gamma}. \tag{4.16}$$

Proof. We remind the reader that $\lim_{n\to\infty} b_n\tau_n^\gamma = 1$ and that $\lim_{n\to\infty} b_n\tau_n^{-\gamma} = 0$. Then, for $\theta \in [a, b]$, we have

$$\begin{aligned}
G_n(\theta\tau_n) &= 2b_n\big(\varphi_\gamma(\tau_n\theta) - \varphi_\gamma(\sigma_n) - \varphi_\gamma(\tau_n(1+\theta)) + \varphi_\gamma(\sigma_n+\tau_n)\big)\\
&= b_n\tau_n^\gamma\theta^\gamma + b_n\tau_n^{-\gamma}\theta^{-\gamma} - 2b_n - 2b_n\varphi_\gamma(\sigma_n) + b_n\tau_n^\gamma\big(1-(1+\theta)^\gamma\big) - b_n\tau_n^{-\gamma}(1+\theta)^{-\gamma}\\
&\quad + b_n\tau_n^\gamma\big((1+\sigma_n\tau_n^{-1})^\gamma - 1\big) + b_n\tau_n^{-\gamma}(1+\sigma_n\tau_n^{-1})^{-\gamma}\\
&= b_n\tau_n^\gamma\theta^\gamma + b_n\tau_n^\gamma\big(1-(1+\theta)^\gamma\big) + b_n\tau_n^{-\gamma}(\theta^{-\gamma} - (1+\theta)^{-\gamma}) + u_n(\eta, \varepsilon),
\end{aligned}$$

where $u_n(\eta, \varepsilon)$ does not depend on θ and satisfies $\lim_{n\to\infty} u_n(\eta, \varepsilon) = 0$. Thus,

$$G_n(\theta\tau_n)\theta^{-\gamma} - 1 = -\frac{(1+\theta)^\gamma - 1}{\theta^\gamma} + R_n(\eta, \gamma, \theta),$$

where

$$R_n(\eta, \gamma, \theta) := (b_n\tau_n^\gamma - 1)\left(1 - \frac{(1+\theta)^\gamma - 1}{\theta^\gamma}\right) + \frac{b_n\tau_n^{-\gamma}(\theta^{-\gamma} - (1+\theta)^{-\gamma}) + u_n(\eta, \varepsilon)}{\theta^\gamma}.$$

Since $\lim_{n\to\infty} b_n\tau_n^\gamma = 1$ by (4.3), we have $\lim_{n\to\infty}\sup_{\theta\in[a,\, b]} |R_n(\eta, \gamma, \theta)| = 0$, which implies that

$$\limsup_{n\to\infty} \sup_{\theta\in[a,\, b]} |G_n(\theta\tau_n)\theta^{-\gamma} - 1| = \sup_{\theta\in[a,\, b]} \frac{(1+\theta)^\gamma - 1}{\theta^\gamma}.$$

Note that $\frac{(1+\theta)^\gamma - 1}{\theta^\gamma} = \gamma\int_0^1 \frac{\mathrm{d}v}{(\theta^{-1}+v)^{1-\gamma}}$, which is increasing in θ. This entails (4.16). □

We now turn to the proof of Lemma 6 in the second case. Applying Lemma 7 to $a = \frac{1}{2}(\eta\varepsilon)^{1/\gamma}$ and $b = \gamma^2$, and since $\lim_{\varepsilon\to 0^+}\gamma^{-3}\gamma^{-2\gamma}((1+\gamma^2)^\gamma - 1) = 1$, it follows that there exists $\varepsilon_4(\eta) \in (0, \varepsilon_3(\eta))$ such that for $\varepsilon \in (0, \varepsilon_4(\eta))$ there is an integer $n_9(\eta, \varepsilon) \ge n_8(\eta, \varepsilon)$ such that for $n \ge n_9(\eta, \varepsilon)$ and $z \in \big(\frac{1}{2}(\eta\varepsilon)^{1/\gamma}\tau_n, \gamma^2\tau_n\big)$,

$$(1-2\gamma^3)\Big(\frac{z}{\tau_n}\Big)^\gamma \le G_n(z) \le (1+2\gamma^3)\Big(\frac{z}{\tau_n}\Big)^\gamma. \tag{4.17}$$

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We recall that $\mathrm{LHS}_{(4.5)}$ stands for the expression on the left-hand side of (4.5). Then

$$\mathrm{LHS}_{(4.5)} = \frac{1}{2}\widetilde{\mathrm{I}}_n(z) + \widetilde{\mathrm{II}}_n(z) + \varepsilon\,\widetilde{\mathrm{III}}_n(z),$$

where:

- $\widetilde{\mathrm{I}}_n(z) := G_n(z)^2 - \mathbb{P}(Z_n + \widehat{Z}_n \le z)$;
- $\widetilde{\mathrm{II}}_n(z) := G_{n+1}\left(\frac{z}{1+\eta\gamma}\right) - G_n(z)$;
- $\widetilde{\mathrm{III}}_n(z) := 2G_n(z) - G_n(z)^2 - \mathbb{P}(Z_n + \widehat{Z}_n \le z) = 2G_n(z) - 2G_n(z)^2 + \widetilde{\mathrm{I}}_n(z)$.

We claim that there exists $\varepsilon_5(\eta) \in (0,\ \varepsilon_4(\eta))$ such that for $\varepsilon \in (0,\ \varepsilon_5(\eta))$ there is an integer $n_{10}(\eta,\ \varepsilon)$ such that for $n \ge n_{10}(\eta,\ \varepsilon)$,

$$\widetilde{\mathrm{I}}_n(z) \le 4(1-\eta)\varepsilon\, G_n(z)^2, \quad z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n). \tag{4.18}$$

To see why (4.18) is true, we recall that $G_n(z) = \int_{\sigma_n}^{z} G_n'(t)\,\mathrm{d}t$ and that $\mathbb{P}(Z_n + \widehat{Z}_n \le z) = \int_{\sigma_n}^{z-\sigma_n} G_n'(t) G_n(z-t)\,\mathrm{d}t$. Since $G_n(\sigma_n) = 0$, by definition of G_n, we thus get

$$\widetilde{\mathrm{I}}_n(z) = G_n(z)\big(G_n(z) - G_n(z-\sigma_n)\big) + \int_{\sigma_n}^{z-\sigma_n} G_n'(t)\big(G_n(z) - G_n(z-t)\big)\,\mathrm{d}t.$$

To get an upper bound for the first term on the right-hand side, we observe that

$$G_n(z) - G_n(z-\sigma_n) = \int_{z-\sigma_n}^{z} G_n'(t)\,\mathrm{d}t = \int_{z-\sigma_n}^{z} 2b_n\big(\varphi_\gamma'(t) - \varphi_\gamma'(t+\tau_n)\big)\,\mathrm{d}t \le \int_{z-\sigma_n}^{z} 2b_n\varphi_\gamma'(t)\,\mathrm{d}t.$$

For $t \in (z-\sigma_n,\ z)$, we have

$$\varphi_\gamma'(t) = \frac{\gamma}{t}\sinh(\gamma\log t) \le \frac{\gamma}{z-\sigma_n}\sinh(\gamma\log z) \le \frac{\gamma}{2(z-\sigma_n)} z^\gamma,$$

so that for $z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$,

$$G_n(z) - G_n(z-\sigma_n) \le \frac{b_n\sigma_n\gamma z^\gamma}{z-\sigma_n} \le \frac{b_n\sigma_n\gamma z^\gamma}{(\eta\varepsilon)^{1/\gamma}\tau_n - \sigma_n} = \frac{\gamma}{\varepsilon\tau_n}\frac{b_n\tau_n^\gamma\sigma_n}{(\eta\varepsilon)^{1/\gamma} - \frac{\sigma_n}{\tau_n}}\varepsilon\left(\frac{z}{\tau_n}\right)^\gamma.$$

By (4.17), this yields that

$$G_n(z) - G_n(z-\sigma_n) \le \frac{\gamma}{(1-2\gamma^3)\varepsilon\tau_n}\frac{b_n\tau_n^\gamma\sigma_n}{(\eta\varepsilon)^{1/\gamma} - \frac{\sigma_n}{\tau_n}}\varepsilon G_n(z).$$

Let $\varepsilon_5(\eta) \in (0,\ \varepsilon_4(\eta))$ be such that $6\gamma^3 < \eta$ for $\varepsilon \in (0,\ \varepsilon_5(\eta))$. Note that

$$4(1-\eta)\left(\eta - 6\gamma^3\right) + \left(1 + 6\gamma^3\right)4(1-\eta)^2 < 4(1-\eta). \tag{4.19}$$

Let $\varepsilon \in (0,\ \varepsilon_5(\eta))$. Since

$$\lim_{n\to\infty}\frac{1}{\varepsilon(1-2\gamma^3)\tau_n} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{b_n\tau_n^\gamma\gamma\sigma_n}{(\eta\varepsilon)^{1/\gamma} - \frac{\sigma_n}{\tau_n}} = \gamma(\eta\varepsilon)^{-\frac{1}{\gamma}}$$

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by (4.3), whereas $4(1-\eta)(\eta-6\gamma^3)>0$, there exists $n_{11}(\eta,\ \varepsilon)\ge n_9(\eta,\ \varepsilon)$ such that for $n\ge n_{11}(\eta,\ \varepsilon)$,

$$G_n(z)\big(G_n(z)-G_n(z-\sigma_n)\big)\le 4(1-\eta)(\eta-6\gamma^3)\,\varepsilon G_n(z)^2,\quad z\in[(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n). \tag{4.20}$$

The specific choice of the factor in front of $\varepsilon G_n(z)^2$ in the right-hand side member of the last inequality is justified further.

To deal with $\int_{\sigma_n}^{z-\sigma_n} G_n'(t)(G_n(z)-G_n(z-t))\,\mathrm{d}t$, we recall that $G_n'(t)\le 2b_n\varphi_\gamma'(t)$ and we note that $G_n(z)-G_n(z-t)\le\int_{z-t}^{z}2b_n\varphi_\gamma'(s)\,\mathrm{d}s=2b_n(\varphi_\gamma(z)-\varphi_\gamma(z-t))$, so

$$\begin{aligned}\int_{\sigma_n}^{z-\sigma_n} G_n'(t)\big(G_n(z)-G_n(z-t)\big)\,\mathrm{d}t &\le 4b_n^2\int_{\sigma_n}^{z-\sigma_n}\varphi_\gamma'(t)\big(\varphi_\gamma(z)-\varphi_\gamma(z-t)\big)\,\mathrm{d}t\\ &\le 4b_n^2\int_{1}^{z-1}\varphi_\gamma'(t)\big(\varphi_\gamma(z)-\varphi_\gamma(z-t)\big)\,\mathrm{d}t.\end{aligned}$$

(Indeed, (4.7) implies that if $z\ge(\eta\varepsilon)^{1/\gamma}\tau_n$, then $z>2\sigma_n>2$.) The integral on the right-hand side is $\mathtt{cvl}_{\gamma,1}(z)$ by our definition in (3.10). By Lemma 3, we get that

$$\int_{\sigma_n}^{z-\sigma_n} G_n'(t)\big(G_n(z)-G_n(z-t)\big)\,\mathrm{d}t\le 4b_n^2\zeta(2)\gamma^2\big(\sinh(\gamma\log z)\big)^2.$$

Observe that

$$4\big(\sinh(\gamma\log z)\big)^2\le z^{2\gamma}=\tau_n^{2\gamma}\left(\frac{z}{\tau_n}\right)^{2\gamma},$$

which is bounded by $(\tau_n^{2\gamma}/(1-2\gamma^3)^2)G_n(z)^2$ (see (4.17)). Therefore,

$$\int_{\sigma_n}^{z-\sigma_n} G_n'(t)\big(G_n(z)-G_n(z-t)\big)\,\mathrm{d}t\le\zeta(2)\gamma^2\,\frac{(b_n\tau_n^{\gamma})^2}{(1-2\gamma^3)^2}\,G_n(z)^2.$$

Recall that $\zeta(2)\gamma^2=4(1-\eta)^2\varepsilon$, and that $\lim_{n\to\infty}b_n\tau_n^\gamma=1$ (see (4.3)). Since $1/(1-2\gamma^3)^2<1+6\gamma^3$ (because $\gamma<\frac{1}{3}$), there exists $n_{10}(\eta,\ \varepsilon)\ge n_{11}(\eta,\ \varepsilon)$ such that for $n\ge n_{10}(\eta,\ \varepsilon)$, we have

$$\int_{\sigma_n}^{z-\sigma_n} G_n'(t)\big(G_n(z)-G_n(z-t)\big)\,\mathrm{d}t\le(1+6\gamma^3)4(1-\eta)^2\,\varepsilon G_n(z)^2,\quad z\in[(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n). \tag{4.21}$$

We obtain (4.18) by (4.20), (4.21), and (4.19).

Let us consider $\widetilde{\mathrm{II}}_n(z)$. Let $\varepsilon\in(0,\ \varepsilon_5(\eta))$, $n\ge n_{10}(\eta,\ \varepsilon)$, and $z\in[(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$. We start with the trivial inequality $\widetilde{\mathrm{II}}_n(z)\le G_{n+1}(z)-G_n(z)$, and look for an upper bound for $G_{n+1}(z)-G_n(z)$. Since $\lim_{\varepsilon\to0^+}(1-(2+\eta)\varepsilon)^{1/\gamma}=1$ and $\lim_{\varepsilon\to0^+}(\gamma^3/\varepsilon)=0$, there exists $\varepsilon_5(\eta)\in(0,\ \min\{\varepsilon_5(\eta),\ \eta/2\})$ such that

$$\frac{1}{2}<(1-(2+\eta)\varepsilon)^{1/\gamma}\quad\text{and}\quad 5\gamma^3<\frac{1}{2}\eta\varepsilon,\quad \varepsilon\in(0,\ \varepsilon_6(\eta)). \tag{4.22}$$

We fix $\varepsilon\in(0,\ \varepsilon_6(\eta))$. Since $\lim_{n\to\infty}\frac{\tau_n}{\tau_{n+1}}=(1-(2+\eta)\varepsilon)^{1/\gamma}$ (by (4.3)), there exists an integer $\varepsilon_6(\eta,\ \varepsilon)\ge n_{11}(\eta,\ \varepsilon)$ such that

$$\frac{\tau_n}{\tau_{n+1}}>\frac{1}{2}$$

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for $n \geq n_{12}(\eta,\ \varepsilon)$, which implies that $z \in \left(\frac{1}{2}(\eta\varepsilon)^{1/\gamma}\tau_{n+1},\ \gamma^2\tau_{n+1}\right)$; so applying (4.17) twice leads to

$$G_{n+1}(z) \leq \left(1+2\gamma^3\right)\left(\frac{z}{\tau_{n+1}}\right)^{\gamma} \leq \frac{1+2\gamma^3}{1-2\gamma^3}\frac{\tau_n^{\gamma}}{\tau_{n+1}^{\gamma}}G_n(z).$$

Note that

$$\frac{1+2\gamma^3}{1-2\gamma^3} < 1+5\gamma^3$$

(because $\gamma < \frac{1}{3}$), and that

$$\lim_{n\to\infty}\frac{\tau_n^{\gamma}}{\tau_{n+1}^{\gamma}} = \lim_{n\to\infty}\frac{b_{n+1}}{b_n} = 1-(2+\eta)\varepsilon$$

(see (4.3)). Thus, there exists an integer $\varepsilon_{13}(\eta,\ \varepsilon) \geq n_{12}(\eta,\ \varepsilon)$ such that $G_{n+1}(z) \leq (1+5\gamma^3)\big(1-(2+\eta)\varepsilon\big)G_n(z)$ for $n \geq n_{13}(\eta,\ \varepsilon)$ and $z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$; consequently,

$$\widetilde{\mathrm{II}}_n(z) \leq G_{n+1}(z) - G_n(z) \leq \left[(1+5\gamma^3)(1-(2+\eta)\varepsilon)-1\right]G_n(z).$$

Since $\left(1+5\gamma^3\right)(1-(2+\eta)\varepsilon)-1 = 5\gamma^3-(2+\eta)\varepsilon-5\gamma^3(2+\eta)\varepsilon < 5\gamma^3-(2+\eta)\varepsilon$, which is smaller than $-(2+\frac{1}{2}\eta)\varepsilon$ (by (4.22)), we deduce that for $n \geq n_{13}(\eta,\ \varepsilon)$ and $z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$,

$$\widetilde{\mathrm{II}}_n(z) \leq -\left(2+\frac{1}{2}\eta\right)\varepsilon G_n(z). \tag{4.23}$$

Finally, we turn to $\widetilde{\mathrm{III}}_n(z)$, which is easy to estimate:

$$\begin{aligned}\widetilde{\mathrm{III}}_n(z) &= 2G_n(z) - 2G_n(z)^2 + \widetilde{\mathrm{I}}_n(z)\\ &\leq 2G_n(z) - 2G_n(z)^2 + 4(1-\eta)\varepsilon G_n(z)^2\\ &\leq 2G_n(z) - 2G_n(z)^2 + 4\varepsilon G_n(z)^2.\end{aligned} \tag{4.24}$$

Assembling (4.18), (4.23), and (4.24) yields that for $\varepsilon \in (0,\ \varepsilon_6(\eta))$, $n \geq n_{13}(\eta,\ \varepsilon)$, and $z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$,

$$\begin{aligned}\mathrm{LHS}_{(4.5)} &= \frac{1}{2}\widetilde{\mathrm{I}}_n(z) + \widetilde{\mathrm{II}}_n(z) + \varepsilon\,\widetilde{\mathrm{III}}_n(z)\\ &\leq 2(1-\eta)\varepsilon G_n(z)^2 - (2+\frac{1}{2}\eta)\varepsilon G_n(z) + 2\varepsilon G_n(z) - 2\varepsilon G_n(z)^2 + 4\varepsilon^2 G_n(z)^2\\ &= -\frac{1}{2}\eta\varepsilon G_n(z) - 2(\eta-2\varepsilon)\varepsilon G_n(z)^2,\end{aligned}$$

which is non-positive since $\varepsilon < \varepsilon_6(\eta) < \frac{\eta}{2}$. This completes the proof of Lemma 6 when $z \in [(\eta\varepsilon)^{1/\gamma}\tau_n,\ \gamma^2\tau_n)$ (i.e. in the second case). □

Proof of Lemma 6: third case $\gamma^2\tau_n \leq z < \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$. We easily check that there exists $q_0 \in (0,\ \frac{1}{9})$ such that for all $q \in (0,\ q_0)$ the following holds true.

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$$\left(e^{1/\sqrt{q}}+1\right)^{2q} \le 1+3\sqrt{q} < 2, \tag{4.25}$$

$$q^2 - e^{-4/\sqrt{q}} > q^3 > e^{-1/\sqrt{q}}, \tag{4.26}$$

$$(1-q)\left(\frac{q^3}{2}\right)^q > 1-\sqrt{q}. \tag{4.27}$$

For any $q \in (0,\ q_0]$, we define the function

$$g_q(\theta) := (\theta+1)^q - \theta^q, \quad \theta \in \mathbb{R}_+^* := (0,\ \infty). \tag{4.28}$$

Since $\lim_{\theta\to 0^+} g_q(\theta) = 1$, $\lim_{\theta\to\infty} g_q(\theta) = 0$, and $g_q'(\theta) = q((\theta+1)^{q-1} - \theta^{q-1}) < 0$ for $\theta \in \mathbb{R}_+^*$, it follows that g_q is a decreasing bijection from $\mathbb{R}_+^*$ onto $(0,\ 1)$.

We collect some elementary properties of g_q.

Lemma 8. *Let $q \in (0,\ q_0]$. Then the following holds true.*

(i) *For $a \in (0,\ e^{-1})$ and $\theta \in [a,\ e^{1/\sqrt{q}}]$,*

$$g_q(\theta) \le \frac{6q \log \frac{1}{a}}{\theta+1}. \tag{4.29}$$

(ii) *For $\theta \in [q^3,\ e^{1/\sqrt{q}})$ and $r \in [1,\ 2]$,*

$$g_q\left(\frac{\theta}{r}\right) - g_q(\theta) \ge \frac{q(1-\sqrt{q})(r-1) - q(r-1)^2}{\theta+1}. \tag{4.30}$$

(iii) *For $\theta \in [0,\ e^{1/\sqrt{q}}]$,*

$$\int_0^\theta |g_q'(t)| \left(g_q(\theta-t) - g_q(\theta)\right) \mathrm{d}t \le (1+3\sqrt{q})q^2 \frac{\zeta(2)}{\theta+1}. \tag{4.31}$$

Proof. See Appendix B. □

We proceed to the proof of Lemma 6 in the third case. Let $q_0 \in (0,\ \frac{1}{9})$ be the small constant ensuring (4.25)–(4.27) hold. Fix $\eta \in (0,\ 1)$. We set

$$c_* = c_*(\eta) := \eta(5-2\eta) + \frac{4\eta(1-\eta)^2}{\zeta(2)} > 0. \tag{4.32}$$

Let $\varepsilon \in (0,\ \varepsilon_7(\eta))$, where $\varepsilon_7(\eta) \in (0,\ \min\{\varepsilon_6(\eta),\ \frac{1}{2}\})$ is such that

$$\gamma = \gamma(\varepsilon,\ \eta) := \frac{2}{\sqrt{\zeta(2)}}(1-\eta)\sqrt{\varepsilon} \in (0,\ q_0), \quad (1+\eta\gamma)\big(1-(2+\eta)\varepsilon\big)^{-\frac{1}{\gamma}} < 2, \tag{4.33}$$

$$\left(\frac{1}{2}+\varepsilon\right)\left(1+3\sqrt{\gamma}\right) \le \frac{1}{2} + 2\sqrt{\gamma}, \tag{4.34}$$

$$c_*\varepsilon > \left(\eta\gamma + \frac{(2+\eta)\varepsilon}{\gamma}\right)(\gamma^{3/2} + \eta\gamma^2 + (2+\eta)\varepsilon) + 2\gamma^{5/2}\zeta(2) + 12\varepsilon\gamma^{1/2} + 16e^{-1/\sqrt{\gamma}}. \tag{4.35}$$

The seemingly complicated form of the right-hand member of (4.35) is justified in the following: it is the sum of several explicit inequalities. We could have used slightly simpler terms, but the price would have been an extra layer of numbered constants, which we wanted to avoid for the sake of clarity.

We want to check (4.5) in Lemma 6 when $z=\tau_n x$, with $x\in[\gamma^2,\ \mathrm{e}^{1/\sqrt{\gamma}})$. To this end we first need to find a lower bound for $\mathbb{P}(Z_{n+1}>x\tau_n/(1+\eta\gamma))$. More precisely, in the third case (4.5) is equivalent to prove that there exists $\varepsilon'\in(0,\infty)$ (depending on η) such that for all $\varepsilon\in(0,\varepsilon')$ there is $n(\varepsilon')$ such that for all $n\geq n(\varepsilon')$ and all $x\in[\gamma^2,\ \mathrm{e}^{1/\sqrt{\gamma}})$:

$$[\Psi_{\frac{1}{2}+\varepsilon}(\nu_n')]((x\tau_n,\ \infty))\leq\mathbb{P}\left(Z_{n+1}>\frac{x\tau_n}{1+\eta\gamma}\right). \tag{4.36}$$

where Ψ_p is the transformation in (2.9).

To this end, we first study the tail probability of $\frac{Z_n}{\tau_n}$. Let $a>0$. By (4.4), for all sufficiently large n such that $a\tau_n\geq\sigma_n$, and all $\theta\geq a$,

$$\begin{aligned}\mathbb{P}(Z_n>\theta\tau_n)&=b_n\tau_n^\gamma\left((\theta+1)^\gamma-\theta^\gamma\right)-b_n\tau_n^{-\gamma}\left(\theta^{-\gamma}-(\theta+1)^{-\gamma}\right)\\&=g_\gamma(\theta)\big(b_n\tau_n^\gamma-b_n\tau_n^{-\gamma}\theta^{-\gamma}(\theta+1)^{-\gamma}\big),\end{aligned}$$

where g_γ is the function in (4.28). Hence,

$$\sup_{\theta\in[a,\ \infty)}\left|g_\gamma(\theta)^{-1}\mathbb{P}(Z_n>\theta\tau_n)-1\right|\leq|b_n\tau_n^\gamma-1|+b_n\tau_n^{-\gamma}a^{-\gamma}.$$

Since $\lim_{n\to\infty}b_n\tau_n^\gamma=1$ and $\lim_{n\to\infty}b_n\tau_n^{-\gamma}=0$ (by (4.3)), this implies that

$$\lim_{n\to\infty}\sup_{\theta\in[a,\ \infty)}\left|g_\gamma(\theta)^{-1}\mathbb{P}(Z_n>\theta\tau_n)-1\right|=0. \tag{4.37}$$

Note that $\sup_{\theta\in[a,\ \infty)}g_\gamma(\theta)=g_\gamma(a)<\infty$; thus, we also have $\lim_{n\to\infty}\sup_{\theta\in[a,\ \infty)}\left|\mathbb{P}(Z_n>\theta\tau_n)-g_\gamma(\theta)\right|=0$. Taking $a=\frac{1}{2}\mathrm{e}^{-4/\sqrt{\gamma}}$ yields the existence of a positive integer $n_{14}(\eta,\ \varepsilon)$ such that for $n\geq n_{14}(\eta,\ \varepsilon)$ and $\theta\in[\frac{1}{2}\mathrm{e}^{-4/\sqrt{\gamma}},\ \infty)$,

$$\left|g_\gamma(\theta)^{-1}\mathbb{P}(Z_n>\theta\tau_n)-1\right|\leq\mathrm{e}^{-2/\sqrt{\gamma}}, \tag{4.38}$$

$$\left|\mathbb{P}(Z_n>\theta\tau_n)-g_\gamma(\theta)\right|\leq\mathrm{e}^{-2/\sqrt{\gamma}}. \tag{4.39}$$

In order to deal with $\mathbb{P}(Z_{n+1}>x\tau_n/(1+\eta\gamma))$ on the right-hand side of (4.36), we write

$$\varrho_n(\gamma):=\ (1+\eta\gamma)\,\frac{\tau_{n+1}}{\tau_n}.$$

By (4.3), $\lim_{n\to\infty}b_n\tau_n^\gamma=1$, whereas by definition, $\frac{b_{n+1}}{b_n}=1-(2+\eta)\varepsilon$. By using the inequality $(1-y)^{-z}\geq1+yz$ for $y\in(0,\ 1)$ and $z\geq0$, we therefore get

$$\lim_{n\to\infty}\rho_n(\gamma)=(1+\eta\gamma)(1-(2+\eta)\varepsilon)^{-\frac{1}{\gamma}}\geq(1+\eta\gamma)\left(1+\frac{(2+\eta)\varepsilon}{\gamma}\right)>1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}.$$

In addition, by (4.33), $\lim_{n\to\infty} \rho_n(\gamma) < 2$. Therefore, there exists an integer $n_{15}(\eta,\ \varepsilon) \geq n_{14}(\eta,\ \varepsilon)$ such that for $n \geq n_{15}(\eta,\ \varepsilon)$,

$$1 + \eta\gamma + \frac{(2+\eta)\varepsilon}{\gamma} < \rho_n(\gamma) < 2.$$

Let $x \in [\gamma^2,\ \mathrm{e}^{1/\sqrt{\gamma}})$. We have

$$\frac{x\tau_n}{1+\eta\gamma} = \frac{x\tau_{n+1}}{\rho_n(\gamma)},$$

and

$$\frac{x}{\rho_n(\gamma)} > \frac{\gamma^2}{2} \geq \frac{1}{2}\mathrm{e}^{-4/\sqrt{\gamma}}$$

(by the elementary inequality $z^2 \leq \mathrm{e}^{4\sqrt{z}}$, for $z \geq 1$), so we are entitled to apply (4.39) to $\theta := x/\rho_n(\gamma)$ to see that

$$\mathbb{P}\left(Z_{n+1} > \frac{x\tau_n}{1+\eta\gamma}\right) \geq g_\gamma\left(\frac{x}{\rho_n(\gamma)}\right) - \mathrm{e}^{-2/\sqrt{\gamma}} \geq g_\gamma\left(\frac{x}{1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}}\right) - \mathrm{e}^{-2/\sqrt{\gamma}}. \quad (4.40)$$

To prove (4.36), we need to find an appropriate upper bound for $[\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)]((x\tau_n,\ \infty))$. Let $Z_n^* := \frac{Z_n}{\tau_n}$, and let $\widehat{Z}_n^*$ denote an independent copy of Z_n^*. For $\theta \in \left(\frac{\gamma^3}{2},\ \mathrm{e}^{1/\sqrt{\gamma}}\right)$, we define the event $B_n(\theta) := \{Z_n^* + \widehat{Z}_n^* \geq \theta\}$, and note that

$$\begin{aligned}\mathbb{P}(B_n(\theta)) &\leq \mathbb{P}\big(Z_n^* \geq \theta\big) + \mathbb{P}\big(\widehat{Z}_n^* \geq \theta; Z_n^* < \theta\big) + \mathbb{P}\big(\widehat{Z}_n^* < \theta; Z_n^* < \theta; B_n(\theta)\big)\\ &= 2\mathbb{P}\big(Z_n^* \geq \theta\big) - [\mathbb{P}\big(Z_n^* \geq \theta\big)]^2 + \mathbb{P}\big(\widehat{Z}_n^* < \theta; Z_n^* < \theta; B_n(\theta)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\big[\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)\big]((x\tau_n,\ \infty)) &= \left(\frac{1}{2}+\varepsilon\right)\mathbb{P}(B_n(x)) + \left(\frac{1}{2}-\varepsilon\right)[\mathbb{P}(Z_n^* \geq x)]^2\\ &\leq (1+2\varepsilon)\mathbb{P}(Z_n^* \geq x) + \left(\frac{1}{2}+\varepsilon\right)\mathbb{P}(\widehat{Z}_n^* < x; Z_n^* < x; B_n(x)). \quad (4.41)\end{aligned}$$

We first get an upper bound for the second term in the right-hand side member of (4.41). To this end, we write $x_1 := x - \mathrm{e}^{-4/\sqrt{\gamma}}$, so by (4.26), $x_1 \geq \gamma^2 - \mathrm{e}^{-4/\sqrt{\gamma}} > \gamma^3 > \mathrm{e}^{-1/\sqrt{\gamma}}$. We consider $\{\widehat{Z}_n^* < x\}$ as the union of $\{\widehat{Z}_n^* < \mathrm{e}^{-4/\sqrt{\gamma}}\}$, $\{\widehat{Z}_n^* \in [\mathrm{e}^{-4/\sqrt{\gamma}},\ x_1]\}$, and $\{\widehat{Z}_n^* \in (x_1,\ x)\}$. On $\{\widehat{Z}_n^* < \mathrm{e}^{-4/\sqrt{\gamma}}\} \cap B_n(x)$, we have $Z_n > x_1$. This implies that

$$\begin{aligned}&\big[\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)\big]((x\tau_n,\ \infty))\\ &\leq (1+2\varepsilon)\mathbb{P}(Z_n^* \geq x) + (1+2\varepsilon)\mathbb{P}(Z_n^* \in (x_1,\ x))\\ &\quad + \left(\frac{1}{2}+\varepsilon\right)\mathbb{P}\big(\widehat{Z}_n^* \in \big[\mathrm{e}^{-4/\sqrt{\gamma}},\ x_1\big]; Z_n^* \in \big(x - \widehat{Z}_n^*,\ x\big)\big)\\ &= (1+2\varepsilon)\mathbb{P}(Z_n^* > x_1) + \left(\frac{1}{2}+\varepsilon\right)\mathbb{P}\big(\widehat{Z}_n^* \in \big[\mathrm{e}^{-4/\sqrt{\gamma}},\ x_1\big]; Z_n^* \in \big(x - \widehat{Z}_n^*,\ x\big)\big). \quad (4.42)\end{aligned}$$

We then estimate the two probability expressions on the right-hand side. Since we have proved that $x_1 > e^{-1/\sqrt{\gamma}}$, (4.39) applies with $\theta = x_1$ and we get $\mathbb{P}(Z_n^* \geq x_1) \leq g_\gamma(x_1) + \mathrm{e}^{-2/\sqrt{\gamma}}$. By the

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mean-value theorem, $g_\gamma(x_1) - g_\gamma(x) = -(x - x_1)g'_\gamma(y) = -\mathrm{e}^{-4/\sqrt{\gamma}} g'_\gamma(y)$ for some $y \in [x_1,\ x]$. Since $-g'_\gamma(y) = \gamma[y^{-(1-\gamma)} - (y+1)^{-(1-\gamma)}] \le \gamma y^{-(1-\gamma)} \le \gamma (x_1)^{-(1-\gamma)} < \gamma^{1-3(1-\gamma)} \le \gamma^{-2}$ (we have used the fact that $x_1 > \gamma^3$), we obtain that

$$g_\gamma(x_1) - g_\gamma(x) \le \gamma^{-2}\,\mathrm{e}^{-4/\sqrt{\gamma}} \le \mathrm{e}^{-2/\sqrt{\gamma}} \tag{4.43}$$

(here, we have used also the elementary inequality $z^2 \le \mathrm{e}^{2\sqrt{z}}$ for $z \ge 1$). Therefore,

$$\mathbb{P}(Z_n^* > x_1) \le g_\gamma(x) + 2\mathrm{e}^{-2/\sqrt{\gamma}}.$$

By (4.29) (applied to $a = \mathrm{e}^{-1/\sqrt{\gamma}}$ and $q = \gamma$, which is possible since $\gamma < q_0$), $g_\gamma(x) \le \frac{6\gamma^{1/2}}{x+1}$. Since $2(1+2\varepsilon) \le 3$, this implies that

$$\begin{aligned}(1+2\varepsilon)\mathbb{P}(Z_n^* > x_1) &\le g_\gamma(x) + 2\varepsilon g_\gamma(x) + 2(1+2\varepsilon)\mathrm{e}^{-2/\sqrt{\gamma}} \\ &\le g_\gamma(x) + \frac{12\varepsilon\gamma^{1/2}}{x+1} + 3\mathrm{e}^{-2/\sqrt{\gamma}}.\end{aligned} \tag{4.44}$$

We next prove an upper bound for $\mathbb{P}(\widehat{Z}_n^* \in [\mathrm{e}^{-4/\sqrt{\gamma}},\ x_1]; Z_n^* \in (x - \widehat{Z}_n^*,\ x))$ thanks to (4.39). We fix $t \in (0,\ x_1)$ and we first observe that $x \ge x - t \ge x - x_1 = e^{-4/\sqrt{\gamma}}$. Then (4.39) applies with θ equal to x and $x - t$ and we get

$$\mathbb{P}(Z_n^* \in (x-t,\ x)) = \mathbb{P}(Z_n^* >\ x - t) - \mathbb{P}(Z_n^* > x) \le g_\gamma(x-t) - g_\gamma(x) + 2\mathrm{e}^{-2/\sqrt{\gamma}}.$$

To simplify, we next denote by $f_{Z_n^*}(\cdot)$ the density of Z_n^* and we set $\gamma_1 :=\ \mathrm{e}^{-4/\sqrt{\gamma}} = x - x_1$. Then

$$\begin{aligned}\mathbb{P}(\widehat{Z}_n^* \in [\gamma_1,\ x_1]; Z_n^* \in (x - \widehat{Z}_n^*,\ x)) &= \int_{\mathbb{R}} f_{Z_n^*}(t)\mathbf{1}_{[\gamma_1,x_1]}(t)\mathbb{P}\big(Z_n^* \in (x-t, x]\big)\,\mathrm{d}t \\ &\le \int_{\mathbb{R}} \mathbf{1}_{[\gamma_1,x_1]}(t) f_{Z_n^*}(t)\big(g_\gamma(x-t) - g_\gamma(x)\big)\,\mathrm{d}t + 2\mathrm{e}^{-2/\sqrt{\gamma}}.\end{aligned} \tag{4.45}$$

Then, by Fubini, observe that

$$\begin{aligned}\int_{\mathbb{R}} \mathbf{1}_{[\gamma_1,x_1]}(t) f_{Z_n^*}(t)\big(g_\gamma(x-t) - g_\gamma(x)\big)\,\mathrm{d}t &= \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}s\, f_{Z_n^*}(t)\mathbf{1}_{[\gamma_1,x_1]}(t)\mathbf{1}_{[0,t]}(x-s)\big(-g'_\gamma(s)\big) \\ &= \int_0^{x_1} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}t\, f_{Z_n^*}(t)\mathbf{1}_{[\gamma_1,x_1]}(t)\mathbf{1}_{[u,\infty)}(t)\big|g'_\gamma(x-u)\big| \\ &= \int_0^{x_1} \big|g'_\gamma(x-u)\big|\mathbb{P}\big(Z_n^* \in [\gamma_1 \vee u, x_1]\big)\,\mathrm{d}u.\end{aligned}$$

Note that if $u \in [0, x_1]$, then $x_1 \ge u \vee \gamma_1 \ge \gamma_1 > \frac{1}{2}\mathrm{e}^{-4/\sqrt{\gamma}}$. Thus, (4.39) applies and we get

$$\begin{aligned}&\int_0^{x_1} \big|g'_\gamma(x-u)\big|\mathbb{P}(Z_n^* \in [\gamma_1 \vee u, x_1])\mathrm{d}u \le \int_0^{x_1} \big|g'_\gamma(x-u)\big|(g_\gamma(u \vee \gamma_1) - g_\gamma(x_1) + 2e^{-2/\sqrt{\gamma}})\mathrm{d}u \\ &\le \int_0^{x} \big|g'_\gamma(x-u)\big|\big(g_\gamma(u) - g_\gamma(x)\big)\,\mathrm{d}u + 2e^{-2/\sqrt{\gamma}} \int_0^{x_1} \big|g'_\gamma(x-u)\big|\,\mathrm{d}u,\end{aligned} \tag{4.46}$$

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since g_γ is decreasing. Then

$$\int_0^{x_1} |g'_\gamma(x-u)|\mathbb{P}\big(Z_n^* \in [\gamma_1 \vee u, x_1]\big)\mathrm{d}u$$
$$\le \int_0^x |g'_\gamma(t)|\big(g_\gamma(x-t)-g_\gamma(x)\big)\mathrm{d}t + 2e^{-2/\sqrt{\gamma}}\big(g_\gamma(\gamma_1)-g_\gamma(x_1)\big)$$
$$\le (1+3\sqrt{\gamma}\,)\gamma^2\,\frac{\zeta(2)}{x+1} + 2e^{-2/\sqrt{\gamma}} \tag{4.47}$$

by (4.31), Lemma 8(iii) (with $q=\gamma<q_0$ and $\theta=x<\mathrm{e}^{1/\sqrt{\gamma}}$), and since $x-x_1=\gamma_1$ and $g_\gamma(\gamma_1)-g_\gamma(x_1)\le g_\gamma(\gamma_1)<1$. By (4.45) and (4.47) we then get

$$\mathbb{P}(\widehat{Z}_n^* \in [\gamma_1,\ x_1]; Z_n^* \in (x-\widehat{Z}_n^*,\ x)) \le (1+3\sqrt{\gamma}\,)\gamma^2\,\frac{\zeta(2)}{x+1} + 4\mathrm{e}^{-2/\sqrt{\gamma}}.$$

Since $(\frac{1}{2}+\varepsilon)(1+3\sqrt{\gamma}\,)\le \frac{1}{2}+2\sqrt{\gamma}$ by (4.34) and $\frac{1}{2}+\varepsilon\le 1$, it follows that

$$\left(\frac{1}{2}+\varepsilon\right)\mathbb{P}(\widehat{Z}_n^* \in [\gamma_1,\ x_1]; Z_n^* \in (x-\widehat{Z}_n^*,\ x)) \le \left(\frac{1}{2}+2\sqrt{\gamma}\right)\gamma^2\frac{\zeta(2)}{x+1} + 4\mathrm{e}^{-2/\sqrt{\gamma}}.$$

Combined with (4.42) and (4.44), we obtain that

$$[\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)]((x\tau_n,\ \infty)) \le g_\gamma(x) + 3\mathrm{e}^{-2/\sqrt{\gamma}} + \frac{12\varepsilon\gamma^{1/2}}{x+1} + \left(\frac{1}{2}+2\sqrt{\gamma}\right)\gamma^2\frac{\zeta(2)}{x+1} + 4\mathrm{e}^{-2/\sqrt{\gamma}}$$
$$= g_\gamma(x) + \frac{12\varepsilon\gamma^{1/2} + \left(\frac{1}{2}+2\sqrt{\gamma}\right)\gamma^2\zeta(2)}{x+1} + 7\mathrm{e}^{-2/\sqrt{\gamma}}.$$

In view of (4.40), this yields that

$$\mathbb{P}\left(Z_{n+1} > \frac{x\tau_n}{1+\eta\gamma}\right) - [\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)]((x\tau_n,\ \infty))$$
$$\ge g_\gamma\left(\frac{x}{1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}}\right) - g_\gamma(x) - \mathrm{e}^{-2/\sqrt{\gamma}} - \frac{12\varepsilon\gamma^{1/2} + (\frac{1}{2}+2\sqrt{\gamma}\,)\gamma^2\zeta(2)}{x+1} - 7\mathrm{e}^{-2/\sqrt{\gamma}}$$
$$= g_\gamma\left(\frac{x}{1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}}\right) - g_\gamma(x) - \frac{\frac{\zeta(2)}{2}\gamma^2 + (2\gamma^{5/2}\zeta(2) + 12\varepsilon\gamma^{1/2})}{x+1} - 8\mathrm{e}^{-2/\sqrt{\gamma}}.$$

We easily check that we can apply (4.30) with $q=\gamma<q_0$, $\theta=x\in[\gamma^3,\mathrm{e}^{1/\sqrt{\gamma}}]$, and $r=1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}\in[1,2]$. Then we get

$$g_\gamma\left(\frac{x}{1+\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}}\right) - g_\gamma(x) \ge \frac{\gamma(1-\gamma^{1/2})\left(\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}\right) - \gamma\left(\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}\right)^2}{x+1}$$
$$= \frac{\eta\gamma^2 + (2+\eta)\varepsilon - \left(\eta\gamma+\frac{(2+\eta)\varepsilon}{\gamma}\right)(\gamma^{3/2}+\eta\gamma^2+(2+\eta)\varepsilon)}{x+1}.$$

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As a consequence,

$$\mathbb{P}\left(Z_{n+1} > \frac{x\tau_n}{1+\eta\gamma}\right) - [\Psi_{\frac{1}{2}+\varepsilon}(\nu'_n)]((x\tau_n,\ \infty)) \geq \frac{\eta\gamma^2 + (2+\eta)\varepsilon - \frac{\zeta(2)}{2}\gamma^2 - R_\varepsilon}{x+1} - 8\mathrm{e}^{-2/\sqrt{\gamma}},$$

where

$$R_\varepsilon := \left(\eta\gamma + \frac{(2+\eta)\varepsilon}{\gamma}\right)(\gamma^{3/2} + \eta\gamma^2 + (2+\eta)\varepsilon) + 2\gamma^{5/2}\zeta(2) + 12\varepsilon\gamma^{1/2}.$$

Recall that $\eta \in (0,\ 1)$ is a constant and recall from (4.32), (4.33), (4.34), and (4.35) that $\gamma = (2/\sqrt{\zeta(2)})(1-\eta)\sqrt{\varepsilon}$. Thus,

$$\eta\gamma^2 + (2+\eta)\varepsilon - \frac{\zeta(2)}{2}\gamma^2 = c_*\,\varepsilon,$$

where

$$c_* = \eta(5-2\eta) + \frac{4\eta(1-\eta)^2}{\zeta(2)} > 0.$$

Since $8(x+1)\mathrm{e}^{-2/\sqrt{\gamma}} \leq 8(\mathrm{e}^{1/\sqrt{\gamma}}+1)\mathrm{e}^{-2/\sqrt{\gamma}} \leq 16\mathrm{e}^{-1/\sqrt{\gamma}}$, it follows from (4.35) that

$$\frac{\eta\gamma^2 + (2+\eta)\varepsilon - \frac{\zeta(2)}{2}\gamma^2) - R_\varepsilon}{x+1} - 8\mathrm{e}^{-2/\sqrt{\gamma}} > 0.$$

Therefore, we have found $\varepsilon_7(\eta) \in (0,\ 1)$ such that for all $\varepsilon \in (0,\ \varepsilon_7(\eta))$, there is $n_{15}(\eta,\ \varepsilon)$ such that (4.36) holds true for all $n \geq n_{15}(\eta,\ \varepsilon)$ and all $x \in [\gamma^2,\ \mathrm{e}^{1/\sqrt{\gamma}}]$, which completes the proof of Lemma 6 in the third case $\gamma^2\tau_n \leq z < \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$.

Proof of Lemma 6: fourth (and last) case $z \geq \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$. Let $\eta, \varepsilon \in (0,\ 1)$ and recall the definition of γ from (4.1): namely, $\gamma = \gamma(\varepsilon,\ \eta) := 2\zeta(2)^{-1/2}(1-\eta)\sqrt{\varepsilon}$. We fix η and we easily check that we can find $\varepsilon_8(\eta) \in (0,\ \varepsilon_7(\eta))$ such for all $\varepsilon \in (0, \varepsilon_8(\eta))$, the following inequalities hold true:

$$\gamma \leq \frac{1}{400}, \quad (1+2\varepsilon)\,\frac{\left(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\right)^4}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}} < \left(1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}\right)^2 (1-(2+\eta)\varepsilon)^{-(1-\gamma)/\gamma}. \tag{4.48}$$

The following lemma gives an estimate of $\mathbb{P}(Z_n \geq \tau_n\theta)$ when θ is sufficiently large.

Lemma 9. *Let $\eta \in (0,\ 1)$. Let $\varepsilon \in (0,\ \varepsilon_8(\eta))$ and $n \geq n_{15}(\eta,\ \varepsilon)$. Then*

$$1 - 2\mathrm{e}^{-1/(2\sqrt{\gamma})} \leq \frac{\theta^{1-\gamma}}{\gamma}\,\mathbb{P}(Z_n \geq \tau_n\theta) \leq 1 + \mathrm{e}^{-2/\sqrt{\gamma}}, \quad \theta \in [\mathrm{e}^{1/(2\sqrt{\gamma})},\ \infty). \tag{4.49}$$

Furthermore,

$$\mathbb{P}(Z_n \geq \mathrm{e}^{3/(4\sqrt{\gamma})}\tau_n) \leq \mathrm{e}^{-3/(4\sqrt{\gamma})}. \tag{4.50}$$

Proof. For $\theta \in \mathbb{R}_+^*$, write as before

$$g_\gamma(\theta) := (\theta+1)^\gamma - \theta^\gamma = \frac{\gamma}{\theta^{1-\gamma}} \int_0^1 \frac{\mathrm{d}u}{(1+\frac{u}{\theta})^{1-\gamma}}.$$

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Thus,

$$\frac{\theta^{1-\gamma}}{\gamma} g_\gamma(\theta) \leq 1.$$

On the other hand,

$$\frac{\theta^{1-\gamma}}{\gamma} g_\gamma(\theta) \geq \int_0^1 \frac{\mathrm{d}u}{1+\frac{u}{\theta}} \geq \int_0^1 \frac{\mathrm{d}u}{1+\frac{1}{\theta}} = \frac{\theta}{1+\theta}.$$

Thus,

$$\text{for all } \theta \in \mathbb{R}_+^*, \qquad \frac{\theta}{1+\theta} \leq \frac{\theta^{1-\gamma}}{\gamma} g_\gamma(\theta) \leq 1.$$

By (4.38), for $n \geq n_{15}(\eta,\ \varepsilon)$ and $\theta \in [\frac{1}{2}\mathrm{e}^{-4/\sqrt{\gamma}},\ \infty)$, we get

$$(1-\mathrm{e}^{-2/\sqrt{\gamma}})\frac{\theta}{1+\theta} \leq \frac{\theta^{1-\gamma}}{\gamma}\, \mathbb{P}(Z_n \geq \tau_n \theta) \leq 1+\mathrm{e}^{-2/\sqrt{\gamma}}. \tag{4.51}$$

If $\theta > \mathrm{e}^{1/(2\sqrt{\gamma})}$, then $\frac{1}{1+\theta} \leq \frac{1}{\theta} \leq \mathrm{e}^{-1/(2\sqrt{\gamma})}$, so $(1-\mathrm{e}^{-2/\sqrt{\gamma}})\frac{\theta}{1+\theta} \geq 1 - \frac{1}{1+\theta} - \mathrm{e}^{-2/\sqrt{\gamma}} \geq 1 - 2\mathrm{e}^{-1/(2\sqrt{\gamma})}$, and (4.49) follows.

To get (4.50), it suffices to take $\theta := \mathrm{e}^{3/(4\sqrt{\gamma})}$, and note that in this case,

$$\frac{\gamma}{\theta^{1-\gamma}} = \frac{\gamma \mathrm{e}^{3\sqrt{\gamma}/4}}{\mathrm{e}^{3/(4\sqrt{\gamma})}} \leq \frac{\gamma \mathrm{e}}{\mathrm{e}^{3/(4\sqrt{\gamma})}} \leq \frac{1}{2}\mathrm{e}^{-3/(4\sqrt{\gamma})},$$

whereas $1+\mathrm{e}^{-2/\sqrt{\gamma}} \leq 2$, so (4.50) follows from the second inequality in (4.51).

We proceed to the proof of Lemma 6 in the fourth case: $z \geq \mathrm{e}^{1/\sqrt{\gamma}}\tau_n$. Let $\varepsilon \in (0,\ \varepsilon_8(\eta))$, $n \geq n_{15}(\eta,\ \varepsilon)$, and $x \in [\mathrm{e}^{1/\sqrt{\gamma}},\ \infty)$. We write as before $Z_n^* := \frac{Z_n}{\tau_n}$ where $\widehat{Z}_n^*$ denotes an independent copy of Z_n^*. We have, for $x' \in (0,\ x)$,

$$\begin{aligned}\mathbb{P}\big(Z_n^* + \widehat{Z}_n^* > x\big) &\leq \mathbb{P}(Z_n^* > x') + \mathbb{P}(\widehat{Z}_n^* > x') + \mathbb{P}(\widehat{Z}_n^* > x - x',\ Z_n^* > x - x') \\ &= 2\mathbb{P}(Z_n^* > x') + \big(\mathbb{P}(Z_n^* > x - x')\big)^2.\end{aligned}$$

We now take $x' := (1-\mathrm{e}^{-1/(4\sqrt{\gamma})})x$, so $x' \in [\mathrm{e}^{1/(2\sqrt{\gamma})},\ \infty)$ (because $1-\mathrm{e}^{-1/(4\sqrt{\gamma})} > \mathrm{e}^{-1/(2\sqrt{\gamma})}$ as $\sqrt{\gamma} \leq \frac{1}{20}$ according to (4.48)) and $x - x' \in [\mathrm{e}^{3/(4\sqrt{\gamma})},\ \infty)$. By (4.50),

$$\mathbb{P}(Z_n^* > x - x') \leq \mathbb{P}\big(Z_n^* > \mathrm{e}^{3/(4\sqrt{\gamma})}\big) \leq \mathrm{e}^{-3/(4\sqrt{\gamma})}. \tag{4.52}$$

Hence,

$$\mathbb{P}\big(Z_n^* + \widehat{Z}_n^* > x\big) \leq 2\mathbb{P}(Z_n^* > x') + \mathrm{e}^{-3/(4\sqrt{\gamma})}\, \mathbb{P}(Z_n^* > x - x').$$

We easily check that we can apply (4.49) to $\theta = x'$ and to $\theta = x - x'$, which yields

$$\mathbb{P}\big(Z_n^* + \widehat{Z}_n^* > x\big) \leq \left(2\left(\frac{x}{x'}\right)^{1-\gamma} + \mathrm{e}^{-3/(4\sqrt{\gamma})}\left(\frac{x}{x-x'}\right)^{1-\gamma}\right)\frac{1+\mathrm{e}^{-2/\sqrt{\gamma}}}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\, \mathbb{P}(Z_n^* > x).$$

We have

$$\left(\frac{x}{x'}\right)^{1-\gamma} \leq \frac{x}{x'} = \frac{1}{1-\mathrm{e}^{-1/(4\sqrt{\gamma})}} < 1+\mathrm{e}^{-1/(5\sqrt{\gamma})}$$

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(noting that $(1+\mathrm{e}^{-1/(5\sqrt{\gamma})})(1-\mathrm{e}^{-1/(4\sqrt{\gamma})})>1$ because $\sqrt{\gamma}\le\frac{1}{20}$ by (4.48)),

$$\left(\frac{x}{x-x'}\right)^{1-\gamma}\le\frac{x}{x-x'}=\mathrm{e}^{1/(4\sqrt{\gamma})}.$$

Thus,

$$2\left(\frac{x}{x'}\right)^{1-\gamma}+\mathrm{e}^{-3/(4\sqrt{\gamma})}\left(\frac{x}{x-x'}\right)^{1-\gamma}\le 2(1+\mathrm{e}^{-1/(5\sqrt{\gamma})})+\mathrm{e}^{-1/(2\sqrt{\gamma})}\le 2\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big),$$

which implies that

$$\begin{aligned}\mathbb{P}\big(Z_n^*+\widehat{Z}_n^*>x\big)&\le 2\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)\,\frac{1+\mathrm{e}^{-2/\sqrt{\gamma}}}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\mathbb{P}(Z_n^*>x)\\&\le\frac{2\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)^2}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\mathbb{P}(Z_n^*>x).\end{aligned}$$

Let $V_n:=(Z_n+\widehat{Z}_n)\mathscr{E}_n+(Z_n\wedge\widehat{Z}_n)(1-\mathscr{E}_n)$, where $\mathscr{E}_n$ denotes a Bernoulli random variable that is independent of $(Z_n,\widehat{Z}_n)$ and such that $\mathbb{P}(\mathscr{E}_n=1)=\frac{1}{2}+\varepsilon$. Then

$$\mathbb{P}(V_n\ge x\tau_n)=\left(\frac{1}{2}+\varepsilon\right)\mathbb{P}(Z_n^*+\widehat{Z}_n^*>x)+\left(\frac{1}{2}-\varepsilon\right)\big(\mathbb{P}(Z_n^*>x)\big)^2.$$

We use the trivial inequality $\frac{1}{2}-\varepsilon\le 1$. By (4.52) $\mathbb{P}(Z_n^*>x)\le\mathbb{P}(Z_n^*>x-x')\le\mathrm{e}^{-3/(4\sqrt{\gamma})}$. Therefore,

$$\begin{aligned}\mathbb{P}(V_n\ge x\tau_n)&\le(1+2\varepsilon)\,\frac{\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)^2}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\mathbb{P}(Z_n^*>x)+\mathrm{e}^{-3/(4\sqrt{\gamma})}\,\mathbb{P}(Z_n^*>x)\\&\le(1+2\varepsilon)\,\frac{\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)^3}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\mathbb{P}(Z_n^*>x).\end{aligned}$$

To complete the proof of the lemma, we observe that

$$\begin{aligned}\mathrm{LHS}_{(4.5)}&:=\mathbb{P}(V_n\ge x\tau_n)-\mathbb{P}\left(Z_{n+1}>\tau_n\frac{x}{1+\eta\gamma}\right)\\&=\mathbb{P}(V_n\ge x\tau_n)-\mathbb{P}\left(Z_{n+1}^*>\frac{\tau_n}{\tau_{n+1}}\frac{x}{1+\eta\gamma}\right)\\&\le(1+2\varepsilon)\,\frac{\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)^3}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\mathbb{P}(Z_n^*>x)-\mathbb{P}\left(Z_{n+1}^*>\frac{\tau_n}{\tau_{n+1}}x\right).\end{aligned}$$

We can apply (4.49) to $x\ge\mathrm{e}^{1/\sqrt{\gamma}}>\mathrm{e}^{1/(2\sqrt{\gamma})}$ and we get $\mathbb{P}(Z_n^*>x)\le(\gamma/x^{1-\gamma})(1+\mathrm{e}^{-2/\sqrt{\gamma}})$, which is bounded by $(\gamma/x^{1-\gamma})\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)$. Thus,

$$\mathrm{LHS}_{(4.5)}\le(1+2\varepsilon)\,\frac{\big(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\big)^4}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}}\,\frac{\gamma}{x^{1-\gamma}}-\mathbb{P}\left(Z_{n+1}^*>\frac{\tau_n}{\tau_{n+1}}x\right).\tag{4.53}$$

We look for a lower bound for $\mathbb{P}(Z_{n+1}^*>(\tau_n/\tau_{n+1})x)$. By definition, $\lim_{n\to\infty}\tau_n/\tau_{n+1}=(1-(2+\eta)\varepsilon)^{1/\gamma}$ and $\lim_{\varepsilon\to 0^+}(1-(2+\eta)\varepsilon)^{1/\gamma}=1$. Therefore, there exists $\varepsilon_9(\eta)\in$

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$(0,\ \varepsilon_8(\eta))$ such that for $\varepsilon \in (0,\ \varepsilon_9(\eta))$, there is an integer $n_7(\eta, \varepsilon) \geq n_{15}(\eta, \varepsilon)$ such that for all $n \geq n_{16}(\eta,\ \varepsilon)$,

$$\frac{\tau_n}{\tau_{n+1}} \geq \mathrm{e}^{-1/(2\sqrt{\gamma})} \quad \text{and} \quad \left(\frac{\tau_{n+1}}{\tau_n}\right)^{1-\gamma} \geq \left(1 - 2\mathrm{e}^{-1/(2\sqrt{\gamma})}\right)(1-(2+\eta)\varepsilon)^{-(1-\gamma)/\gamma}.$$

Since $\frac{\tau_n}{\tau_{n+1}} x \geq \mathrm{e}^{1/(2\sqrt{\gamma})}$, (4.49) applies to $\theta := \frac{\tau_n}{\tau_{n+1}} x$ and for all $\varepsilon \in (0,\ \varepsilon_9(\eta))$ and all $n \geq n_{16}(\eta,\ \varepsilon)$, we get

$$\mathbb{P}\left(Z^*_{n+1} > \frac{\tau_n}{\tau_{n+1}} x\right) \geq \left(1 - 2\mathrm{e}^{-1/(2\sqrt{\gamma})}\right)\left(\frac{\tau_{n+1}}{\tau_n}\right)^{1-\gamma} \frac{\gamma}{x^{1-\gamma}}$$
$$\geq \left(1 - 2\mathrm{e}^{-1/(2\sqrt{\gamma})}\right)^2 (1-(2+\eta)\varepsilon)^{-(1-\gamma)/\gamma} \frac{\gamma}{x^{1-\gamma}}.$$

Going back to (4.53), for all $\varepsilon \in (0,\ \varepsilon_9(\eta))$, all $n \geq n_{16}(\eta, \varepsilon)$, and all $x \in [\mathrm{e}^{1/\sqrt{\gamma}},\ \infty)$, we obtain

$$\mathrm{LHS}_{(4.5)} \leq \left((1+2\varepsilon)\, \frac{\left(1+2\mathrm{e}^{-1/(5\sqrt{\gamma})}\right)^4}{1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}} - \left(1-2\mathrm{e}^{-1/(2\sqrt{\gamma})}\right)^2 (1-(2+\eta)\varepsilon)^{-(1-\gamma)/\gamma}\right) \frac{\gamma}{x^{1-\gamma}},$$

which is negative according to (4.48). This implies Lemma 6 in the fourth case, and thus completes the proof of the lemma. □

5. Proof of Theorem 2

Let $(\texttt{Graph}_n(p))_{n\in\mathbb{N}}$ be the sequence of graphs that are constructed as explained in the introduction, i.e. $\texttt{Graph}_{n+1}(p)$ is obtained by replacing each edge of $\texttt{Graph}_n(p)$, either by two edges in series with probability $p = \frac{1}{2} + \varepsilon$, or by two parallel edges with probability $1 - p = \frac{1}{2} - \varepsilon$, whereas $\texttt{Graph}_0(p)$ is the graph of two vertices connected by an edge.

Let m be a non-negative integer. We construct a Galton–Watson branching process $(Z^{(m)}_k)_{k\in\mathbb{N}}$ whose offspring distribution is the law of $D_m(p)$, such that $Z^{(m)}_1 = D_m(p)$ and such that

$$\mathbb{P}\text{-a.s. for all } k \in \mathbb{N}, \quad D_{km}(p) \leq Z^{(m)}_k. \tag{5.1}$$

Indeed, suppose that $D_{km}(p) \leq Z^{(m)}_k$. Then $\texttt{Graph}_{(k+1)m}(p)$ is obtained by replacing each edge of $\texttt{Graph}_{km}(p)$ by an independent copy of $\texttt{Graph}_m(p)$. Choose a geodesic path in $\texttt{Graph}_{km}(p)$ and denote by $D_{m,j}(p)$ the distance joining the extreme vertices of the graph $\texttt{Graph}_{m,j}(p)$ which replaces the jth edge of the specified geodesic path of $\texttt{Graph}_{km}(p)$ in the recursive construction of $\texttt{Graph}_{(k+1)m}(p)$ from $\texttt{Graph}_{km}(p)$. Conditionally given $\texttt{Graph}_{km}(p)$, the $\texttt{Graph}_{m,j}(p)$ are independent and identically distributed (i.i.d.) with the same law as $\texttt{Graph}_m(p)$, as mentioned previously. It entails $D_{(k+1)m}(p) \leq \sum_{1\leq j\leq D_{km}(p)} D_{m,j}(p)$. Let $(\Delta(k, j))_{k,j\in\mathbb{N}}$ be an array of i.i.d. random variables with the same law as $D_m(p)$. Assume, furthermore, that $(\Delta(k, j))_{k,j\in\mathbb{N}}$ is independent of the graphs $(\texttt{Graph}_n(p))_{n\in\mathbb{N}}$. Then, we set $D_{m,j}(p) = \Delta(k, j - D_{km}(p))$ for all integers $j > D_{km}(p)$ and we set $Z^{(m)}_{k+1} = \sum_{1\leq j\leq Z^{(m)}_k} D_{m,j}(p)$. Therefore, $D_{(k+1)m}(p) \leq Z^{(m)}_{k+1}$ and conditionally given $Z^{(m)}_k$, $Z^{(m)}_{k+1}$ is distributed as the sum of $Z^{(m)}_k$ i.i.d. random variables having the same law as $D_m(p)$. This completes the proof of (5.1).

For all $k \in \mathbb{N}$, we set $W^{(m)}_k = Z^{(m)}_k / \mathbb{E}[D_m(p)]^k$. Then $(W^{(m)}_k)_{k\in\mathbb{N}}$ is a martingale which is bounded in L^2 (here the support of the law of $D_m(p)$ is finite). Therefore, $\mathbb{P}$-a.s. $\lim_{k\to\infty} W^{(m)}_k =$

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$W_\infty^{(m)} > 0$ (because there is no extinction since $\mathbb{P}(D_m(p) = 0) = 0$). Since $(D_n(p))_{n\in\mathbb{N}}$ is a non-decreasing sequence of random variables, we $\mathbb{P}$-a.s. get for all $n \in \mathbb{N}$ that

$$D_n(p) \le D_{k(n)m}(p) \le W_{k(n)}^{(m)} \mathbb{E}[D_m(p)]^{k(n)} \quad \text{where we have set } k(n) = \lfloor n/m \rfloor + 1.$$

Therefore, for all $m \ge 2$,

$$\mathbb{P}\text{-a.s.} \quad \limsup_{n\to\infty} \frac{1}{n} \log D_n(p) \le \frac{1}{m} \log \mathbb{E}[D_m(p)],$$

which implies the last inequality in (1.4) in Theorem 2.

Let us prove the first inequality in (1.4). To this end, let Y_n be a random variable with law ν_n as defined in Lemma 1. As already explained in the proof of the lower bound of Theorem 1, for all $\eta \in (0, \eta_0)$, there exists $\varepsilon_\eta \in (0, 1/2)$ such that for all $\varepsilon \in (0, \varepsilon_\eta)$, there is $n_{\eta,\varepsilon} \in \mathbb{N}$ and $\theta_{\eta,\varepsilon}$ that satisfy $\theta_{\eta,\varepsilon} Y_n \overset{\text{st}}{\le} D_n$ for all $n \ge n_{\eta,\varepsilon}$ and, thus,

$$\mathbb{P}\big(D_n \le \theta_{\eta,\varepsilon} n^{-2/\delta} \lambda_n\big) \le \mathbb{P}\big(Y_n \le n^{-2/\delta} \lambda_n\big) = 2a_n \varphi_\delta\big(n^{-2/\delta} \lambda_n\big),$$

where we recall from (3.2) that $\varphi_\delta(x) = \frac{1}{2}(x^\delta + x^{-\delta}) - 1$, that

$$\delta = 2\zeta(2)^{-\frac{1}{2}}(1+\eta)\sqrt{\varepsilon}, \quad \text{that } a_n = \frac{1}{4}\big(1 - 2\varepsilon + \eta\delta^2\big)^n, \quad n \in \mathbb{N}.$$

and that λ_n is such that $2a_n\varphi_\delta(\lambda_n) = 1$. Since $\lim_{n\to\infty} a_n \lambda_n^\delta = 1$ (by Lemma 1(i)) we get $2a_n\varphi_\delta(n^{-2/\delta}\lambda_n) \sim_{n\to\infty} n^{-2}$ and $\sum_{n\ge1} \mathbb{P}(D_n \le \theta_{\eta,\varepsilon} n^{-2/\delta}\lambda_n) < \infty$, which implies by Borel–Cantelli that

$$\mathbb{P}\text{-.a.s. for all sufficiently large } n, \quad D_n \ge \theta_{\eta,\varepsilon} n^{-2/\delta} \lambda_n.$$

Since $\lim_{n\to\infty} a_n\lambda_n^\delta = 1$, it implies for all $\eta \in (0, \eta_0)$ and for all $\varepsilon \in (0, \varepsilon_\eta)$ that

$$\mathbb{P}\text{-.a.s.} \quad \liminf_{n\to\infty} \frac{1}{n} \log D_n(p) \ge -\frac{1}{\delta} \log\left(1 - 2\varepsilon + \eta\delta^2\right) =: \psi_\eta(\varepsilon).$$

Then observe that

$$\frac{\psi_\eta(\varepsilon)}{\sqrt{\varepsilon}} \sim_{\varepsilon\to0^+} \frac{\sqrt{\zeta(2)}}{1+\eta}\left(1 - \frac{2}{\zeta(2)}\eta(1+\eta)^2\right) \xrightarrow[\eta\to0^+]{} \sqrt{\zeta(2)}.$$

This easily entails the existence of a function $\widetilde{\alpha}(\cdot)$ as in the statement of Theorem 2.

Then we derive (1.4) from (1.5) by noticing that for all $p, p' \in [0, 1]$ such that $p < p'$, we have

$$D_n(p) \overset{\text{st}}{\le} D_n(p'),$$

which is an easy consequence of the equation (1.1): we leave the details to the reader.

Appendix A. Heuristic derivation of (2.3) from (2.1) using the scaling form (2.2)

For $p = \frac{1}{2} + \varepsilon$, we can rewrite (2.1) as

$$a_{n+1}(k) - a_n(k) = S_1 + \varepsilon S_2 + S_3,$$

where

$$S_1 := \frac{1}{2} \sum_{1 \le i < k} a_n(i)\big(a_n(k-i) - 2a_n(k)\big),$$

$$S_2 := -2a_n(k) + \sum_{1 \le i < k} a_n(i)\big(a_n(k-i) + 2a_n(k)\big) \quad \text{and} \quad S_3 := -(1-p)a_n(k)^2.$$

Using the scaling form (2.2)

$$a_n(k) = \frac{1}{k\sqrt{n}} f\Big(n,\ \frac{\log k}{\sqrt{n}}\Big)$$

with f regular and bounded, and the fact that

$$\frac{1}{i(k-i)} = \frac{1}{ki} + \frac{1}{k(k-i)},$$

we have

$$\sum_{1 \le i < k} a_n(i)a_n(k-i) = \frac{2}{kn} \sum_{1 \le i < k} \frac{1}{i} f\Big(n,\ \frac{\log i}{\sqrt{n}}\Big) f\Big(n,\ \frac{\log (k-i)}{\sqrt{n}}\Big),$$

and, thus,

$$S_1 = \frac{1}{kn} \sum_{1 \le i < k} \frac{f\Big(n,\ \frac{\log i}{\sqrt{n}}\Big)}{i} \left[f\left(n,\ \frac{\log (k-i)}{\sqrt{n}}\right) - f\left(n,\ \frac{\log k}{\sqrt{n}}\right)\right].$$

Writing $\log k = x\sqrt{n}$, we get that

$$\begin{aligned}
&f\left(n,\ \frac{\log i}{\sqrt{n}}\right)\left[f\left(n,\ \frac{\log (k-i)}{\sqrt{n}}\right) - f\left(n,\ \frac{\log k}{\sqrt{n}}\right)\right] \\
&\approx f\left(n,\ x + \frac{\log \frac{i}{k}}{\sqrt{n}}\right)\left[f\left(n,\ x + \frac{\log (1-\frac{i}{k})}{\sqrt{n}}\right) - f\big(n,\ x\big)\right] \\
&\approx f\big(n,\ x\big)\, \partial_x f\big(n,\ x\big)\, \frac{\log (1-\frac{i}{k})}{\sqrt{n}}.
\end{aligned}$$

By taking $i = ku$, we get that, for large n,

$$\begin{aligned}
S_1 &\approx \frac{1}{k\, n\sqrt{n}} f\big(n,\ x\big)\, \partial_x f\big(n,\ x\big)\, \frac{1}{k} \sum_{1 \le i < k} \frac{\log (1-\frac{i}{k})}{i/k} \\
&\approx \frac{1}{k\, n\sqrt{n}} f\big(n,\ x\big)\, \partial_x f\big(n,\ x\big) \int_0^1 \frac{\log (1-u)}{u}\, \mathrm{d}u.
\end{aligned}$$

Similarly, we can show that

$$S_2 \approx \frac{1}{k\sqrt{n}}\left[-2f\big(n,\ x\big) + 4f\big(n,\ x\big)\int_0^x f\big(n,\ y\big)\, \mathrm{d}y\right].$$

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On the other hand,

$$a_n(k)^2 = \frac{1}{n}\mathcal{O}\left(\frac{1}{k^2}\right),$$

and

$$a_{n+1}(k) - a_n(k) \approx \frac{1}{k\sqrt{n}}\partial_n f(n, x) - \frac{1}{2kn\sqrt{n}} f(n, x) - \frac{x}{2kn\sqrt{n}}\partial_x f(n, x).$$

By neglecting terms of order $\mathcal{O}(k^{-2})$ and of order $k^{-1}o\big(n^{-3/2}\big)$, we can rewrite (2.1) as

$$\begin{aligned}
&\frac{1}{k\,n\,\sqrt{n}}\left[n\partial_n f(n,\ x) - \frac{1}{2} f(n,\ x) - \frac{1}{2}\partial_x f(n,\ x)\right] \\
&= \frac{1}{k\,n\,\sqrt{n}} f\big(n,\ x\big)\ \partial_x f\big(n,\ x\big)\ \int_0^1 \frac{\log\left(1-u\right)}{u}\,\mathrm{d}u \\
&\quad + \frac{\varepsilon}{k\,\sqrt{n}}\left[-2f\big(n,\ x\big) + 4f\big(n,\ x\big)\int_0^x f\big(n,\ y\big)\,\mathrm{d}y\right],
\end{aligned}$$

and this leads to (2.3).

Appendix B. Proofs of Lemmas 4 and 8

Proof of Lemma 4

(i) Observe that $\varphi_q'(x) = (q/x)\sinh\left(q\log x\right)$ that is non-negative on $[1, \infty)$. Since $\varphi_q'(1) = 0 = \lim_{x\to\infty}\varphi_q'(x)$, there exists $x_q \in (1, \infty)$ such that $\varphi_q'(x_q) = \sup_{x\in[1,\infty)}\varphi_q'(x) =: M_q$. Then

$$\varphi_q''(x) = \frac{q}{x^2}\cosh\left(q\log x\right)\big(q - \tanh\left(q\log x\right)\big) = \frac{q}{x^2}\cosh\left(q\log x\right)\left(\frac{2}{x^{2q}+1} - (1-q)\right).$$

Thus, $x_q = \left(\frac{1+q}{1-q}\right)^{1/(2q)}$, and (i) follows immediately.

(ii) Note that φ_q'' is positive on $[1, x_q)$ and negative on $(x_q,\ \infty)$, which implies the existence of the inverse functions ℓ_q and r_q as in (ii). As $y \to 0^+$, $\ell_q(y) \to 1$ and $r_q(y) \to \infty$. Set $\lambda = \ell_q(y) - 1$ and observe that

$$y = \varphi_q'(1+\lambda) = \frac{q\big((1+\lambda)^q - (1+\lambda)^{-q}\big)}{2(1+\lambda)} = q^2\lambda\big(1 + \mathcal{O}_q(\lambda)\big),$$

which implies the estimate in (ii) for $\ell_q(y)$ as $y \to 0^+$. Similarly, observe that

$$y = \varphi_q'\big(r_q(y)\big) \sim_{y\to 0^+} \frac{q}{r_q(y)}\frac{1}{2}r_q(y)^q \sim_{y\to 0^+} \frac{q}{2}r_q(y)^{-(1-q)},$$

which implies the estimate in (ii) for $r_q(y)$ as $y \to 0^+$.

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(iii) Observe that $\ell_q(M_q)=r_q(M_q)=x_q$ and, thus, $\Phi_q(M_q)=0$. Note for all $y\in(0,\ M_q)$ that $\Phi'_q(y)=y\big(r'_q(y)-\ell'_q(y)\big)<0$. Since $\Phi_q(y)\sim\frac{1}{2}r_q(y)^q\sim\frac{1}{2}(2y/q)^{-\frac{q}{1-q}}$ as $y\to 0^+$, the function $\Phi_q:(0,\ M_q]\to\mathbb{R}_+$ is a C^1 decreasing bijection and the estimates for $\Phi_q^{-1}(x)$, $\ell_q(\Phi_q^{-1}(x))$, and $r_q(\Phi_q^{-1}(x))$ as $x\to\infty$ are immediate consequence of the previous equivalence and of (ii).

(iv) First note that

$$2g(x)=x^q\big((1+ax^{-1})^q-1\big)+x^{-q}\big((1+ax^{-1})^{-q}-1\big)\sim_{x\to\infty} qax^{-(1-q)}.$$

Thus, $g>0$ and $g(x)\to 0$ as $x\to\infty$. Suppose there exists $x^*\in[1,\ \infty)$ such that $g'(x^*)=0$. Then with $y^*:=\varphi'_q(x^*)$, we have $x^*=\ell_q(y^*)$ and $x^*+a=r_q(y^*)$. Let us check that $g'<0$ on $(x^*,\ \infty)$. Assume there exists $x'>x^*$ such that $g'(x')\geq 0$. Since

$$\frac{2}{q}g'(x)=x^{-(1-q)}\big((1+ax^{-1})^{-(1-q)}-1\big)-x^{-(1+q)}\big((1+ax^{-1})^{-(1+q)}-1\big)$$
$$\sim_{x\to\infty} -(1-q)ax^{-(2-q)},$$

which would imply that $g'(x)<0$ for all sufficiently large x. Therefore, there would exist $x''\in[x',\ \infty)$ such that $g'(x'')=0$. This would imply that $x''=\ell_q(y'')$ and $x''+a=r_q(y'')$, with $y'':=\varphi'_q(x'')$. But $y\mapsto r_q(y)-\ell_q(y)$ being increasing, we would have $y''=y^*$ and, thus, $x^*=x''$, which would be absurd. Consequently, $g'(x)<0$ for all $x\in(x^*,\ \infty)$, which proves (iv).

Proof of Lemma 8

Fix $q\in(0,\ q_0]$. (i) Let $a\in(0,\ \mathrm{e}^{-1})$ and $\theta\in[a,\ \mathrm{e}^{1/\sqrt{q}}]$. By definition, $g_q(\theta)=(\theta+1)^q\,[1-\mathrm{e}^{-q\log(\frac{1}{\theta}+1)}]$. Since $1-\mathrm{e}^{-x}\leq x$ (for $x\in\mathbb{R}_+^*$), this yields that $g_q(\theta)\leq(\theta+1)^q q\log(1/a+1)$. By observing that $\log(\frac{1}{a}+1)=\log\frac{1}{a}+\log(1+a)\leq\log(1/a)+1<2\log(1/a)$ (for $a\in(0,\ \mathrm{e}^{-1})$), we get that

$$g_q(\theta)\leq 2(\theta+1)^q q\log\frac{1}{a}.$$

On the other hand, let us write

$$g_q(\theta)=q\int_0^1\frac{\mathrm{d}u}{(\theta+u)^{1-q}}.$$

Since

$$\frac{1}{(\theta+u)^{1-q}}\leq\frac{(\theta+1)^q}{\theta+u}\leq\frac{(\theta+1)^q}{\theta}$$

for $u\in[0,\ 1]$, we get that

$$g_q(\theta)\leq(\theta+1)^q\,\frac{q}{\theta}\leq 2(\theta+1)^q\,\frac{q}{\theta}.$$

Therefore,

$$g_q(\theta)\leq 2(\theta+1)^q\,q\min\left\{\log\frac{1}{a},\ \frac{1}{\theta}\right\}=\frac{2(\theta+1)^q\,q}{\theta+1}\,(\theta+1)\min\left\{\log\frac{1}{a},\ \frac{1}{\theta}\right\}.$$

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By (4.25), $(\theta+1)^q \le (\mathrm{e}^{1/\sqrt{q}}+1)^q < \sqrt{2}$. We claim that $(\theta+1)\min\{\log(1/a),\ 1/\theta\} \le 2\log(1/a)$. This is obvious if $\theta < 1$ because in this case, $(\theta+1)\log(1/a) < 2\log(1/a)$; this is also obvious if $\theta \ge 1$, in which case $(\theta+1)(1/\theta) \le 2 < 2\log(1/a)$ (recalling that $a < \mathrm{e}^{-1}$). The inequality (4.29) is proved because $4\sqrt{2} < 6$.

(ii) Let $r \in [1,\ 2]$ and $\theta \in [q^3,\ \mathrm{e}^{1/\sqrt{q}}]$. Then

$$
\begin{aligned}
g_q\left(\frac{\theta}{r}\right) - g_q(\theta) &= q(1-q)\int_0^1 \mathrm{d}u \int_{r^{-1}\theta}^{\theta} \frac{\mathrm{d}t}{(t+u)^{2-q}} \\
&\ge q(1-q)(r^{-1}\theta)^q \int_0^1 \mathrm{d}u \int_{r^{-1}\theta}^{\theta} \frac{\mathrm{d}t}{(t+u)^2} \\
&= q(1-q)(r^{-1}\theta)^q \log\left(\frac{\theta+r}{\theta+1}\right).
\end{aligned}
$$

Since $(1-q)(r^{-1}\theta)^q \ge (1-q)(q^3/2)^q > 1-\sqrt{q}$ (by (4.27)). We next apply the inequality $\log(1+x) \ge x - x^2/2$, which holds true for all for $x \in [0,\ 1]$, to $x := (r-1)/(\theta+1)$ and we get

$$
g_q\left(\frac{\theta}{r}\right) - g_q(\theta) \ge q(1-\sqrt{q})\,\frac{r-1}{\theta+1} - q(1-\sqrt{q})\,\frac{(r-1)^2}{2(\theta+1)^2}.
$$

This yields (4.30) because

$$
(1-\sqrt{q})\frac{(r-1)^2}{2(\theta+1)^2} \le \frac{(r-1)^2}{\theta+1}.
$$

(iii) We set $h_q(\theta) := -g_q'(\theta) = q\big(\theta^{q-1} - (\theta+1)^{q-1}\big) = q(1-q)\int_\theta^{\theta+1} (\mathrm{d}w/w^{2-q})$. Then we get

$$
\begin{aligned}
\int_0^\theta h_q(t)\big(g_q(\theta-t) - g_q(\theta)\big)\mathrm{d}t &= q\int_0^\theta \mathrm{d}t\, h_q(t)\int_0^1 \mathrm{d}u\,\big((\theta+1-t-u)^{q-1} - (\theta+1-u)^{q-1}\big) \\
&= q(1-q)\int_0^\theta \mathrm{d}t\, h_q(t)\int_0^1 \mathrm{d}u\int_0^t \mathrm{d}v\,\frac{(\theta+1-v-u)^q}{(\theta+1-v-u)^2} \\
&\le q(\theta+1)^q\int_0^\theta \mathrm{d}t\, h_q(t)\int_0^1 \mathrm{d}u\int_0^t \frac{\mathrm{d}v}{(\theta+1-v-u)^2}.
\end{aligned}
$$

For $t \in (0,\ \theta]$, we have

$$
h_q(t) = q(1-q)\int_t^{t+1}\frac{w^q\mathrm{d}w}{w^2} \le q(\theta+1)^q\int_t^{t+1}\frac{\mathrm{d}w}{w^2} = \frac{q(\theta+1)^q}{t(t+1)}.
$$

This leads to

$$
\int_0^\theta h_q(t)\big(g_q(\theta-t) - g_q(\theta)\big)\,\mathrm{d}t \le q^2(\theta+1)^{2q}J(\theta),
$$

where

$$
J(\theta) := \int_0^\theta \frac{\mathrm{d}t}{t(t+1)}\int_0^t \mathrm{d}v\int_0^1 \frac{\mathrm{d}u}{(\theta+1-v-u)^2}.
$$

By (4.25), $(\theta+1)^{2q} \le (\mathrm{e}^{1/\sqrt{q}}+1)^{2q} \le 1+3\sqrt{q}$.

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It remains to check that $J(\theta) \le \zeta(2)/(\theta+1)$ for all $\theta \in \mathbb{R}_+^*$. By definition,

$$J(\theta) = \int_0^\theta \frac{1}{t(t+1)} \left(\log\left(1 - \frac{t}{\theta+1}\right) - \log\left(1 - \frac{t}{\theta}\right)\right) \mathrm{d}t = \int_0^\theta \frac{-\log\left(1 - \frac{\lambda^2 t}{(1-\lambda)(1-\lambda t)}\right)}{t(t+1)} \mathrm{d}t,$$

where we have set $\lambda := 1/(\theta+1) \in (0,\ 1]$. By means of the change of variables

$$v = 1 - \frac{\lambda^2 t}{(1-\lambda)(1-\lambda t)},$$

we get that

$$J(\theta) = \int_0^1 \frac{\log 1/v}{1-v} \frac{\lambda^2}{1-(1-\lambda^2)v} \mathrm{d}v.$$

Since

$$\int_0^1 \frac{\log 1/v}{1-v} \mathrm{d}v = \sum_{n\ge 0} \int_0^1 \mathrm{d}v\, v^n \log 1/v = \sum_{n\ge 0} (n+1)^{-2} = \zeta(2),$$

this implies that

$$\begin{aligned}\lambda\,\zeta(2) - J(\theta) &= \int_0^1 \frac{\log \frac{1}{v}}{1-v} \left(\lambda - \frac{\lambda^2}{1-(1-\lambda^2)v}\right) \mathrm{d}v \\ &= (1-\lambda^2) \int_0^1 \frac{\log \frac{1}{v}}{1-(1-\lambda^2)v} \mathrm{d}v \ -\ (1-\lambda) \int_0^1 \frac{\log \frac{1}{v}}{1-v} \mathrm{d}v.\end{aligned}$$

For all $r \in (0,\ 1)$,

$$\int_0^1 \frac{\log \frac{1}{v}}{1-rv} \mathrm{d}v = \sum_{n\ge 0} r^n \int_0^1 v^n \log \frac{1}{v} \mathrm{d}v = \frac{1}{r} \sum_{n\ge 0} \frac{r^{n+1}}{(n+1)^2} = \frac{1}{r} \int_0^r \frac{\log \frac{1}{1-v}}{v} \mathrm{d}v.$$

Therefore, $\lambda\,\zeta(2) - J(\theta) = K(\lambda)$, where

$$K(x) := \int_0^{1-x^2} \frac{\log \frac{1}{1-v}}{v} \mathrm{d}v - (1-x) \int_0^1 \frac{\log \frac{1}{v}}{1-v} \mathrm{d}v, \quad x \in [0,\ 1].$$

We want to prove that $K(x) \ge 0$ for all $x \in (0,\ 1]$. Since $K(0) = K(1) = 0$, it suffices to show that K is concave:

$$K''(x) = 4\,\frac{1+x^2}{1-x^2} \left(\frac{1-x^2}{1+x^2} + \log x\right) = 4\,\frac{1+x^2}{1-x^2} (\tanh(y) - y) \le 0,$$

where $y := -\log x \in \mathbb{R}_+$. This proves that

$$\frac{\zeta(2)}{\theta+1} - J(\theta) \ge 0$$

for all $\theta \in \mathbb{R}_+$, which yields (4.31).

Acknowledgements

We wish to thank an anonymous referee for their comments that helped improve the first version of the article.

Funding information

This work was partially supported by NSFC grant No. 12271351.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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