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PAPER

Pattern formation with jump discontinuity in a predator–prey model with Holling-II functional response

Gaihui Guo[1], Xiaoyi Yang[1], Conghui Zhang[2] and Shanbing Li[3]

[1]School of Mathematics and Data Science, Shaanxi University of Science and Technology, Xi'an, China
[2]School of Science, Beijing University of Civil Engineering and Architecture, Beijing, China
[3]School of Mathematics and Statistics, Xidian University, Xian, China
Corresponding author: Gaihui Guo; Email: guogaihui@sust.edu.cn

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Abstract
This paper is focused on the existence and uniqueness of nonconstant steady states in a reaction–diffusion–ODE system, which models the predator–prey interaction with Holling-II functional response. Firstly, we aim to study the occurrence of regular stationary solutions through the application of bifurcation theory. Subsequently, by a generalized mountain pass lemma, we successfully demonstrate the existence of steady states with jump discontinuity. Furthermore, the structure of stationary solutions within a one-dimensional domain is investigated and a variety of steady-state solutions are built, which may exhibit monotonicity or symmetry. In the end, we create heterogeneous equilibrium states close to a constant equilibrium state using bifurcation theory and examine their stability.

1. Introduction

In the seminal paper [26], Turing proposed the concept of diffusion-driven instability (DDI), which may explain the spontaneous formation of the pattern in developmental biology. Here, DDI is the spatial homogeneous instability caused by the interaction of two chemical substances with different diffusion rates. Since then, the Turing notion has become a paradigm for the pattern generation and inspired the emergence of various theoretical models, but its biological verification has remained elusive [1, 10].

However, not all patterns are formed as a result of DDI. Some models incorporate a combination of a reaction-diffusion equation and an ordinary differential equation. A common example of migration involves macroalgae and herbivores, particularly since macroalgae are stationary and exist solely within the environment inhabited by herbivore species. Furthermore, the pattern occurs not just in classical reaction–diffusion systems in which all species diffuse [11, 12, 14, 24] but also in degenerate systems in which certain species do not diffuse. The latter systems were modelled by reaction–diffusion–ODE systems, see [2, 17, 25]. A model consisting of free receptors, bound receptors and ligands was proposed by Sherrat et al. [25], which described the coupling of cell-localized processes with cell to cell communication via diffusion in a cell assembly. Free and bound receptors are located on the surface of the cell and therefore do not diffuse. Ligand diffuses and acts by binding itself to receptors, thereby triggering an intracellular response that leads to cell differentiation. Their model has a built-in spatial heterogeneity that triggers patterning. Marciniak–Czochra [19, 20] later extended their model and demonstrated that nonlinear interactions of hysteresis type can result in the spontaneous emergence of the pattern, without the need for spatial heterogeneity. For a detailed mathematical analysis of their work, please refer to [15, 16, 18, 21].

In order to analyse the contribution of non-diffusive components in the pattern development procedure, we concentrate on the following system

$$\begin{cases} u_t = r\left(1-\dfrac{u}{K}\right)u - \dfrac{cuv}{m+bu}, & t>0,\ x\in\overline{\Omega}, \\ v_t = d_2\Delta v - av + \dfrac{\beta cuv}{m+bu}, & t>0,\ x\in\Omega, \\ \partial_\tau v = 0, & t>0,\ x\in\partial\Omega, \\ u(x,0)=u_0(x)\ge 0, \not\equiv 0 \quad v(x,0)=v_0(x)\ge 0, \not\equiv 0 & x\in\Omega, \end{cases} \tag{1.1}$$

where u and v represent the population density of the prey and predator, respectively; d_2 represents the predator diffusion rates; Ω is a bounded domain in the Euclidean space R^N with smooth boundary, denoted as $\partial\Omega$; Δ is the Laplace operator in R^N; τ is the unit outer normal vector on $\partial\Omega$. The parameters r, m, c, b, a, β, K are positive constants.

The case is interesting because a scalar reaction–diffusion equation typically cannot produce stable spatially heterogeneous patterns [5]. While it is true that problem (1.1) does not exhibit stable Turing-type patterns, it is worth noting that DDI (Diffusion-Driven Instability) can still occur by selecting suitable parameters. Interestingly, under certain conditions, the stationary problem associated with equation (1.1) can be simplified into a boundary value problem for a single reaction–diffusion equation featuring a discontinuous nonlinearity, which leads to the emergence of positive solutions with jump discontinuity. There is a lot of work on such issues, such as [6, 27, 32]. Hence, the focus of this paper is studying the stationary problem associated with equation (1.1).

$$\begin{cases} r\left(1-\dfrac{u}{K}\right)u - \dfrac{cuv}{m+bu} = 0, \ x\in\overline{\Omega}, \\ d_2\Delta v - av + \dfrac{\beta cuv}{m+bu} = 0, \quad x\in\Omega, \\ \partial_\tau v = 0, \quad x\in\partial\Omega. \end{cases} \tag{1.2}$$

For convenience, we let

$$f_1(u,v) = r\left(1-\frac{u}{K}\right)u - \frac{cuv}{m+bu}, \qquad f_2(u,v) = -av + \frac{\beta cuv}{m+bu}.$$

The main findings of our current work can be summarized as follows. To begin, we carefully select suitable coefficients a, c, m, b and β in order to ensure that the kinetic system (without considering diffusion) of problem (1.1) possesses only one positive equilibrium (u_2^*, v_2^*) located on the right branch, as depicted in Figure 1(b). Then, by a variational approach to bifurcation methods, we show the existence of regular stationary solutions of problem (1.2). Next, by transforming the problem into a boundary value problem for a single equation involving $v(x)$, we establish the existence of a discontinuous solution $(u(x), v(x))$ for problem (1.2) using the generalized mountain pass lemma (Theorem 4.1). This solution is characterized by a jump discontinuity in $u(x)$ and $\Delta v(x)$. The innovation of our current research is from the presence of a discontinuous nonlinearity in the reduced problem for $v(x)$, which results in invalidating the general mountain pass lemma introduced by Ambrosetti and Rabinowitz [3]. Fortunately, Chang [7] expanded the existing theory to handle problems involving partial differential equations that contain discontinuous nonlinearities. It appears that this approach is suitable for our specific issue, allowing us to solve challenges we faced.

We analyse problem (1.2) within the one-dimensional domain [0, 1] to know the structure of pattern formation. Under certain conditions on the coefficients, the equation $f_1(u,v)=0$, where $u\ge 0$, has three distinct branches. These branches can be represented as $u=h_0(v) :\equiv 0$, $u=h_1(v)$, and $u=h_2(v)$, with the feature that $h_0(v)<h_1(v)<h_2(v)$ (see Figure 1). To begin, we select a non-negative constant $\gamma\in(0, v_2^*)$ and utilize the functions $u=h_0(v)$ and $u=h_2(v)$ in the following manner: $u=h_0(v)$ for $v<\gamma$ and $u=h_2(v)$ for $v>\gamma$. Subsequently, the equation (1.2) is transformed into a boundary value problem for $v(x)$, which has discontinuous nonlinearity.

Next, by considering all values of the diffusion coefficient d_2, we are able to construct monotone solutions for this particular equation, and they are then used to construct symmetric solutions through

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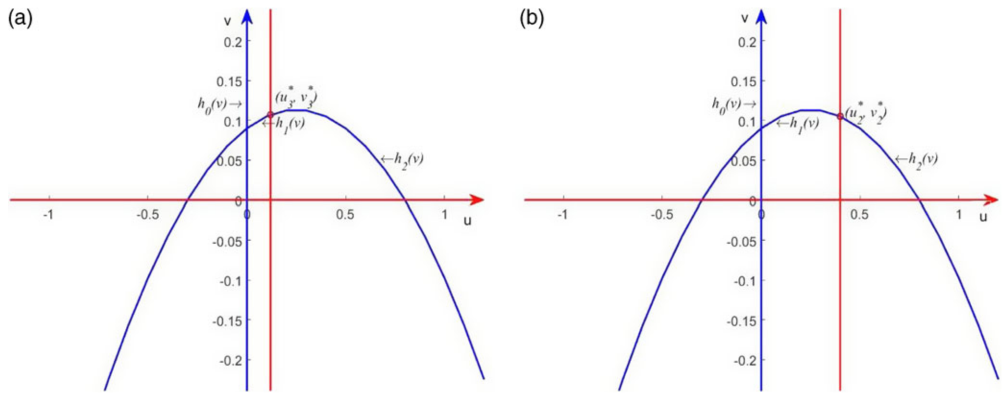

Figure 1. *Nullclines for $f_1(u, v) = f_2(u, v) = 0$. The blue curve represents the solution of $f_1(u, v) = 0$, while the red curve represents the solution of $f_2(u, v) = 0$. In (a), we select $a = 0.4$, $b = 1$, $m = 0.3$, $K = 0.8$, $c = 1$, $\beta = 1.4$, $r = 0.3$. In (b), we select $a = 0.4$, $b = 1$, $m = 0.3$, $K = 0.8$, $c = 1$, $\beta = 0.6$, $r = 0.3$.*

the process of reflecting the monotone solutions, as described in Theorem 5.1. In order to demonstrate Theorem 5.1, we employ the shooting way, which was used in the research of Takagi and Zhang [28]. Furthermore, by selecting a smaller interval for β within the range of $(0, v_2^*)$, we can establish the uniqueness of solutions for any given d_2. This is accomplished by employing another form of shooting method [22], as demonstrated in Theorems 5.2 and 5.3. Moreover, the mode of $(u(x), v(x))$ refers to the number of points at which $v''(x)$ is discontinuous. Specifically, an n-mode solution $v_n(x)$ $(n \geq 2)$ implies that there are exactly n points of discontinuity in $v_n''(x)$. Notably, a one-mode solution $v_1(x)$ shows that $v_1(x)$ is either monotone increasing or monotone decreasing on $[0, 1]$.

Finally, with the aid of bifurcation theory [8], we create nonconstant continuous stable states close to (u_3^*, v_3^*) within the one-dimensional space domain of [0,1] and investigate their instability.

The paper is divided into five sections: In Section 2, we present preliminary results on nonlinear functions f_1 and f_2 that will be utilized in the subsequent sections of this paper. In Section 3, we construct regular stationary solutions of problem (1.2) utilizing the bifurcation theory. In Section 4, we prove the existence of discontinuous stationary solutions of problem (1.2). In Section 5, we not only construct steady states with jump discontinuities but also explore various types of these states, and these steady states can exhibit monotonic or symmetric behaviour. Additionally, we verify the uniqueness of these steady states under certain additional conditions. In Section 6, the investigation focuses on the stability of stationary solutions.

2. Preliminaries

We shall discuss certain properties of the functions f_1 and f_2 that will be applied in this paper.

Proposition 2.1. *If $Kb > m$ and $u \geq 0$ hold, then $f_1(u, v) = 0$ has three distinct branches: $u = h_0(v) = 0$ for $v \in (-\infty, +\infty)$, $u = h_1(v)$ for $v \in (rm/c, v_M)$ and $u = h_2(v)$ for $v \in (-\infty, v_M)$, where*

$$u_M = \frac{Kb - m}{2b}, \quad v_M = \frac{r\left(1 - \dfrac{u_M}{K}\right)(m + bu_M)}{c} > 0.$$

Proof. If $f_1(u, v) = 0$, then

$$u = 0 \quad \text{or} \quad r\left(1 - \frac{u}{K}\right) - \frac{cv}{m + bu} = 0.$$

It is easy to obtain that $v = r\left(1 - \dfrac{u}{K}\right)(m + bu)/c$ has a maximum point (u_M, v_M), where

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$$u_M = \frac{Kb - m}{2b}, \quad v_M = \frac{r\left(1 - \frac{u_M}{K}\right)(m + bu_M)}{c}. \tag{2.1}$$

When $Kb > m$, then $u_M > 0$, which shows that

$$v = p(u) = r\left(1 - \frac{u}{K}\right)(m + bu)/c \tag{2.2}$$

is monotone increasing in $(-\infty, u_M)$, while monotone decreasing in $(u_M, +\infty)$. As a result, for $v \in (p(0), v_M) = (rm/c, v_M)$, $u = h_1(v)$ is monotone increasing with respect to v, and for $v \in (-\infty, v_M)$, $u = h_2(v)$ is monotone decreasing with respect to v. Direct calculations provide $u_M < K$. Based on the expression of v_M in (2.1), we easily deduce that $v_M > 0$. □

Proposition 2.2. *Assume that $Kb > m$ and $\beta c > ab$ hold. Then,*

(i) (u_2^*, v_2^*) is a positive solution of $f_1(u, v) = f_2(u, v) = 0$ for $u_M < am/(\beta c - ab) < K$ and it is on the branch $u = h_2(v)$, where $u_2^* = am/(\beta c - ab)$ and $v_2^* = r\left(1 - \frac{u_2^*}{K}\right)(m + bu_2^*)/c$;

(ii) (u_3^*, v_3^*) is a positive solution of $f_1(u, v) = f_2(u, v) = 0$ for $0 < am/(\beta c - ab) < u_M$ and it is on the branch $u = h_1(v)$, where $u_3^* = am/(\beta c - ab)$ and $v_3^* = r\left(1 - \frac{u_3^*}{K}\right)(m + bu_3^*)/c$.

Proof. We omit the details because the proof is elementary. □

Proposition 2.3. *Assume that the Proposition 2.2 (i) holds, we have the following results.*

(i) $f_2(h_0(v), v) < 0$ for $v \in (0, +\infty)$ and $f_2(h_2(v), v) > 0$ for $v \in (0, v_2^*)$.

(ii) $\frac{d}{dv} f_2(h_0(v), v) < 0$ for $v \in (-\infty, +\infty)$ and there is a constant $\widetilde{d} \in (0, v_2^*)$ that ensures $\frac{d}{dv} f_2(h_2(v), v) < 0$ for $(\widetilde{d}, v_2^*]$.

Proof. (i) Obviously,

$$f_2(h_0(v), v) = -av < 0 \quad \text{for} \quad v \in (0, +\infty).$$

Then, we find that $u = h_2(v) > u_2^*$ for $v \in (-\infty, v_2^*)$ and $\beta cu/(m + bu)$ is monotone increasing with respect to u. So

$$\frac{\beta ch_2(v)}{m + bh_2(v)} - a > \frac{\beta cu_2^*}{m + bu_2^*} - a = 0.$$

This shows that $f_2(h_2(v), v) > 0$ for $v \in (0, v_2^*)$.

(ii) By direct calculations, we get

$$\frac{d}{dv} f_2(h_0(v), v) = -a < 0 \quad \text{for} \quad v \in (-\infty, +\infty),$$

$$\frac{d}{dv} f_2(h_2(v), v) = \left(\frac{\beta ch_2(v)}{m + bh_2(v)} - a\right) + v\frac{\beta cmh_R'(v)}{(m + bh_2(v))^2}.$$

Recall that $h_2'(v) < 0$ for $v \in (-\infty, v_M)$ from the Proposition 2.1 proof. Combining this with $-a + \beta ch_2(v_2^*)/(m + bh_2(v_2^*)) = 0$, then we get $\frac{d}{dv} f_2(h_2(v), v)|_{v=v_2^*} < 0$. By continuity, there exists a constant $\widetilde{d} \in (0, v_2^*)$ such that $\frac{d}{dv} f_2(h_2(v), v) < 0$ for $(\widetilde{d}, v_2^*]$. □

3. Existence of regular stationary solutions

In this section, we mainly show the existence of regular stationary solutions of problem (1.2). Firstly, we review the results of [4] and express them in a form that [4] is already suitable to deal with a system consisting of PDEs and ODEs. Thus, we deal with a solution $(u, v) = (u(x), v(x))$ to the boundary value problem

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$$\begin{cases} f(u,v)=0, \ x\in\overline{\Omega}, \\ d_2\Delta_\tau v+g(u,v)=0, \ x\in\Omega, \end{cases} \tag{3.1}$$

with arbitrary C^2-functions f and g, with constant $d_2>0$, and in an open bounded domain $\Omega\subseteq R^N$ with a C^2-boundary. Δ_τ represents the Laplacian operator and Neumann boundary conditions.

Definition 1. *([4]) A solution $(u,v)=(u(x),v(x))$ of problem (3.1) is called weak if*

(i) u is measurable,

(ii) $v\in W^{1,2}(\Omega)$,

(iii) $g(u,v)\in(W^{1,2}(\Omega))^*$ (the dual of the space $W^{1,2}(\Omega)$),

(iv) the equation $f(u(x),v(x))=0$ is satisfied for almost all $x\in\Omega$,

(v) the equality

$$-d_2\int_\Omega \nabla v(x)\cdot\nabla\zeta(x)dx+\int_\Omega g(u(x),v(x))\zeta(x)dx=0$$

holds for all test functions $\zeta\in W^{1,2}(\Omega)$.

Definition 2. *([4]) The weak solution of problem (3.1) in the sense of Definition 1 is called a regular solution, if there is a C^2-function $\theta:R\to R$ such that $u(x)=\theta(v(x))$ for all $x\in\Omega$.*

Remark 1. It is easy to find that every regular solution of problem (3.1) satisfies

$$f(u(x),v(x))=f(\theta(v(x)),v(x))=0 \quad \text{for all} \quad x\in\Omega,$$

where $v=v(x)$ is a solution of the elliptic Neumann problem

$$d_2\Delta_\tau v+P(v)=0 \quad \text{for} \quad x\in\Omega \tag{3.2}$$

with $P(v)=g(\theta(v),v)$.

Proposition 3.1. *Assume that $N\le 6$. Let $y\in C_b^2(R)$ satisfy $y(0)=y'(0)=0$. There is a sequence of numbers $d_s\to d_2$ and a sequence of non-constant functions $v_s\in W^{1,2}(\Omega)$ such that $||v_s||_{W^{1,2}}\to 0$ and which satisfies the boundary value problem*

$$d_s\Delta_\tau v_s+(\lambda_k+d_2-d_s)v_s+y(v_s)=0 \quad \textit{for} \quad x\in\Omega. \tag{3.3}$$

Proof. We prove this lemma using the Rabinowitz bifurcation theorem of the variational equation [23]. Then, assume that

(i) M is a real Hilbert space,

(ii) $X\in C^2(M,R)$ with $X'(u)=Lu+Z(u)$,

(iii) L is linear and $Z(u)=o(||u||)$ at $u=0$,

(iv) λ is an isolated eigenvalue of L of a finite multiplicity. If these assumptions hold, by [23], we know that $(\lambda,0)\in R\times M$ is a bifurcation point of

$$A(\mu,v)\equiv Lv+Z(v)-\mu v=0. \tag{3.4}$$

Thus, for $||v||\neq 0$, the solution (μ,v) of the equation (3.4) is present in each neighbourhood of $(\lambda,0)$. And we apply the usual Sobolev space $M=W^{1,2}(\Omega)$ with the equivalent scalar product

$$\langle u,v\rangle_{W^{1,2}(\Omega)}=\int_\Omega \nabla u\cdot\nabla v dx+\int_\Omega uvdx$$

and the functional

$$X(v)=\frac{\lambda_k+d_2}{2}\int_\Omega v^2dx+\int_\Omega S(v)dx$$

with $S(v)=\int_0^v y(s)ds$. It is easy to find that $X\in C(W^{1,2}(\Omega),R)$ and $X(v)$ is differentiable in the *Fréchet* sense for each $v\in W^{1,2}(\Omega)$. By simple calculation, we have

$$DX(v)\zeta=(\lambda_k+d_2)\int_\Omega v\zeta dx+\int_\Omega y(v)\zeta dx$$

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with $DX \in C(W^{1,2}(\Omega), \text{Lin}(W^{1,2}(\Omega), R))$. The second *Fréchet* derivative at the point $v \in W^{1,2}(\Omega)$ is represented by the bilinear form

$$\langle D^2X(v)\zeta, \kappa \rangle = (\lambda_k + d_2) \int_\Omega \zeta\kappa\, dx + \int_\Omega y'(v)\zeta\kappa\, dx.$$

Next, we prove that $D^2X(v) \in C\left(W^{1,2}(\Omega), \text{Lin}(W^{1,2}(\Omega), \text{Lin}(W^{1,2}(\Omega), R))\right)$. For $v_n \to v$ in $W^{1,2}(\Omega)$ and $\zeta, \kappa \in W^{1,2}(\Omega)$, we estimate

$$\begin{aligned} |\langle (D^2X(v_n) - D^2X(v))\zeta, \kappa \rangle| &\le \int_\Omega |y'(v_n) - y'(v)||\zeta||\kappa|dx \\ &\le ||y''||_\infty \int_\Omega |v_n - v||\zeta||\kappa|dx \\ &\le ||y''||_\infty ||v_n - v||_3 ||\zeta||_3 ||\kappa||_3 \\ &\le ||y''||_\infty ||v_n - v||_{W^{1,2}} ||\zeta||_{W^{1,2}} ||\kappa||_{W^{1,2}}. \end{aligned}$$

The last inequality comes from the Sobolev embedding assuming $N \le 6$.

Particularly, we have for each test function $\zeta \in W^{1,2}(\Omega)$

$$X'(v)(\zeta) = (\lambda_k + d_2) \int_\Omega v\zeta\, dx + \int_\Omega y(v)\zeta\, dx \equiv Lv(\zeta) + H(v)(\zeta).$$

Therefore, we can find that $H(v) = o(||v||_{W^{1,2}})$ as $||v||_{W^{1,2}} \to 0$ by assuming $y = y(v)$. It is easy to find that $\mu = 1$ is an isolated eigenvalue of the operator L with finite multiplicity. So, we obtain

$$Lv(\zeta) = \langle v, \zeta \rangle_{W^{1,2}(\Omega)} \quad \text{for all} \quad \zeta \in W^{1,2}(\Omega),$$

that is

$$(\lambda_k + d_2) \int_\Omega v\zeta\, dx = \int_\Omega \nabla v \cdot \nabla\zeta\, dx + \int_\Omega v\zeta\, dx \quad \text{for all} \quad \zeta \in W^{1,2}(\Omega).$$

Obviously, we can reduce to the eigenvalue problem for Δ_τ. Now, the property that λ_k is an isolated eigenvalue with finite multiplicity is applied. So, by the Rabinowitz Theorem [23], we can find that $(1, 0)$ is a bifurcation point of (3.4) which means that there is a sequence of numbers $d_s \to d_2$ and nonzero $\{v_s\} \subset W^{1,2}(\Omega)$ such that $||v_s||_{W^{1,2}} \to 0$, satisfying

$$Lv_s(\zeta) + Z(v_s)(\zeta) - d_s \langle v_s, \zeta \rangle_{1,2} = 0 \quad \text{for all} \quad \zeta \in W^{1,2}(\Omega),$$

which is equivalent to the equation satisfied by the weak solutions $v_s \in W^{1,2}(\Omega)$ to problem (3.3)

$$-d_s \int_\Omega \nabla v_s \cdot \nabla\zeta\, dx + (\lambda_k + d_2 - d_s) \int_\Omega v_s\zeta\, dx + \int_\Omega y(v_s)\zeta\, dx = 0$$

for all $\zeta \in W^{1,2}(\Omega)$. □

Proposition 3.2. *Assume that $N \le 6$. Suppose that $(\bar{u}, \bar{v}) \in R^2$ is a constant solution of problem (1.2) such that $f(\bar{u}, \bar{v}) = 0$ and $g(\bar{u}, \bar{v}) = 0$. We use the following notation*

$$a_0 = f_u(\bar{u}, \bar{v}), \quad b_0 = f_v(\bar{u}, \bar{v}), \quad c_0 = g_u(\bar{u}, \bar{v}), \quad d_0 = g_v(\bar{u}, \bar{v}) \tag{3.5}$$

and assume that

$$a_0 \ne 0 \quad \text{and} \quad \frac{1}{a_0} det\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = d_2\lambda_k > 0, \tag{3.6}$$

for some λ_k eigenvalues of $-\Delta_\tau$. Then, there is a sequence of real numbers $d_s \to d_2$ such that the following perturbed problem

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$$\begin{cases} f(u,v)=0, & x\in\overline{\Omega}, \\ d_s\Delta_\tau v+(d_2-d_s)(v-\overline{v})+g(u,v)=0, & x\in\Omega, \end{cases} \tag{3.7}$$

has a non-constant regular solution.

Proof. We construct non-constant solutions to the reaction–diffusion–ODE system by Proposition 3.1. In the following, we define an open ball with a radius of $\rho>0$ that is centred at $\overline{v}$ as $B_\rho(\overline{v})$. First, we show that only a finite number of v_s can be constant in Proposition 3.1. If there is a constant subsequence $\{v_{l_n}\}$ that satisfies the equation (3.3) such that $v_{l_n}\to 0$, then we know that $y'(0)=-\lambda_k$, which is obviously a contradiction.

Next, since $\det f_u(\overline{u},\overline{v})\neq 0$, for all $V\in(B_\rho(\overline{v}))$, we obtain that there is a $\rho>0$ and a function $\theta\in C^2(B_\rho(\overline{v}))$ such that $\theta(\overline{v})=\overline{u}$ and $f(\theta(V),V)=0$. Then, for all $V\in(B_\rho(\overline{v}))$, we prove that $P(V)\equiv g(\theta(V),V)$ satisfies $P(\overline{v})=0$ and $P'(\overline{v})=d_2\lambda_k>0$. It is easy to find that $P(\overline{v})=g(\theta(\overline{v}),\overline{v})=g(\overline{u},\overline{v})=0$. In addition, differentiating the function $P(V)=g(\theta(V),V)$ gets

$$P'(V)=g_u(\theta(V),V)\theta'(V)+g_v(\theta(V),V). \tag{3.8}$$

On the other hand, we differentiate the equation $f(\theta(V),V)=0$ to obtain $f_u(\theta(V),V)\theta'(V)+f_v(\theta(V),V)=0$, or, equivalently,

$$\theta'(V)=-f_u^{-1}(\theta(V),V)f_v(\theta(V),V). \tag{3.9}$$

In the end, choosing $V=\overline{v}$, substituting equation (3.9) into equation (3.8) and by (3.5) we have

$$P'(\overline{v})=-g_u(\theta(\overline{v}),\overline{v})f_u^{-1}(\theta(\overline{v}),\overline{v})f_v(\theta(\overline{v}),\overline{v})+g_u(\theta(\overline{v}),\overline{v})=-c_0a_0^{-1}b_0+d_0.$$

Notice that

$$\begin{aligned} -c_0a_0^{-1}b_0+d_0 &= \det\begin{pmatrix}1 & 0\\ 0 & -c_0a_0^{-1}b_0+d_0\end{pmatrix}=\frac{1}{a_0}\det\begin{pmatrix}a_0 & 0\\ c_0 & -c_0a_0^{-1}b_0+d_0\end{pmatrix} \\ &= \frac{1}{a_0}\det\left(\begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}\begin{pmatrix}1 & -a_0^{-1}b_0\\ 0 & 1\end{pmatrix}\right)=\frac{1}{a_0}\det\begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}. \end{aligned} \tag{3.10}$$

So, $P'(\overline{v})=d_2\lambda_k>0$.

Next, $\overline{P}$ represents an arbitrary extension of the function P to the whole line R that satisfies

$$\overline{P}\in C_b^2(R) \quad \text{and} \quad \overline{P}(V)=P(V) \quad \text{for all} \quad V\in(B_\rho(\overline{v})). \tag{3.11}$$

By Proposition 3.1, we have a sequence $d_s\to d_2$ such that

$$d_s\Delta_\tau v_s+(d_2-d_s)(v_s-\overline{v})+\overline{P}(v_s)=0 \quad \text{for} \quad x\in\Omega \tag{3.12}$$

has a non-constant solution $v_s\in W^{1,2}(\Omega)$. Indeed, It is sufficient to search for these solutions in the form $v_s=\overline{v}+m_s$, where m_s satisfies

$$d_s\Delta_\tau m_s+(\overline{P}'(\overline{v})+d_2-d_s)m_s+y(m_s)=0 \quad \text{in} \quad \Omega \tag{3.13}$$

with $\overline{h}'(\overline{v})=d_2\lambda_k$ and $y(m_s)=\overline{h}(\overline{v}+m_s)-\overline{h}'(\overline{v})m_s$ satisfies $y\in C_b^2(R)$, $y(0)=0$, and $y'(0)=0$.

Next, we can find the solutions m_s of problem (3.13) by Proposition 3.1. Therefore, according to the standard elliptic theory, we have $||m_s||_{W^{2,2}(\Omega)}\to 0$. By the bootstrap arguments using the elliptic L_p estimates and the Sobolev embedding theorem, we know that $||m_s||_{W^{2,q}(\Omega)}\to 0$ for $q>\frac{N}{2}$ and hence $||m_s||_{L^\infty(\Omega)}\to 0$. Particularly, by (3.11), if $||m_s||_\infty\le\rho$, we get $\overline{h}(v_s)=\overline{h}(\overline{v}+m_s)=h(\overline{v}+m_s)=h(v_s)$. So, the nontrivial solution of problem (3.12) is $v_s=\overline{v}+m_s$, where $\overline{h}$ is changed to ρ. In the end, we define $u_s=\theta(v_s)$ to get a nontrivial regular solution of problem (3.7). □

Now we apply the previous results to the specific reaction–diffusion–ODE model (1.2). So, (1.2) may be rewritten as

$$\begin{cases} r\left(1-\dfrac{u}{K}\right)u-\dfrac{cuv}{m+bu}=0, & x\in\overline{\Omega}, \\ d_2\Delta_\tau v-av+\dfrac{\beta cuv}{m+bu}=0, & x\in\Omega. \end{cases} \tag{3.14}$$

By simple calculation, it is easy to see that problem (3.14) has trivial equilibrium $(\overline{u}_1, \overline{v}_1) = (0, 0)$, semi-trivial equilibrium $(\overline{u}_2, \overline{v}_2) = (K, 0)$, and if $0 < \dfrac{am}{\beta c - ab}$, then $(\overline{u}_3, \overline{v}_3) = (u^*, v^*)$, where

$$u^* = \frac{am}{\beta c - ab}, \qquad v^* = \frac{\beta mr[K(\beta c - ab) - am]}{K(\beta c - ab)^2} = \frac{\beta r u^*(K - u^*)}{aK}.$$

We always assume that $a < \dfrac{\beta cK}{m + bK}$ and $u^* < K$.

Theorem 3.3. *Assume that $2\overline{u}_3 > K$. For a, b, c, r, m, β, K are all positive constants and for a discrete sequence of the diffusion coefficients $d_2 > 0$ problem (3.14) has a regular stationary solution.*

Proof. We consider a solution $(\overline{u}_3, \overline{v}_3)$ of problem (3.14) and use Proposition 3.2 with the constant stationary solution $(\overline{u}, \overline{v}) = (\overline{u}_3, \overline{v}_3)$. Since $2\overline{u}_3 > K$, $1 - \dfrac{2\overline{u}_3}{K} < 0$, that is, $a_0 < 0$. By simple calculation, we can find that

$$\det\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{vmatrix} r - \dfrac{2r\overline{u}_3}{K} - \dfrac{cm\overline{v}_3}{(m + b\overline{u}_3)^2} & -\dfrac{c\overline{u}_3}{m + b\overline{u}_3} \\ \dfrac{\beta cm\overline{v}_3}{(m + b\overline{u}_3)^2} & 0 \end{vmatrix} = \frac{\beta cm\overline{v}_3}{(m + b\overline{u}_3)^2} \cdot \frac{c\overline{u}_3}{m + b\overline{u}_3} > 0.$$

As a result, for some eigenvalue $\lambda_k > 0$, we may select $d_2 > 0$ to satisfy equation (3.6). □

4. Existence of steady states with jump discontinuous

In this section, we prove the existence of non-constant solutions of problem (1.2) by applying a generalized mountain pass lemma due to Chang [7, 31]. According to the first equation of (1.2), we obtain $u = h_0(v)$, $u = h_1(v)$ and $u = h_2(v)$. Applying the functions $u = h_0(v)$ and $u = h_2(v)$, we get the following single boundary value problem for v alone

$$\begin{cases} d_2 \Delta v + f_2^{\gamma}(v) = 0, & x \in \Omega, \\ \partial_\tau v = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where

$$f_2^{\gamma}(v) = \begin{cases} f_2(h_0(v), v) = -av, & v < \gamma, \\ f_2(h_2(v), v) = -av + \dfrac{\beta c h_2(v) v}{m + bh_2(v)}, & \gamma < v \le v_M, \end{cases} \tag{4.2}$$

and $\gamma \in (\xi, v_2^*)$. Note that $\xi = p(0)$, where $p(0)$ has been defined in (2.2).

Theorem 4.1. *Assume that the hypotheses of Proposition 2.2 (i) hold, then problem 4.1 has at least one classical nontrivial solution $v(x)$ so that $0 \le v(x) \le v_2^*$. Particularly, $v(x)$ must cross γ.*

Remark 2. By classical nontrivial solution, we mean a solution $v(x)$ of (4.1) such that $v(x) \not\equiv 0$, $v(x) \not\equiv v_2^*$, $v(x) \in C^1(\bar{\Omega})$ and $\Delta v(x)$ on the $\bar{\Omega}$ has jump discontinuity.

To prove Theorem 4.1, we first generalize $f_2^{\gamma}(v)$ to $\widetilde{f}_2^{\gamma}(v)$, as follows

$$\widetilde{f}_2^{\gamma}(v) = \begin{cases} f_2^{\gamma}(v) & \text{for } v \le v_M, \\ -a(v - v_M) + v_M\left(-a + \dfrac{\beta c h_2(v_M)}{m + bh_2(v_M)}\right) & \text{for } v > v_M, \end{cases} \tag{4.3}$$

and consider

$$\begin{cases} d_2 \Delta v + \widetilde{f}_2^{\gamma}(v) = 0, & x \in \Omega, \\ \partial_\tau v = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

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Since $\widetilde{f}_2^\gamma(v)$ discontinues at $v=\gamma$, we search for the solution of (4.4) in $W^{1,2}(\Omega)$. The energy functional $J_{d_2}(v)$ connected to (4.4) is described as follows

$$J_{d_2}(v)=\frac{d_2}{2}\int_\Omega|\nabla v|^2dx-\int_\Omega G^\gamma(v)dx, \tag{4.5}$$

where $G^\gamma(v)=\int_0^v\widetilde{f}_2^\gamma(s)ds$. Moreover, we endow $W^{1,2}(\Omega)$ with the norm

$$||v||=\left(\int_\Omega|\nabla v|^2dx+\int_\Omega v^2dx\right)^{\frac{1}{2}}.$$

Note that $v=0$ and $v=v_2^*$ are two constant solutions of (4.4). As a result, the proof will be divided into two cases: Case 1 $J_{d_2}(0)\le J_{d_2}(v_2^*)$ and Case 2 $J_{d_2}(0)>J_{d_2}(v_2^*)$.

Firstly, we consider Case 1. Let $s=v-v_2^*$ and $Q_{d_2}(s)=J_{d_2}(s+v_2^*)-J_{d_2}(v_2^*)$. Then

$$Q_{d_2}(s)=\frac{d_2}{2}\int_\Omega|\nabla s|^2dx-\int_\Omega(G^\gamma(s+v_2^*)-G^\gamma(v_2^*))dx \tag{4.6}$$

Here, we rephrase the definitions of the generalized gradient, Palais-Smale condition (henceforth denoted by (PS)), and the generalized mountain pass lemma in the context of our problem.

Definition 3. *If $Q:W^{1,2}(\Omega)\to R$ is a locally Lipschitz continuous function, then for each $\phi\in W^{1,2}(\Omega)$, we can define the generalized directional derivative $Q^o(s;\phi)$ in the direction ϕ by*

$$Q^o(s;\phi)=\overline{\lim_{u\to0,r\downarrow0}}\frac{Q(s+u+r\phi)-Q(s+u)}{r},$$

and the generalized gradient of $Q(s)$ at s, denoted by $\partial Q(s)$, is defined to be the subdifferential of the function $Q^o(s;\phi)$ at 0. That is, $\psi\in\partial Q(s)\subset(W^{1,2}(\Omega))^*$ if and only if $\langle\psi,\phi\rangle\le Q^o(s;\phi)$ for all $\phi\in W^{1,2}(\Omega)$, where $(W^{1,2}(\Omega))^*$ is the dual space of $W^{1,2}(\Omega)$.

Definition 4. *We say that a locally Lipschitz continuous function Q satisfies the Palais–Smale condition* (P.S.) *if any sequence $\{s_n\}\subset W^{1,2}(\Omega)$ for which $\{Q(s_n)\}$ is bounded and $\lambda(s_n)=\min_{\psi\in\partial Q(s_n)}||\psi||_{(W^{1,2}(\Omega))^*}\to0$ possesses a convergent subsequence.*

Theorem 4.2. *([7]) Let $Q(s)$ be a locally Lipschitz continuous function on $W^{1,2}(\Omega)$ which satisfies (P.S.) and assume that*

(i) $Q(0)=0$ and there exist positive constants ρ, α such that $Q>0$ in $B_\rho\backslash\{0\}$ and $Q>\alpha$ on ∂B_ρ;
(ii) there is an $e\in W^{1,2}(\Omega)$, $e\neq0$ such that $Q(e)\le0$.
Then $Q(s)$ has a critical point. Here, $B_\rho=\left\{s\in W^{1,2}(\Omega)|\ ||s||_{W^{1,2}(\Omega)}\le\rho\right\}$.

Next, we prove that $Q_{d_2}(s)$ satisfies all the assumptions of Theorem 4.2.

Remark 3. If $0\in\partial Q_{d_2}(s)$, then the critical point of Q_{d_2} is $s\in W^{1,2}(\Omega)$, as stated in [7]. Once we obtain a critical point s of $Q_{d_2}(s)$, then $v=s+v_2^*$ is a critical point of $J_{d_2}(v)$.

Proposition 4.3. *Assume that hypotheses of Proposition 2.2 (i) hold. Then $Q_{d_2}(s)$ is a locally Lipschitz continuous function on $W^{1,2}(\Omega)$.*

Proof. According to the definition of $\widetilde{f}_2^\gamma(v)$ in (4.3), we rewrite equation (4.6) as

$$Q_{d_2}(s)=R^*-\int_\Omega\int_0^s(h^\gamma(w+v_2^*)-h^\gamma(v_2^*))dwdx, \tag{4.7}$$

where $R^*=\frac{d_2}{2}\int_\Omega|\nabla s|^2dx+\frac{a}{2}\int_\Omega s^2dx$. Clearly, R^* is C^1 on $W^{1,2}(\Omega)$ and hence locally Lipschitz continuous. On the other hand, we know that $h^\gamma(v)=0$ if $v<\gamma$, $h^\gamma(v)=\beta ch_2(v)v/(m+bh_2(v))$ if $\gamma<v\le v_M$ and $h^\gamma(v)=\beta ch_2(v_M)v_M/(m+bh_2(v_M))$ if $v>v_M$. Thus, we find that there exist a constant a_1 such that $|h^\gamma(v)-h^\gamma(v_2^*)|<a_1$ for all $v\in R$, which means that $|h^\gamma(s+v_2^*)-h^\gamma(v_2^*)|<a_1$ for all $s\in R$. Let $H(s)=\int_0^s(h^\gamma(w+v_2^*)-h^\gamma(v_2^*))dw$ and $B(s)=\int_\Omega H(s)dx$. Then

$$|H(s_1) - H(s_2)| \leq |\int_{s_2}^{s_1} \left| h^\gamma(w + v_2^*) - h^\gamma(v_2^*)|dw \right| < a_1|s_1 - s_2|;$$

so that

$$|B(s_1) - B(s_2)| < a_1 \int_\Omega |s_1 - s_2|dx \leq a_1 mes(\Omega)^{1/2}||s_1 - s_2||_{L^2(\Omega)}$$
$$\leq a_1 mes(\Omega)^{1/2}||s_1 - s_2||_{W^{1,2}(\Omega)}. \tag{4.8}$$

Therefore, $B(s)$ is a locally Lipschitz continuous function on $L^2(\Omega)$ and $W^{1,2}(\Omega)$. From this, it can be concluded that $Q_{d_2}(s)$ is locally Lipschitz continuous on $W^{1,2}(\Omega)$. □

Proposition 4.4. *Assume Proposition 2.2 (i) holds. Let $\{s_n\} \subset W^{1,2}(\Omega)$ be a sequence such that $\{Q_{d_2}(s_n)\}$ is bounded and $\lambda(s_n) = \min_{\psi \in \partial Q_{d_2}(s_n)} ||\psi||_{(W^{1,2}(\Omega))^*} \to 0$ as $n \to \infty$. Then $\{s_n\}$ possesses a convergent subsequence.*

Proof. By Proposition 4.3, we know

$$Q_{d_2}(s_n) = \frac{d_2}{2}\int_\Omega |\nabla s_n|^2 dx + \frac{a}{2}\int_\Omega s_n^2 dx - \int_\Omega H(s_n)dx$$

$$\geq \frac{1}{2}\min\{d_2, a\} ||s_n||^2_{W^{1,2}(\Omega)} - a_1 \mathrm{mes}(\Omega)^{1/2}||s_n||_{W^{1,2}(\Omega)}.$$

Thus, $\{s_n\}$ is bounded in $W^{1,2}(\Omega)$ and there is a weakly convergent subsequence $\{s_{n_i}\}$ with limit s_0 in $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega) \to L^2(\Omega)$ is compact, $\{s_{n_i}\}$ is strongly convergent in $L^2(\Omega)$. Recalling Proposition 4.3, both $B(s)$ and $Q_{d_2}(s)$ are locally Lipschitz continuous. So, according to Definition 3, the generalized gradients of $B(s)$ and $Q_{d_2}(s)$ with respect to s do exist and they are denoted by $\partial B(s)$ and $\partial Q_{d_2}(s)$, respectively. Note that

$$\partial Q_{d_2}(s_n) \subset \{Ls_n\} - \partial B(s_n), \tag{4.9}$$

where L is an elliptic differential operator such that $Ls = -d_2\Delta s + as$. We have applied Propositions (4) and (3) of [7] to prove (4.9). Since $\lambda(s_n) \to 0$ as $n \to \infty$, there is a sequence $\rho_{n_i} \in \partial B(s_n)$ such that

$$Ls_{n_i} - \rho_{n_i} \to 0 \quad \text{in} \quad (W^{1,2}(\Omega))^*.$$

Because of $\rho_{n_i} \in \partial B(s_n)$, $\{\rho_{n_i}\}$ is bounded in $L^2(\Omega)$. This can be demonstrated by noting that $B(s_{n_i})$ is locally Lipschitz continuous on $L^2(\Omega)$ according to (4.8). Therefore, there is a subsequence $\{n_i'\}$ of $\{n_i\}$, which satisfies that $\{\rho_{n_i'}\}$ is weakly convergent to ρ_0 in $L^2(\Omega)$. Therefore, it is strongly convergent in $(W^{1,2}(\Omega))^*$. So,

$$Ls_{n_i'} \to \rho_0 \quad \text{in} \quad (W^{1,2}(\Omega))^*,$$

which shows that $s_{n_i'} \to (L)^{-1}\rho_0$ in $W^{1,2}(\Omega)$. □

Proposition 4.5. *Assume that Proposition 2.2 (i) holds. We get*

(i) $Q_{d_2}(0) = 0$ and there are constants $\rho > 0$, $\alpha > 0$, which satisfy that $Q_{d_2} \geq \alpha$ if $||s||_{W^{1,2}(\Omega)} = \rho$;

(ii) there is an $e \in W^{1,2}(\Omega)$, $||e||_{W^{1,2}(\Omega)} > \rho$, which satisfies that $Q_{d_2}(e) \leq 0$.

Proof. (i) By (4.7), we know $Q_{d_2}(0) = 0$. Moreover, we can rewrite (4.7) as follows

$$Q_{d_2}(s) = \frac{d_2}{2}\int_\Omega |\nabla s|^2 dx + \frac{1}{2}\int_\Omega (a - (h^\gamma)_v(v_2^*))s^2 dx$$
$$- \int_\Omega \int_0^s (h^\gamma(w + v_2^*) - h^\gamma(v_2^*) - (h^\gamma)_v(v_2^*)w)dwdx \tag{4.10}$$

From the definitions of $f_2(h_2(v), v)$ in (4.2) and $h^\gamma(v)$ after (4.7), we know $f_2(h_2(v), v) = -av + h^\gamma(v)$, which implies $\partial_v f_2(h_2(v), v) = -a + (h^\gamma)_v(v)$. By the proof of Proposition 2.3 (ii), we see $\partial_v f_2(h_2(v_2^*), v_2^*) < 0$, so there is a constant $a_2 > 0$ such that $a - (h^\gamma)_v(v_2^*) > a_2$ for all $x \in \overline{\Omega}$.

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Let $\Phi(s) = h^\gamma(s + v_2^*) - h^\gamma(v_2^*) - (h^\gamma)_v(v_2^*)s$ and $\Psi(s) = \int_0^s \Phi(w)dw$. It is easy to see that $\Phi(s) = o(|s|)$ at $s = 0$ uniformly in $x \in \overline{\Omega}$. Thus, for any $\iota > 0$, there exists a $\delta > 0$ such that $|\Psi(s)| \le \iota s^2$ if $|s| \le \delta$. In addition, in the proof of Proposition 4.3, we can recall the following that $|h^\gamma(w + v_2^*) - h^\gamma(v_2^*)| < a_1$, which leads to the conclusion that for every $\varepsilon \in (1, (N+2)/(N-2))$, there exists a constant $a_3 > 0$ such that $|h^\gamma(w + v_2^*) - h^\gamma(v_2^*)| < a_1 + a_3|w|^\varepsilon$ for all $w \in R$. This means that there exists a constant $a_4 > 0$ such that $|H(s)| = |\int_0^s (h^\gamma(w + v_2^*) - h^\gamma(v_2^*))dw| \le a_4|s|^{\varepsilon+1}$ for $|s| > \delta$. Thanks to the Sobolev embedding theorem, we get

$$
\begin{aligned}
Q_{d_2}(s) &= \frac{d_2}{2}\int_{|s|>\delta} |\nabla s|^2 dx + \frac{a}{2}\int_{|s|>\delta} s^2 dx - \int_{|s|>\delta} H(s)dx \\
&\quad + \frac{d_2}{2}\int_{|s|\le\delta} |\nabla s|^2 dx + \frac{1}{2}\int_{|s|\le\delta} (a - (h^\gamma)_v(v_2^*))s^2 dx - \int_{|s|\le\delta} \Psi(s)dx \\
&\ge \frac{d_2}{2}\int_{|s|>\delta} |\nabla s|^2 dx + \frac{a}{2}\int_{|s|>\delta} s^2 dx - c_5||s||^{\varepsilon+1}_{W^{1,2}(\Omega)} \\
&\quad + \frac{d_2}{2}\int_{|s|\le\delta} |\nabla s|^2 dx + (\frac{a_2}{2} - \iota)\int_{|s|\le\delta} s^2 dx \\
&\ge a_6||s||^2_{W^{1,2}(\Omega)} - a_5||s||^{\varepsilon+1}_{W^{1,2}(\Omega)} = (a_6 - a_5||s||^{\varepsilon-1}_{W^{1,2}(\Omega)})||s||^2_{W^{1,2}(\Omega)}
\end{aligned}
$$

with some positive constants a_5 and a_6. Therefore, let $\rho = (a_6/2a_5)^{1/(\varepsilon-1)}$, we can see that $Q_{d_2}(s) = 0$ for $||s||_{W^{1,2}(\Omega)} \le \rho$ if and only if $s = 0$ and that $Q_{d_2}(s) \ge (a_6/2)\rho^2 = \alpha$ for $||s||_{W^{1,2}(\Omega)} = \rho$.

(ii) By selecting $e = -v_2^*$, we derive $Q_{d_2}(e) = J_{d_2}(0) - J_{d_2}(v_2^*) \le 0$ from Case 1. □

Using Theorem 4.2, we find that Q_{d_2} has a critical point $s(x)$. Then $v(x) = s(x) + v_2^*$ is a critical point of Q_{d_2}. Now, we examine Case 2. Similarly to Propositions 4.3 and 4.4, we can show that $Q_{d_2}(v)$ is also locally Lipschitz continuous on $W^{1,2}(\Omega)$ and satisfies (PS). Hence, it is also crucial to establish the following Proposition.

Proposition 4.6. *Assume that Proposition 2.2 (i) holds. We conclude that*

(i) $Q_{d_2}(0) = 0$ and there are constants $\rho_1 > 0$, $\alpha_1 > 0$, which satisfy that $Q_{d_2} \ge \alpha_1$ if $||v||_{W^{1,2}(\Omega)} = \rho_1$;

(ii) there is an $e_1 \in W^{1,2}(\Omega)$, $||e_1||_{W^{1,2}(\Omega)} > \rho_1$, which satisfies that $Q_{d_2}(e_1) \le 0$.

Proof. (i) Obviously, we know $Q_{d_2}(0) = 0$ by (4.7). Similarly to (4.10), (4.5) can be rewritten as follows

$$
Q_{d_2}(v) = \frac{d_2}{2}\int_\Omega |\nabla v|^2 dx + \frac{a}{2}\int_\Omega v^2 dx - \int_\Omega \int_0^v h^\gamma(w)dwdx. \tag{4.11}
$$

Since $h^\gamma(v) = 0$ if $v < \gamma$, we know that there exists a $\delta_1 \in (0, \gamma)$ such that $\int_0^v h^\gamma(w)dw = 0$ for $|v| \le \delta_1$. Then, by the definition of $h^\gamma(v)$, we obtain $|h^\gamma(w)| < a_7$ with a constant $a_7 > 0$, which implies that for every $\varepsilon \in (1, (N+2)/(N-2))$, there exists a constant $a_8 > 0$ such that $|h^\gamma(w)| < a_7 + a_8|w|^\varepsilon$ for all $w \in R$. This shows that there is a constant $a_9 > 0$ such that $|\int_0^v h^\gamma(w)dw| \le a_9|v|^{\varepsilon+1}$ for $|v| > \delta_1$. Then repeating the proof of Proposition 4.5, we can show that there exist positive constants ρ_1, α_1 such that $Q_{d_2} \ge \alpha_1$ if $||v||_{W^{1,2}(\Omega)} = \rho_1$. (ii) Let $e_1 = v_2^*$. Then it is easy to know that $Q_{d_2}(e_1) = J_{d_2}(v_2^*) - J_{d_2}(0) < 0$ according to Case 2. □

Using Theorem 4.2 again, we can show that Q_{d_2} has a critical point $v(x)$.

Proof of Theorem 4.1. A critical point $v(x)$ of Q_{d_2} has been found. In fact, $v(x)$ is a weak solution of (4.4). We can obtain that $v(x)$ is a classical solution of (4.4) by the elliptic regularity theorem [29].

Now, we demonstrate that $0 \le v(x) \le v_2^*$. Because the proof for $0 \le v(x)$ and $v(x) \le v_2^*$ are similar, we only show that $v(x) \le v_2^*$. Let $v(x_1) = \max_{x\in\overline{\Omega}} v(x) > v_2^*$. If $x_1 \in \Omega$, then $d_2\Delta v|_{x=x_1} \le 0$, which means

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that $\widetilde{f}_2^\gamma(v(x_1)) \geq 0$. But by the definition of $\widetilde{f}_2^\gamma(v)$, we find $\widetilde{f}_2^\gamma(v(x_1)) < 0$, which is a contradiction. If $x_1 \in \Omega$, there is a ball $B_R(r_0) \subset \Omega$ centred at $r_0 \in \Omega$ of radius R, which satisfies $\partial\Omega \cap \overline{B}_R(r_0) = \{x_1\}$ and $v(x) < v(x_1)$ in $B_R(r_0)$. Since $v(x_1) > v_2^*$, it follows from continuity that $v(x) > v_2^*$ in $B_R(r_0)$, provided r_0 is sufficiently close to x_1 and R is small enough. So, $\widetilde{f}_2^\gamma(v(x)) < 0$ for $x \in B_R(r_0)$, which shows that $d_2\Delta v > 0$ in $B_R(r_0)$. In addition, $v(x) < v(x_1)$ in $B_R(r_0)$. By employing the Hopf boundary point Proposition (see, e.g., Chapters 8 and 9 of [13]), we observe that $\partial_\tau v > 0$. This is in contradiction with the boundary condition $\partial_\tau v = 0$. Thus, $v(x) \leq v_2^*$ on $\overline{\Omega}$. Since $f_2^\gamma(v) = \widetilde{f}_2^\gamma(v)$ for all $v \leq v_M$, any classical solution $v(x)$ of (4.4) is also a classical solution of (4.1).

Finally, we demonstrate that $v(x)$ crosses γ. Otherwise, we can suppose that $\gamma < v(x) \leq v_2^*$ for all $x \in \overline{\Omega}$. According to Proposition 2.3 (i), we find that $f_2^\gamma(v) = f_2(h_2(v), v) \geq 0$. So, $\int_\Omega f_2(h_2(v), v)dx \geq 0$. In addition, integrating the first equation of (4.1) over Ω, we get $\int_\Omega f_2^\gamma(v) = \int_\Omega f_2(h_2(v), v)dx = 0$. This only holds if $v(x) \equiv v_2^*$. So we have a contradiction. Similarly, we can demonstrate that $0 \leq v(x) < \gamma$ for all $x \in \overline{\Omega}$ is also invalid. □

Remark 4. Suppose $v(x)$ is a solution of (4.1) and define

$$u(x) = \begin{cases} 0 & v(x) < \gamma, \\ h_2(v(x)) & v(x) > \gamma. \end{cases}$$

Then, $(u(x), v(x))$ forms a stationary solution of problem (1.2).

5. Monotone and symmetric solutions

In this section, we focus on the construction of monotonic and symmetric solutions of (4.1) in the one-dimensional space domain [0, 1] through the method in [31]. Therefore, (4.1) can be expressed as

$$\begin{cases} d_2 v'' + f_2^\gamma(v) = 0, \; x \in (0, 1), \\ v'(0) = v'(1) = 0. \end{cases} \tag{5.1}$$

Firstly, to recall that in the introduction, a solution $V_{n,+}(x)(n \geq 2)$ for an n-mode means that the number of points of discontinuity for $V''_{n,+}(x)$ is n.

First of all, we fix $\gamma \in (\xi, v_2^*)$ and follow the method in [28] to construct monotonically increasing and symmetric solutions to (5.1) for each $d_2 > 0$,where $\xi = p(0)$ has been defined in (2.2).

Theorem 5.1. *Assume that Proposition 2.2 (i) and $d_2 > 0$ hold. For every $\gamma \in (\xi, v_2^*)$, problem (5.1) has a monotonic increasing solution $V_{1,+}(x; d_2)$. Furthermore, the equation (5.1) has an n-mode symmetric solution $V_{n,+}(x; \overline{d}_2)$ with $\overline{d}_2 = n^2 d_2$ for each value of $n > 2$.*

Proof. We divide the proof into two steps.

Step 1. We consider the following initial value problems

$$\begin{cases} d_2 v'' - av = 0, & x \in (0, n_0), \\ v'(0) = 0, \quad v(n_0) = \gamma, \end{cases} \tag{5.2}$$

$$\begin{cases} d_2 v'' + f_2(h_2(v), v) = 0, & x \in (n_0, 1), \\ v(n_0) = \gamma, \quad v'(1) = 0, \end{cases} \tag{5.3}$$

where $n_0 \in (0, 1)$. It is easy to find that $W_0(x; n_0, d_2)$ is a unique monotone increasing solution to problem (5.2) for every $d_2 > 0$, where

$$W_0(x; n_0, d_2) = \frac{\gamma}{\cosh\sqrt{a/d_2}\,n_0}\cosh\sqrt{a/d_2}\,x.$$

Next, we prove that problem (5.3) has a unique monotone increasing solution $W_1(x; n_0, d_2)$ for every $d_2 > 0$. Since $f_2(h_2(v_2^*), v_2^*) = 0$, $f_2(h_2(v), v) > 0$ for all $v \in (0, v_2^*)$ and $\partial_v f_2(h_2(v_2^*), v_2^*) < 0$ by Proposition 2.3,

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there exist $0<n_1<n_2$ such that $-n_1(v-v_2^*)<f_2(h_2(v),v)<-n_2(v-v_2^*)$ for $v<v_2^*$. In order to find a solution of (5.3), we study the following initial value problems for $i=1,2$

$$\begin{cases} d_2v''-n_i(v-v_2^*)=0 & x\in(n_0,1),\\ v(n_0)=\gamma, & v'(1)=0.\end{cases} \tag{5.4}$$

Let $Y_i(x;n_0,d_2)$ be respective solutions of problems (5.4) for $i=1,2$. Then it is easy to find that $Y_1(x;n_0,d_2)$ is a lower solution of (5.3) and $Y_2(x;n_0,d_2)$ is an upper solution of (5.3). For simplicity, we let $G_{n_0}(x;a)=\cosh(a(1-x))/\cosh(a(1-n_0))$. Simple calculations yield that for $i=1,2$, $Y_i(x;n_0,d_2)=(\gamma-v_2^*)G_{n_0}(x;\sqrt{n_i/d_2})+v_2^*$. We claim that $Y_1(x;n_0,d_2)<Y_2(x;n_0,d_2)$, let $Y(x;n_0,d_2)=Y_1(x;n_0,d_2)-Y_2(x;n_0,d_2)$. Then $Y(x;n_0,d_2)$ satisfies

$$\begin{cases} d_2Y''=n_1(Y_1-v_2^*)-n_2(Y_2-v_2^*)>n_1Y & x\in(n_0,1),\\ Y(k;n_0,d_2)=0, & Y'(1;n_0,d_2)=0.\end{cases}$$

If $Y(x_M;n_0,d_2)=\max_{x\in[n_0,1]}Y(x;n_0,d_2)>0$ at some $x_M\in(n_0,1)$, then

$$0\geq d_2Y''(x_M;n_0,d_2)>n_1Y(x_M;n_0,d_2)>0.$$

This is a contradiction. So, $Y_1(x;n_0,d_2)\leq Y_2(x;n_0,d_2)$. We verify that (5.3) has a solution $W_1(x;n_0,d_2)$ by using the upper and lower solution approach.

Next, we show the uniqueness of a solution for (5.3). By the comparison method, we can guarantee the existence of a maximal solution $W_M(x)$ and a minimal solution $W_m(x)$ such that

$$Y_1(x)<W_m(x)<W_M(x)<Y_2(x).$$

Let $\widetilde{f}_2(v)=f_2(h_2(v),v)/v$, then $\widetilde{f}_2(v)$ is strictly decreasing in v. Due to $W_M(x)$ and $W_m(x)$ satisfying (5.3), we see

$$d_2W_M''+\widetilde{f}_2(W_M)W_M=0,\qquad x\in(n_0,1), \tag{5.5}$$

and

$$d_2W_m''+\widetilde{f}_2(W_m)W_m=0,\qquad x\in(n_0,1). \tag{5.6}$$

Multiply (5.5) by W_m and multiply (5.6) by W_M. Then we obtain

$$\begin{aligned}0&=\int_{n_0}^1((d_2W_M''+\widetilde{f}_2(W_M)W_M)W_m-(d_2W_m''+\widetilde{f}_2(W_m)W_m)W_M)dx\\ &=d_2(W_M'(x)W_m(x)-W_m'(x)W_M(x))\Big|_{n_0}^1+\int_{n_0}^1(\widetilde{f}_2(W_M)-\widetilde{f}_2(W_m))W_MW_mdx\\ &=d_2\gamma(W_m'(n_0)-W_M'(n_0))+\int_{n_0}^1(\widetilde{f}_2(W_M)-\widetilde{f}_2(W_m))W_MW_mdx.\end{aligned} \tag{5.7}$$

Since $W_M(x)>W_m(x)$ in $(n_0,1]$ and $\widetilde{f}_2(v)$ is strictly decreasing in v, then

$$W_m'(n_0)-W_M'(n_0)\leq0,\quad \widetilde{f}_2(W_M)-\widetilde{f}_2(W_m)\leq0,\quad n_0\leq x\leq1.$$

Due to $W_MW_m>0$, $W_M\equiv W_m$ can be seen from equation (5.7). This shows that we have completed the proof of the uniqueness of a solution for (5.3). Moreover, combining $d_2W_1''+f_2(h_2(W_1),W_1)=0$ with $f_2(h_2(W_1),W_1)>0$ and $W_1'(1;n_0,d_2)=0$, we get that $W_1(x;n_0,d_2)$ is monotone increasing in x.

Step 2. By simple calculation, we get

$$W_0'(n_0;n_0,d_2)=\gamma\sqrt{a/d_2}\tanh(\sqrt{a/d_2}n_0),\ W_1'(n_0;n_0,d_2)=\frac{1}{d_2}\int_{n_0}^1 f_2(h_2(W_1),W_1)dx. \tag{5.8}$$

Define ρ_0 to be a sufficiently small positive number. If $n_0=1-\rho_0$, then we find $W_0'(1-\rho_0;1-\rho_0,d_2)>W_1'(1-\rho_0;1-\rho_0,d_2)$ by (5.8), since $f_2(h_2(v),v)$ is bounded for all $x\in[0,v_2^*]$. In addition, if $n_0=\rho_0$, then $W_0'(\rho_0;\rho_0,d_2)<W_1'(\rho_0;\rho_0,d_2)$ for sufficiently small $\rho_0>0$. Assume $\Theta(n_0,d_2)=W_0'(n_0;n_0,d_2)-W_1'(n_0;n_0,d_2)$. Thus, we obtain

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$$\Theta(1-\rho_0, d_2) > 0, \quad \Theta(\rho_0, d_2) < 0. \tag{5.9}$$

It is easy to see that $\Theta(n_0, d_2)$ is continuous with respect to n_0 for all $d_2 > 0$. The combination of this and (5.9), there exists a n_0^* such that $\Theta(n_0^*, d_2) = 0$ and

$$V_{1,+}(x, d_2) = \begin{cases} W_0(x; n_0^*, d_2) \text{ for } & x \in [0, n_0^*], \\ W_1(x; n_0^*, d_2) \text{ for } & x \in [n_0^*, 1], \end{cases}$$

is a monotone increasing solution of (5.1) for all $d_2 > 0$.

Following, using $V_{1,+}(x, d_2)$ and its reflection, we create symmetric solutions to (5.1) starting from the monotone increasing solution $V_{1,+}(x, d_2)$. For all $n \geq 2$, define a function $V_{n,+}(x, \overline{d}_2)$ on $0 \leq x \leq 1$ by

$$V_{n,+}(x, \overline{d}_2) = \begin{cases} V_{1,+}(nx - 2j; \overline{d}_2) & \text{for } \quad x \in [2j/n, (2j+1)/n], \\ V_{1,+}(2(j+1) - nx; \overline{d}_2) & \text{for } \quad x \in [(2j+1)/n, 2(j+1)/n], \end{cases}$$

where $j = 0, 1, 2, \cdots, [n/2]$ and $\overline{d}_2 = n^2 d_2$. Then $V_{n,+}(x, \overline{d}_2)$ is a symmetric solution of (5.1). This completes the proof. □

Next, by using the shooting approach developed in the work of Mimura, Tabata and Hosono [22], we further fix $\gamma \in (d^*, v_2^*)$ and demonstrate the existence and uniqueness of monotone increasing and symmetric solutions to problem (5.1). We consider the following two initial value problems

$$\begin{cases} d_2 v'' - av = 0, \ x > 0, \\ v'(0) = 0, \ v(0) = b_0^*, \end{cases} \tag{5.10}$$

and

$$\begin{cases} d_2 v'' + f_2(h_2(v), v) = 0, \ x > 0, \\ v'(0) = 0, \ v(0) = b_1^*, \end{cases} \tag{5.11}$$

where $d^* < b_0^* < \gamma < b_1^* < v_2^*$.

Let $V_0(x; b_0^*)$ and $V_1(x; b_1^*)$ be unique solutions of (5.10) and (5.11). We can see that $V_0(x; b_0^*)$ is monotonically increasing and $V_1(x; b_1^*)$ is monotonically decreasing (see Proposition 5.4). For $j = 0, 1$, let $x = l_j$ be the unique solution of $V_j(x; b_j^*) = \gamma$ and $\psi_j(l_j)$ satisfies $\psi_j(l_j) = \dfrac{\partial V_j}{\partial x}(l_j; b_j^*(l_j))$. Next we set

$$\begin{aligned} &\bar{l}_0 = \limsup_{b_0^* \to d^*} l_0(b_0^*), \bar{l}_1 = \limsup_{b_1^* \to v_2^*} l_1(b_1^*), \overline{\psi}_0 = \lim_{l_0 \to \bar{l}_0} \psi_0(l_0), \overline{\psi}_1 = \lim_{l_1 \to \bar{l}_1} \psi_1(l_1), \\ &z_0(\alpha_0) = \psi_0^{-1}(\alpha_0), z_1(\alpha_1) = \psi_1^{-1}(\alpha_1), \bar{z}_0 = \lim_{\alpha_0 \to \overline{\alpha}} z_0(\alpha), \bar{z}_1 = \lim_{\alpha_1 \to -\overline{\alpha}} z_1(\alpha), \end{aligned} \tag{5.12}$$

where $\alpha_j = \psi_j(l_j)$ and $\overline{\alpha} = \min\left\{\overline{\psi}_0, -\overline{\psi}_1\right\}$. Here, we have utilized the facts that the inverses $b_j^*(l_j)$ of $l_j(b_j^*)$ and the inverses $\psi_j^{-1}(\alpha_j)$ of α_j $(j = 0, 1)$ indeed exist. We will establish these findings in Proposition 5.5. Then the following two theorems are main results of this section.

Theorem 5.2. *Assume that Proposition 2.2 (i) holds. For each $\gamma \in (d^*, v_2^*)$, let $1 \leq \bar{z}_0 + \bar{z}_1$. If $d_2 > 0$, $T_{1,+}(x)$ is a unique increasing solution of problem (5.1) and for every integer $n \geq 2$, $T_{n,+}(x)$ is a unique n-mode symmetric solution of problem (5.1).*

Theorem 5.3. *Assume that Proposition 2.2 (i) holds. For each $\gamma \in (d^*, v_2^*)$, let $1 > \bar{z}_0 + \bar{z}_1$. If $d_2 > 0$, $\widetilde{B}_{n,+}(x)$ is a unique n-mode symmetric solution of problem (5.1) for every integer $n \geq N_0$, where N_0 is the smallest positive integer greater than $1/(\bar{z}_0 + \bar{z}_1)$.*

We start our discussion with the following propositions.

Proposition 5.4. *Assume that Proposition 2.2 (i) holds. For all b_0^* such that (5.10) has a unique positive and strictly increasing solution $V_0(x; b_0^*)$ defined for $x > 0$, and (5.11) has a unique positive and strictly decreasing solution $V_1(x; b_1^*)$ for $0 \leq x < x_{b_1^*}$, where $x_{b_1^*}$ is the solution of $V_1(x; b_1^*) = 0$.*

Proof. By $f_2(h_L(V_0), V_0) = f_2(0, V_0) = -aV_0$, we know $d_2V_0'' = aV_0$. By simple calculation

$$V_0(x; b_0^*) = b_0^* \cosh\left(\sqrt{a/d_2}x\right) \quad \text{and} \quad V_0'(x; b_0^*) = b_0^*\sqrt{a/d_2}\sinh\left(\sqrt{a/d_2}x\right). \tag{5.13}$$

So, for all $x > 0$, we can get $V_0(x; b_0^*)$ is positive and strictly increasing. By Proposition 2.3 (i), we have $d_2V_1'' = -f_2(h_2(V_1), V_1) < 0$. Thus, $V_1(x; b_1^*)$ is decreasing with respect to x. Since $V_1'(0; b_1^*) = 0$, so $V_1'(x; b_1^*) < 0$ for $0 \le x < x_{b_1^*}$. These results show that $V_1(x; b_1^*) > 0$ is strictly monotone decreasing on $[0, x_{b_1^*})$ and $V_1(x_{b_1^*}; b_1^*) = 0$. □

Proposition 5.5. *Assume that Proposition 2.2 (i) holds. For $\gamma \in (d^*, v_2^*)$, we have the following conclusions*

(i) $\dfrac{\partial l_0}{\partial b_0^*} < 0$, $\lim_{b_0^* \to \gamma} l_0(b_0^*) = 0$ *and* $\dfrac{\partial l_1}{\partial b_1^*} > 0$, $\lim_{b_1^* \to \gamma} l_1(b_1^*) = 0$;

(ii) $\dfrac{\partial \psi_0(l_0)}{\partial l_0} > 0$, $\lim_{l_0 \to 0} \psi_0(l_0) = 0$ *and* $\dfrac{\partial \psi_1(l_1)}{\partial l_1} < 0$, $\lim_{l_1 \to 0} \psi_1(l_1) = 0$,

where for $j = 0, 1$, l_j and $\psi_j(l_j)$ are already defined in the previous section.

Proof. Since the assertions for l_0 and $\psi_0(l_0)$ may be treated similarly, it is sufficient to verify the statements for l_1 and $\psi_1(l_1)$.

(i) Recall that $V_1(l_1(b_1^*); b_1^*) = \gamma$. We differentiate it with respect to both sides with respect to b_1^* and get

$$\frac{\partial V_1}{\partial x}\frac{dl_1}{db_1^*} + \frac{\partial V_1}{\partial b_1^*} = 0,$$

so

$$\frac{dl_1}{db_1^*} = -\frac{\partial V_1}{\partial b_1^*} \Big/ \frac{\partial V_1}{\partial x}.$$

Moreover, let $\xi_1(x; b_1^*) = \dfrac{\partial V_1}{\partial x}(x; b_1^*)$ and $\eta_1(x; b_1^*) = \dfrac{\partial V_1}{\partial b_1^*}(x; b_1^*)$. We get $\xi_1(x; b_1^*) < 0$ for all $x \in (0, l_1)$ by the proof of Proposition 5.4. It is easy to find that $\eta_1(x; b_1^*)$ is a solution of

$$\begin{cases} d_2\eta_1'' + \dfrac{d}{dv}f_2(h_2(V_1), V_1)\eta_1 = 0 \text{ for} \quad x \in [0, 1], \\ \eta_1'(0; b_1^*) = 0, \quad \eta_1(0; b_1^*) = 1. \end{cases}$$

We find that if $0 \le x \le l_1$, then $V_1(x; b_1^*) \in [\gamma, b_1^*] \subset (d^*, v_2^*]$. By Proposition 2.3 (ii), we have $\dfrac{d}{dv}f_2(h_2(V_1), V_1) < 0$ for $V_1(x; b_1^*) \in [\gamma, b_1^*] \subset (d^*, v_2^*]$. So,

$$\eta_1''(0; b_1^*) = -\frac{1}{d_2}\frac{d}{dv}f_2(h_2(b_1^*), b_1^*)\eta_1(0; b_1^*) > 0.$$

Expanding $\eta_1(x; b_1^*)$ near $x = 0$, we get

$$\eta_1(x; b_1^*) = \eta_1(0; b_1^*) + \eta_1'(0; b_1^*)x + \frac{1}{2}\eta_1''(0; b_1^*)x^2 + \cdots > 0.$$

Thus, we have $d_2\eta_1'' = -\dfrac{d}{dV_1}f_2(h_2(V_1), V_1)\eta_1 > 0$ for $\eta_1(x; b_1^*) > 0$. Then, $\eta_1'(x; b_1^*)$ is strictly increasing with respect to x. Since $\eta_1'(0; b_1^*) = 0$, we know $\eta_1'(x; b_1^*) > 0$ for $x \in (0, l_1]$. Combining this with $\eta_1(0; b_1^*) = 1$, we get $\eta_1(x; b_1^*) \ge 1$ for all $x \in [0, l_1]$. So,

$$\frac{dl_1}{db_1^*} = -\frac{\partial V_1}{\partial b_1^*} \Big/ \frac{\partial V_1}{\partial x} = -\frac{\eta_1(l_1(b_1^*); b_1^*)}{\xi_1(l_1(b_1^*); b_1^*)} > 0,$$

and l_1 is strictly increasing in b_1^*. Next, we show that $\lim_{b_1^* \to \gamma} l_1(b_1^*) = 0$. Since $e^* := f_2(h_2(\gamma), \gamma)/d_2 > 0$ and $V_1(x; b_1^*)$ satisfies

$$V_1(x; b_1^*) = b_1^* - \frac{e^*}{2}x^2 + O(x^3) \quad \text{as} \quad x \to 0. \tag{5.14}$$

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Therefore, by simple calculation, we have $l_1(b_1^*) = \sqrt{2(b_1^* - \gamma)/e^*}(1 + o(1))$ as $b_1^* \to \gamma$, which shows that $\lim_{b_1^* \to \gamma} l_1(b_1^*) = 0$. As a result, the inverse of $l_1(b_1^*)$ exists and is represented as $b_1^*(l_1)$.

(ii) Since $\psi_1(l_1) = \frac{\partial V_1}{\partial x}(l_1; b_1^*(l_1)) = \xi_1(l_1; b_1^*(l_1))$, from Proposition 4.2, we get

$$\frac{d\psi_1(l_1)}{dl_1} = \frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_1}{\partial b_1^*}\frac{db_1^*}{dl_1} = \frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_1}{\partial b_1^*}\frac{\xi_1(l_1; b_1^*(l_1))}{\eta_1(l_1; b_1^*(l_1))}$$

$$= \frac{1}{\eta_1(l_1; b_1^*(l_1))}\left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right)(l_1; b_1^*(l_1))$$

$$= \frac{1}{\eta_1(l_1; b_1^*(l_1))}\left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right)(0; b_1^*(l_1))$$

$$= -\frac{f_2(h_2(b_1^*), b_1^*)}{d_2\eta_1(l_1; b_1^*(l_1))} < 0.$$

Since $\left(\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1\right)' = \xi_1''\eta_1 - \eta_1''\xi_1 = 0$, we obtain $\frac{\partial \xi_1}{\partial x}\eta_1 - \frac{\partial \eta_1}{\partial x}\xi_1$ is a constant. And by (5.14), we have $V_1'(x; b_1^*) = -e^*x + O(x^2)$ as $x \to 0$, so

$$\psi_1(l_1) = -\sqrt{2e^*(b_1^* - \gamma)}(1 + o(1)) \quad \text{as} \quad l_1 \to 0.$$

Hence, $\lim_{l_1 \to 0} \psi_1(l_1) = -\lim_{b_1^* \to \gamma} \sqrt{2e^*(b_1^* - \gamma)} = 0$. As a result, $\psi_1(l_1)$ is strictly decreasing with respect to l_1 and the inverse $\psi_1^{-1}(\alpha_1)$ of α_1 does exist. □

Proposition 5.6. *For $\alpha \in [0, \overline{\alpha}]$, $z_0(\alpha) + z_1(-\alpha)$ are a strictly increasing functions of class C^1 such that*

$$z_0(0) + z_1(0) = 0 \quad \textit{and} \quad z_0(\overline{\alpha}) + z_1(-\overline{\alpha}) = \overline{z}_0 + \overline{z}_1.$$

Proof. By the implicit function theorem and Proposition 5.5, we know that $z_0(\alpha)$ and $z_1(-\alpha)$ are a strictly increasing functions of class C^1. Then, it follows from Proposition 5.5 (ii) that $\lim_{l_0 \to 0} \psi_0(l_0) = 0$ and $\lim_{l_1 \to 0} \psi_1(l_1) = 0$. Thus, $\lim_{\alpha \to 0} z_0(\alpha) = 0$ and $\lim_{\alpha \to 0} z_1(-\alpha) = 0$, which show $z_0(0) + z_1(0) = 0$. Since $\overline{z}_0 = \lim_{\alpha_0 \to \overline{\alpha}} z_0(\alpha)$, $\overline{z}_1 = \lim_{\alpha_1 \to -\overline{\alpha}} z_1(\alpha)$, we have $z_0(\overline{\alpha}) + z_1(-\overline{\alpha}) = \overline{z}_0 + \overline{z}_1$. □

Proof of Theorem 5.2. Since $1 \le \overline{z}_0 + \overline{z}_1$, $z_0(\alpha) + z_1(-\alpha)$ increases in α according to Proposition 5.6, so there is a unique $\alpha^* \in (0, \overline{\alpha}]$ that satisfies $z_0(\alpha^*) + z_1(-\alpha^*) = 1$. Let $z_0(\alpha^*) = \Upsilon^*$ for such α^*. Then the definitions of z_0 and z_1 yield $\psi_0(\Upsilon^*) = \alpha^*$ and $\psi_1(1 - \Upsilon^*) = -\alpha^*$. So, $\psi_0(\Upsilon^*) + \psi_1(1 - \Upsilon^*) = 0$. Define

$$T_{1,+}(x) = \begin{cases} V_0(x; a_0^*(\Upsilon^*)) & \text{for } x \in [0, \Upsilon^*], \\ V_1(1 - x; a_1^*(1 - \Upsilon^*)) & \text{for } x \in [\Upsilon^*, 1], \end{cases}$$

where $a_0^*(\Upsilon^*) = a_0^*(z_0(\alpha^*))$ and $a_0^*(1 - \Upsilon^*) = a_1^*(z_1(-\alpha^*))$.

Then it becomes a unique increasing solution of (5.1). Then, for each integer $n \ge 2$, we demonstrate the existence of $T_{n,+}(x)$. Let $\overline{\alpha}_n = \alpha_0 = -\alpha_1$ such that $z_0(\overline{\alpha}_n) + z_1(-\overline{\alpha}_n) = \frac{1}{n}$. Since $n \ge 2$, we know $\frac{1}{n} < 1 \le \overline{z}_0 + \overline{z}_1$. As a result, we can construct a unique monotone increasing solution $\overline{Z}_{1,+}(x)$ on $[0, 1/n]$, where $\overline{T}_{1,+}(x) = V_0(x; \overline{b}_0^*)$ with $\overline{b}_0^* = b_0^*(z_0(\overline{\alpha}_n))$ for $x \in [0, z_0(\overline{\alpha}_n)]$ and $\overline{T}_{1,+}(x) = V_1((1/n) - x; \overline{b}_1^*)$ with $\overline{b}_1^* = b_1^*(z_1(-\overline{\alpha}_n))$ for $x \in [z_0(\overline{\alpha}_n), 1/n]$. Now define

$$T_{n,+}(x) = \begin{cases} \overline{T}_{1,+}((2j/n) + x) & \text{for } x \in [2j/n, (2j+1)/n], \\ \overline{T}_{1,+}([2(j+1)/n] - x) & \text{for } x \in [(2j+1)/n, 2(j+1)/n], \end{cases}$$

where $j=0,1,2,\cdots,[n/2]$. Then $T_{n,+}(x)$ is an n-mode symmetric solution of (5.1). This completes the proof. □

Proof of Theorem 5.3 Since $z_0(\alpha)+z_1(-\alpha)$ is a strictly increasing function for $\alpha\in[0,\overline{\alpha}]$ from Proposition 5.6, whose range is the same as $[0,\overline{z}_0+\overline{z}_1]\subset[0,1]$. If N_0 is the smallest positive integer greater than $1/(\overline{z}_0+\overline{z}_1)$, it follows that there is a unique α_n^* such that $z_0(\alpha_n^*)+z_1(-\alpha_n^*)=1/n(n\geq N_0)$. Similar to the proof of Theorem 5.5, we can create a unique monotone increasing solution $\widetilde{B}_{1,+}(x)$ on $[0,z_0(\alpha_n^*)+z_1(-\alpha_n^*)]$, where

$$\widetilde{B}_{1,+}(x)=\begin{cases}V_0(x;b_0^*) & \text{for } x\in[0,z_0(\alpha_n^*)],\\ V_1(z_0(\alpha)+z_1(-\alpha)-x;b_1^*) & \text{for } x\in[z_0(\alpha_n^*),E]\end{cases}$$

with $b_0^*=b_0^*(z_0(\alpha_n^*))$, $b_1^*=b_1^*(z_1(-\alpha_n^*))$ and $E=z_0(\alpha_n^*)+z_1(-\alpha_n^*)$. Then we extend $\widetilde{B}_{1,+}(x)$ to the interval $[0,2E]$ by

$$\widetilde{B}_{2,+}(x)=\begin{cases}\widetilde{B}_{1,+}(x) & \text{for } x\in[0,E],\\ \widetilde{B}_{1,+}(2E-x) & \text{for } x\in[E,2E].\end{cases}$$

We continue this process until x reaches $x=1$. Then $\widetilde{B}_{n,+}(x)$ is an n-mode symmetric solution of (5.1). □

6. Existence and stability of bifurcation solutions

Firstly, in order to study the stability of this equilibrium solution for system (1.1) on one-dimensional domain $[0,1]$, we analyse the spectrum of the linearized operator through the method in [31]. Let $(\widetilde{u},\widetilde{v})$ be any constant solution of system (1.2) and

$$f(u,v)=(f_1(u,v),d_2v''+f_2(u,v)),$$

then the Fre'chet derivative with respect to (u,v) of F at $(\widetilde{u},\widetilde{v})$ is expressed as follows

$$L=\begin{pmatrix}f_{11} & f_{12}\\ f_{21} & d_2\frac{d^2}{dx^2}+f_{22}\end{pmatrix},$$

where $f_{11}=f_{1u}(\widetilde{u},\widetilde{v})$, $f_{12}=f_{1v}(\widetilde{u},\widetilde{v})$, $f_{21}=f_{2u}(\widetilde{u},\widetilde{v})$, $f_{22}=f_{2v}(\widetilde{u},\widetilde{v})$. Suppose that λ is an eigenvalue of L. Then we find that λ satisfies the characteristic equation

$$\lambda^2-(f_{11}+f_{22}-d_2L_j)\lambda+f_{11}f_{22}-f_{12}f_{21}-d_2f_{11}L_j=0 \tag{6.1}$$

for some $j\geq0$, where $L_j=(\pi j)^2, j=0,1,2,\cdots$, are the eigenvalues for $\frac{d^2}{dx^2}$ subject to Neumann boundary conditions. In addition, $\cos(\pi jx)$ is an eigenfunction corresponding to L_j and $\{\cos(\pi jx)\}_{j=0}^{\infty}$ forms a basis of $L^2(0,1)$.

Theorem 6.1. *For $d_2>0$, the following assertions are true.*

(i) The trivial solution $(\widetilde{u},\widetilde{v})=(0,0)$ and the semi-trivial solution $(\widetilde{u},\widetilde{v})=(K,0)$ are unstable.

(ii) The positive solution $(\widetilde{u},\widetilde{v})=(u_2^*,v_2^*)$ is locally asymptotically stable under the assumption of Proposition 2.2 (i).

(iii) The positive solution $(\widetilde{u},\widetilde{v})=(u_3^*,v_3^*)$ is locally asymptotically stable under the assumption of Proposition 2.2 (ii).

Proof. The proof is simple. As a result, we omit the detail. □

Then, we consider d_2 as a bifurcation parameter and study the bifurcation problem near the constant steady state (u_3^*,v_3^*) in the boundary value problem

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$$\begin{cases} r\left(1-\frac{u}{K}\right)u-\frac{cuv}{m+bu}=0, x\in[0,1],\\ d_2v''-av+\frac{\beta cuv}{m+bu}=0, \quad x\in(0,1),\\ v'(0)=0, \quad v'(1)=0. \end{cases} \tag{6.2}$$

Let

$$\widetilde{X}=\left\{(u,v)|u\in C^0([0,1]), v\in C^2([0,1]), v'(0)=v'(1)=0\right\},$$

$$\widetilde{Y}=C^0([0,1])\times C^0([0,1]).$$

Then the *Fréchet* derivative $\widetilde{L}$ with respect to (u,v) of F at (u_3^*, v_3^*) can be written as follows

$$\widetilde{L}=\begin{pmatrix}\widetilde{f_{11}} & \widetilde{f_{12}}\\ \widetilde{f_{21}} & d_2\frac{d^2}{dx^2}+\widetilde{f_{22}}\end{pmatrix},$$

where

$$\widetilde{f_{11}}=\frac{r(K-2u_3^*)}{K}-\frac{mr(K-u_3^*)}{K(m+bu_3^*)}>0, \quad \widetilde{f_{12}}=-\frac{cu_3^*}{m+bu_3^*}<0,$$

$$\widetilde{f_{21}}=\frac{\beta cmv_3^*}{(m+bu_3^*)^2}>0, \quad \widetilde{f_{22}}=0.$$

Therefore, the characteristic equation (6.1) is transformed into

$$\lambda^2-(\widetilde{f_{11}}-d_2L_j)\lambda-\widetilde{f_{12}}\widetilde{f_{21}}-d_2\widetilde{f_{11}}L_j=0. \tag{6.3}$$

Let $\widetilde{d_2}=-\widetilde{f_{12}}\widetilde{f_{21}}/\widetilde{f_{11}}L_j$. Then $\widetilde{d_2}>0$ for every $j\geq 1$. If we assume that $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}\neq 0$, then at $d_2=\widetilde{d_2}$ for every $j\geq 1$, $\widetilde{f_{11}}-d_2L_j=(\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}})/\widetilde{f_{11}}\neq 0$. So, we see that the zero is a simple eigenvalue of (6.3). Therefore, the following conclusions can be drawn.

Theorem 6.2. *Assume that Proposition 2.2 (ii) holds. If* $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}\neq 0$, *then* $(\widetilde{d_2},(u_3^*,v_3^*))$ *is a bifurcation point of* $F=0$. *Furthermore, there is a* δ_0, *which satisfies that (6.2) admits a one-parameter family of non-constant solutions* $\left\{(\widetilde{d_2}+d_2(l),(u(l),v(l))), |l|<\delta_0\right\}$ *of the form* $u(l)=u_3^*+l\phi_j+o(l)$, $v(l)=v_3^*+l\psi_j+o(l)$, *where* $\phi_j=cos(\pi jx)$, $\psi=-(\widetilde{f_{11}}\phi_j)/\widetilde{f_{12}}$ *and* $d_2(0)=0$. *Particularly, there exist no solutions other than* $\left\{(\widetilde{d_2}+d_2(l),(u(l),v(l))), |l|<\delta_0\right\}\cup\left\{(d_2,(u_3^*,v_3^*)), |d_2-\widetilde{d_2}|<\delta_0\right\}$ *in a small neighbourhood of* $(\widetilde{d_2},(u_3^*,v_3^*))$ *in* $R\times X$.

Proof. Assume that $\widetilde{\Phi}=(\widetilde{\phi},\widetilde{\psi})\in \ker\widetilde{L}$ and let $\widetilde{\phi}=\sum_j\widetilde{c_j}\phi_j$, $\widetilde{\psi}=\sum_j\widetilde{d_j}\phi_j$, then $\sum_j^\infty D_j\begin{pmatrix}\widetilde{c_j}\\ \widetilde{d_j}\end{pmatrix}\phi_j=0$, where

$$D_j=\begin{pmatrix}\widetilde{f_{11}} & \widetilde{f_{12}}\\ \widetilde{f_{21}} & -d_2L_j\end{pmatrix}.$$

Obviously, $\det D_j=0\Leftrightarrow d_2=\widetilde{d_2}$. Let $d_2=\widetilde{d_2}$, we get

$$\ker\widetilde{L}=\text{span}\{\Phi_0\}, \quad \Phi_0=\begin{pmatrix}\phi_0\\ \psi_0\end{pmatrix}=\begin{pmatrix}\cos(\pi jx)\\ -\frac{\widetilde{f_{11}}}{\widetilde{f_{12}}}\cos(\pi jx)\end{pmatrix}.$$

Similar to this, it is simple to calculate an eigenvector Φ_0^* of $\widetilde{L}^*$ associated with 0 having the following form

$$\ker\widetilde{L}^*=\text{span}\{\Phi_0^*\}, \quad \Phi_0^*=\begin{pmatrix}\phi_0\\ \psi_0\end{pmatrix}=\begin{pmatrix}\cos(\pi jx)\\ -\frac{\widetilde{f_{11}}}{\widetilde{f_{21}}}\cos(\pi jx)\end{pmatrix},$$

where $\widetilde{L}^*$ is the adjoint operator of $\widetilde{L}$, which is obtained by

$$\widetilde{L}^* = \begin{pmatrix} \widetilde{f}_{11} & \widetilde{f}_{21} \\ \widetilde{f}_{12} & d_2 \frac{d^2}{dx^2} \end{pmatrix}.$$

Because $\text{rang}\widetilde{L} = (\ker\widetilde{L}^*)^{\perp}$, we have $\dim\ker\widetilde{L}^* = \text{codim rang}\,\widetilde{L} = 1$. Finally, since

$$\hat{L} = \frac{\partial \hat{L}}{\partial d_2} \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}$$

and $\hat{L}\Phi_0 \notin \text{rang}\,\widetilde{L}$, we get that the conditions required for the standard bifurcation theorem to apply, based on the presence of a simple eigenvalue [8], are satisfied. □

Next, we study the stability of bifurcation solutions. Suppose that $L(l)$ represent the linearized operator $\partial_U F(\widetilde{d_2} + d_2(l), \overline{U} + l\Phi_0 + o(l))$, where $U = (u, v)$, $\overline{U} = (u_3^*, v_3^*)$ and $(\widetilde{d_2} + d_2(l), \overline{U} + l\Phi_0 + o(l))$ is a bifurcation solution obtained by Theorem 6.2.

Definition 5. *Let $B(X, Y)$ denote the set of bounded linear maps of X into Y. Let $T, K \in B(X, Y)$. Then $\mu \in R$ is a K-simple eigenvalue of T if*

(i) $\dim N(T - \mu K) = \text{codim}R(T - \mu K) = 1$ and, if $N(T - \mu K) = \text{span}\,\{x_0\}$,

(ii) $Kx_0 \in R(T - \mu K)$.

Proposition 6.3. *For $d_2 = \widetilde{d_2}$, 0 is an i-simple eigenvalue of $\widetilde{L}$ and i is the inclusion mapping $\widetilde{X} \to \widetilde{Y}$.*

Proof. According to the proof procedure of theorem 6.2, we can get $\dim\ker\widetilde{L} = \text{codim rang}\,\widetilde{L} = 1$. Then $i\Phi \notin \text{rang}\,\widetilde{L}$, where Φ_0 satisfies $\ker\widetilde{L} = \{\Phi_0\}$. Thus, it is clear that $\widetilde{L}$ possesses 0 as an i-simple eigenvalue according to the definition of a K-simple eigenvalue presented in [30]. □

We have identified an i-simple eigenvalue $\lambda_j(d_2)$ for $\widetilde{L}$ near $d_2 = \widetilde{d_2}$, as well as an i-simple eigenvalue $\lambda(l)$ for $L(l)$ when $|l|$ is small enough. By making use of the well-known theorem by Crandall and Rabinowitz [9], we obtain

$$\lim_{s\to 0, \lambda(l)\neq 0} -\frac{l d_2'(l)\lambda_j'(\widetilde{d_2})}{\lambda(l)} = 1.$$

Proposition 6.4. *If $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} < 0$, then both $\lambda(l)$ and $-ld_2'(l)$ possess the same sign. But if $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} > 0$, then both $\lambda(l)$ and $ld_2'(l)$ possess the same sign.*

Proof. From (6.3) we get $(\lambda_j(d_2))^2 - (\widetilde{f_{11}} - d_2L_j)\lambda_j(d_2) - \widetilde{f_{12}}\widetilde{f_{21}} - d_2\widetilde{f_{11}}L_j = 0$. Taking the derivative of both sides with respect to d_2, we get $2\lambda_j(d_2)\lambda_j'(d_2) + L_j\lambda_j(d_2) - (\widetilde{f_{11}} - d_2L_j)\lambda_j'(d_2) - \widetilde{f_{11}}L_j = 0$. So, $\lambda_j(\widetilde{d_2}) = 0$ shows

$$\lambda_j'(d_2) = \frac{-L_j\widetilde{f_{11}}}{\widetilde{f_{11}} - L_j\widetilde{d_2}} = \frac{-\widetilde{f_{11}}^2 L_j}{\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}}}.$$

Thus, if $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} < 0$, then $\lambda_j'(d_2) > 0$, which shows that both $\lambda(l)$ and $-ld_2'(l)$ possess the same sign. But if $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} > 0$, then $\lambda_j'(d_2) < 0$, which implies that both $\lambda(l)$ and $ld_2'(l)$ possess the same sign. □

Then we analyse the sign of $\lambda(l)$. Because (u_3^*, v_3^*) is on the branch $u = h_1(v)$, we study the following boundary value problem

$$\begin{cases} d_2 v'' + d(v) = 0, & x \in (0, 1), \\ v'(0) = 0, \quad v'(1) = 0, \end{cases} \tag{6.4}$$

where $d(v) = f_2(h_1(v), v)$.

Theorem 6.5. *Let $C = d'(v_3^*)$, $D = d''(v_3^*)$, $E = d'''(v_3^*)$ and define $N = 3CE - 5D^2$, then we have the following conclusions.*

(i) If $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}<0$ *and* $N>0$*, then* $\lambda(l)<0$ *for* $0<l\ll 1$.
(ii) If $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}<0$ *and* $N<0$*, then* $\lambda(l)>0$ *for* $0<l\ll 1$.
(iii) If $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}>0$ *and* $N>0$*, then* $\lambda(l)>0$ *for* $0<l\ll 1$.
(iv) If $\widetilde{f_{11}}^2+\widetilde{f_{12}}\widetilde{f_{21}}>0$ *and* $N<0$*, then* $\lambda(l)<0$ *for* $0<l\ll 1$.

Proof. We only demonstrate the assertion (i) here, as we can approach the proof of the other assertions in a similar manner. By Proposition 6.4, it is sufficient to compute $d_2'(l)$ to reveal the sign of $\lambda(l)$. We expand $v(x,l)$ and $d_2(l)$ in l to obtain

$$v(x,l)=v_3^*+l\widetilde{n}_1(x)+l^2\widetilde{n}_2(x)+l^3\widetilde{n}_3(x)+\cdots,$$
$$d_2(l)=\widetilde{d}_2+l\widetilde{k}_1+l^2\widetilde{k}_2+l^3\widetilde{k}_3+\cdots.$$

Due to $d(v_3^*)=f_2(h_1(v_3^*),v_3^*)=0$, we obtain

$$d(v)=C(v-v_3^*)+\frac{1}{2}D(v-v_3^*)^2+\frac{1}{6}E(v-v_3^*)^3+\cdots,$$

where

$$C=\frac{\beta cmv_3^*}{(m+bh_1(v_3^*))^2}h_1'(v_3^*),$$

$$D=\frac{-2\beta cmbv_3^*}{(m+bh_1(v_3^*))^3}(h_1'(v_3^*))^2+\frac{2\beta cm}{(m+bh_1(v_3^*))^2}h_1'(v_3^*)+\frac{\beta cmv_3^*}{(m+bh_1(v_3^*))^2}h_1''(v_3^*),$$

$$E=\frac{6\beta cmb^2v_3^*}{(m+bh_1(v_3^*))^4}(h_1'(v_3^*))^3-\frac{6\beta cmb}{(m+bh_1(v_3^*))^3}(h_1'(v_3^*))^2-$$

$$\frac{6\beta cmbv_3^*}{(m+bh_1(v_3^*))^3}h_1'(v_3^*)h_1''(v_3^*)+\frac{3\beta cm}{(m+bh_1(v_3^*))^2}h_1''(v_3^*)+\frac{\beta cmv_3^*}{(m+bh_1(v_3^*))^2}h_1'''(v_3^*).$$

In addition, it is easy to find that $f_1(h_1(v_3^*),v_3^*)=0$. So, we get $h_1'(v_3^*)=-\widetilde{f_{12}}/\widetilde{f_{11}}$. Consequently, based on the previous definition of $\widetilde{f_{21}}$ and $\widetilde{d_2}$, we know

$$C=\frac{\beta cmv_3^*}{(m+bh_1(v_3^*))^2}h_1'(v_3^*)=\frac{\beta cmv_3^*}{(m+bh_1(v_3^*))^2}(\frac{-\widetilde{f_{12}}}{\widetilde{f_{11}}})=-\frac{\widetilde{d_2}\widetilde{f_{11}}L_j}{\widetilde{f_{12}}}(\frac{-\widetilde{f_{12}}}{\widetilde{f_{11}}})=\widetilde{d_2}\pi^2j^2.$$

By substituting these expressions into (6.4), we can obtain a sequence of equations by assigning a value of zero to the coefficient of each power of l.

$$\widetilde{d}_2\widetilde{n}_1''+C\widetilde{n}_1=0, \tag{6.5}$$

$$\widetilde{d}_2\widetilde{n}_2''+\widetilde{k}_1\widetilde{n}_1''+\frac{1}{2}D\widetilde{n}_1^2+C\widetilde{n}_2=0, \tag{6.6}$$

$$\widetilde{d}_2\widetilde{n}_3''+\widetilde{k}_1\widetilde{n}_2''+\widetilde{k}_2\widetilde{n}_1''+C\widetilde{n}_3+D\widetilde{n}_1\widetilde{n}_2+\frac{1}{6}E\widetilde{n}_1^3=0. \tag{6.7}$$

By (6.5), we know that $\widetilde{d}_2\frac{d^2}{dx^2}+C$ has an eigenvalue of 0. So, equation (6.6) can be solved only if

$$\widetilde{k}_1\int_0^1\widetilde{n}_1''\widetilde{n}_1\mathrm{d}x+\frac{1}{2}D\int_0^1\widetilde{n}_1^3\mathrm{d}x=0. \tag{6.8}$$

When $\widetilde{n}_1(x)=\cos(\pi jx)$, equation (6.5) is satisfied. By simple calculations, it can be deduced that $\widetilde{k}_1=0$ when it is substituted into (6.8). Therefore, (6.6) is rewritten as

$$\widetilde{d}_2\widetilde{n}_2''+\frac{1}{2}D\widetilde{n}_1^2+C\widetilde{n}_2=0.$$

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Since $\widetilde{n}_1^2 = (1+\cos(2\pi jx))/2$, then we know

$$\widetilde{d}_2\widetilde{n}_2'' + C\widetilde{n}_2 = -\frac{1}{2}D\frac{1+\cos(2\pi jx)}{2}.$$

By simple calculations, it can be deduced that

$$\widetilde{n}_2(x) = -\frac{D}{4C} - \frac{D}{4(C-4\widetilde{d}_2\pi^2 j^2)}\cos(2\pi jx) = -\frac{D}{4C} + \frac{D}{12C}\cos(2\pi jx).$$

Now we consider equation (6.7). Since $\widetilde{k}_1 = 0$, it follows that (6.7) has a solution if and only if

$$\widetilde{k}_2\int_0^1 \widetilde{n}_1''\widetilde{n}_1\mathrm{d}x + D\int_0^1 \widetilde{n}_1^2\widetilde{n}_2\mathrm{d}x + \frac{1}{6}E\int_0^1 \widetilde{n}_1^4\mathrm{d}x = 0. \tag{6.9}$$

It can be directly calculated that

$$\int_0^1 \widetilde{n}_1''\widetilde{n}_1\mathrm{d}x = -\frac{1}{2}L_j, \quad \int_0^1 \widetilde{n}_1^2\widetilde{n}_2\mathrm{d}x = -\frac{5D}{48C}, \quad \int_0^1 \widetilde{n}_1^4\mathrm{d}x = \frac{3}{8}.$$

So, (6.9) becomes

$$-\frac{1}{2}\widetilde{k}_2L_j - \frac{5D^2}{48C} + \frac{E}{16} = 0; \quad \text{so that} \quad \widetilde{k}_2 = \frac{1}{24CL_j}(3CE - 5D^2).$$

Since $\widetilde{k}_1 = 0$ and $ld_2{}'(l) = l(\widetilde{k}_1 + 2l\widetilde{k}_2 + O(l^2))$, we have $ld_2{}'(l) = 2l^2\widetilde{k}_2 + O(l^3))$. Thus, for $|l|$ sufficiently small, the sign of $ld_2{}'(l)$ is the same as that of $\widetilde{k}_2$. This implies that the sign of $-\lambda(l)$ is the same as that of $\widetilde{k}_2$ if $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} < 0$, according to Proposition 6.4. So, if $\widetilde{f_{11}}^2 + \widetilde{f_{12}}\widetilde{f_{21}} < 0$ and $N > 0$, then $\widetilde{k}_2 > 0$; so that $\lambda(l) < 0$. □

7. Conclusions

In this paper, we examine a mechanism of pattern formation that occurs in a predator–prey model with Holling-II functional response. The model consists of a single reaction–diffusion equation coupled with an ordinary differential equation. The value of this paper reflects in three aspects.

7.1. Existence of non-constant regular solution

We prove the existence of regular stationary solutions for system (3.14), using the method in [4]. If the internal equilibrium $2\overline{u}_3$ system (3.14) is greater than the carrying capacity K, and the other parameters are non-negative. Furthermore, if the diffusion coefficient $d_2 > 0$ of the predator, then the system (3.14) produces a non-constant regular solution (Theorem 3.3).

7.2. Existence and uniqueness of steady states with jump discontinuity

We apply various approaches to demonstrate the existence of steady states with jump discontinuities and investigate their characteristics on a one-dimensional spatial domain (refer to Theorem 4.1 for domains of higher dimensions and Theorems 5.1–5.3 for one-dimensional domains). These results show the existence of discontinuous steady-state solutions $(u(x), v(x))$ for system (1.1), where $u(x)$ displays a jump discontinuity while $v(x)$ is either monotonic or symmetric, depending on a fixed parameter γ. Furthermore, it is observed that by selecting a smaller range for γ, the solution becomes unique. This uniqueness stems from the fact that $f_2(h_2(v), v)$ is a strictly decreasing function in relation to v within this interval. It should be emphasized that these phenomena differ significantly from those observed in systems where both species exhibit diffusion or non-diffusion.

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7.3. Existence and stability of bifurcation solutions

In Section 6, we focus on the bifurcating solutions of the system (1.1). It has been observed that stable patterns emerge near the constant equilibrium state in a partial differential equation (*PDE*) system with diffusion-driven instability (*DDI*) property. However, the system (1.1) analysed in this paper exhibits the characteristic of *DDI* (Theorem 6.2), but all Turing-type patterns are unstable (Theorem 6.5). This is significantly different from the classical diffusive model, exhibiting a notable difference.

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