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KAZHDAN CONSTANTS FOR COMPACT GROUPS

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Abstract

It is shown that for the computation of the Kazhdan constant for a compact group only the regular representation restricted to the orthogonal complement of the constant functions needs to be taken into account.

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Kazhdan constants are a quantitative version of property T, which was introduced by Kazhdan [8] in 1967. This property is representation theoretic with remarkable applications, see [7] for an account. The related constants yield a sort of distance between the trivial representation and those not containing it. The question of calculating Kazhdan constants appears as a natural question in [7, page 133]. Explicit Kazhdan constants can be useful, for example, in connection with expanding graphs [9], random walks [12], or the product replacement algorithm [10].

Although it is an easy observation that a compact group has property T, the computation of Kazhdan constants is nevertheless not trivial even for this class of groups, compare with, for example, [1–4, 11]. The purpose of the theorem in this note is to facilitate in some sense further computations of Kazhdan constants for compact groups.

Let G be a locally compact group. For a subset Q of G and a strongly continuous unitary representation π of G on the representation space H_π, let

$$\kappa_G(Q,\pi) = \inf_{\xi\in S_\pi} \sup_{g\in Q} \|\pi(g)\xi - \xi\|,$$

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where $S_\pi = \{\xi \in H_\pi : \|\xi\| = 1\}$ is the unit sphere in H_π. The *Kazhdan constant* is defined by $\kappa_G(Q) = \inf_{\pi \in r(G)} \kappa_G(Q, \pi)$, where $r(G)$ is the set of all equivalence classes of representations of G on separable Hilbert spaces not containing the trivial representation. Another constant depending only on the irreducible representations can be defined by $\hat{\kappa}_G(Q) = \inf_{\pi \in \widehat{G}\setminus\{1\}} \kappa_G(Q, \pi)$, where $\widehat{G}$ denotes the set of equivalence classes of irreducible representations of G.

Note that if σ is a subrepresentation of π then $\kappa_G(Q, \pi) \le \kappa_G(Q, \sigma)$. Let $m \in \mathbb{N} \cup \{\infty\}$, and denote by $m\pi$ the m-fold direct sum of the representation π on H_π^m. Then in general only $\kappa_G(Q, m\pi) \le \kappa_G(Q, \pi)$, but equality need not hold necessarily. An explicit example where equality does not hold is given in [11]. There, $G = \mathrm{SU}(2)$, Q is any conjugacy class of a non-central element and π_2 is the unique (up to equivalence) irreducible representation of degree 3. In this case $\kappa_G(Q, \pi_2) > \kappa_G(Q, 3\pi_2)$.

Let now G be compact and denote by $L_0^2(G)$ the orthogonal complement of the constant functions in $L^2(G)$ where the compact group G is naturally equipped with the unique normalised Haar measure. Let ρ be the regular representation of G restricted to $L_0^2(G)$. Obviously $\kappa_G(Q) \le \kappa_G(Q, \rho) \le \hat{\kappa}_G(Q)$ holds in general. An easy consequence of the Peter-Weil theorem, see, for example, [6, page 133], is $\kappa_G(Q) = \kappa_G(Q, \infty\rho)$. The following result, which will be proven below, states in fact that ∞ can be omitted.

THEOREM. *Let ρ be the regular representation of the compact group G restricted to $L_0^2(G)$ and Q a subset of G. Then $\kappa_G(Q) = \kappa_G(Q, \rho)$.*

For special cases of G and Q, this appears in [1–4, 11].

To be more precise, [2] states that the result holds in the special case of the dihedral group $G = D_n = \langle a, b : a^2, b^2, (ab)^n \rangle$ and $Q = \{a, b\}$, as well as for any abelian compact group G with a compact generating set Q. In the first case also $\kappa_G(Q) = \hat{\kappa}_G(Q)$. The result for abelian G appears likewise in [3, page 463]. For the instance, where G is the cyclic group of order n and $Q = G$, the result can be found again in [1] and furthermore for G the symmetric group and $Q = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$. Note also that in the first case $\kappa_G(Q) < \hat{\kappa}_G(Q)$ if $n \ge 4$ while in the second $\kappa_G(Q) = \hat{\kappa}_G(Q)$. Moreover, in the latter case the Kazhdan constant is equal to $\kappa_G(Q, \pi)$ where π is the irreducible representation corresponding to the natural action of S_n on $\mathbf{C}^n$, that is, the representation corresponding to the partition $(n-1, 1)$. It is observed in [1, page 496] that $\kappa_G(Q) = \hat{\kappa}_G(Q)$ or $\kappa_G(Q) < \hat{\kappa}_G(Q)$ really depends not only on G but also on Q. For any compact group G with $Q = G$ the theorem is included in [4, page 309]. It is contained in [11] for any compact group G and Q a conjugacy class.

For the proof of the theorem note that by definition $\kappa_G(Q) \le \kappa_G(Q, \rho)$. Hence it suffices to demonstrate that $\kappa_G(Q, \rho) \le \kappa_G(Q, \pi)$ for any $\pi \in r(G)$. As noted before, by the Peter-Weil theorem a restriction to the case $\pi = \infty\rho$ would be possible.

However, this would not significantly simplify the proof presented below.

PROOF. Let π be a representation of the compact group G not containing the trivial representation and $\xi \in H_\pi$. Then the function $g \mapsto \langle \pi(g)\xi, \xi \rangle$ is continuous, and thus square-integrable, as G is compact. By [5, page 309] and the fact that π does not contain the trivial representation, there exists an $f \in L_0^2(G)$ such that $\langle \pi(g)\xi, \xi \rangle = \langle \rho(g)f, f \rangle$ for all $g \in G$. Then $\|\xi\| = \|f\|$ can be read off for $g = 1$. Hence

$$\begin{aligned}
\|\pi(g)\xi - \xi\|^2 &= 2\|\xi\|^2 - 2\,\mathrm{Re}\langle \pi(g)\xi, \xi \rangle \\
&= 2\|f\|^2 - 2\,\mathrm{Re}\langle \rho(g)f, f \rangle \\
&= \|\rho(g)f - f\|^2.
\end{aligned}$$

Thus

$$\kappa_G(Q, \pi) = \inf_{\xi \in S_\pi} \sup_{g \in Q} \|\pi(g)\xi - \xi\| \geq \inf_{f \in S_\rho} \sup_{g \in Q} \|\rho(g)f - f\| = \kappa_G(Q, \rho),$$

and this proves the theorem. □

Finally note that in general the statement of the theorem does not hold for non-compact locally compact groups as for example a compactly generated group which is not amenable and does not have property T with a compact generating set Q satisfies $\kappa_G(Q) = 0 < \kappa_G(Q, \rho)$. A specific example would be the free group on two generators. Here, of course, ρ is just the regular representation as $L_0^2(G) = L^2(G)$ because there are no non-zero constant functions.

A remark pointed out by A. Żuk is that the theorem also holds for non-compact amenable groups G and compact subsets Q, since both constants are then 0. Even more generally, this holds for any subset Q of G, since for an amenable group G any representation of G is weakly contained in the regular representation see, for example, [5, page 358] which implies $\kappa_G(Q, \rho) \leq \kappa_G(Q)$.

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References

[1] R. Bacher and P. de la Harpe, 'Exact values of Kazhdan constants for some finite groups', *J. Algebra* **163** (1994), 495–515.

[2] A. Deutsch, *Kazhdan's property (T) and related properties of locally compact and discrete groups* (Ph.D. Thesis, University of Edinburgh, 1992).
[3] ———, 'Kazhdan constants for the circle', *Bull. London Math. Soc.* **26** (1994), 459–464.
[4] A. Deutsch and A. Valette, 'On diameters of orbits of compact groups in unitary representations', *J. Austral. Math. Soc. (Series A)* **59** (1995), 308–312.
[5] J. Dixmier, *C^*-algebras*, North-Holland Math. Library 15 (North-Holland, Amsterdam, 1982).
[6] G. B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1995).
[7] P. de la Harpe and A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, Astérisque 175 (Société Mathématique de France, 1989).
[8] D. A. Kazhdan, 'Connection of the dual space of a group with the structure of its close subgroups', *Funct. Anal. Appl.* **1** (1967), 63–65.
[9] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics 125 (Birkhäuser, Basel, 1994).
[10] A. Lubotzky and I. Pak, 'The product replacement algorithm and Kazhdan's property (T)', *J. Amer. Math. Soc.* **14** (2000), 347–363.
[11] M. Neuhauser, 'Kazhdan constants for conjugacy classes of compact groups', *J. Algebra* **270** (2003), 564–582.
[12] I. Pak and A. Żuk, 'On Kazhdan constants and mixing of random walks', *Int. Math. Res. Not.* **36** (2002), 1891–1905.

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