ON THE DIVISIBILITY OF $r_2(n)$

by E. J. SCOURFIELD

(Received 13 October, 1975; revised 21 November, 1975)

During the past few years, some papers of P. Deligne and J.-P. Serre (see **[2]**, **[9]**, **[10]** and other references cited there) have included an investigation of certain properties of coefficients of modular forms, and in particular Serre **[10]** (see also **[11]**) obtained the divisibility property (1) below. Let

$$f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n z/M} \qquad (M \geqq 1)$$

be a modular form of integral weight $k \geqq 1$ on a congruence subgroup of $SL_2(Z)$, and suppose that each c_n belongs to the ring R_K of integers of an algebraic number field K finite over Q. For $c \in R_K$ and $m \geqq 1$ an integer, write $c \equiv 0 \pmod{m}$ if $c \in mR_K$ and $c \not\equiv 0 \pmod{m}$ otherwise. Then Serre showed that there exists $\alpha > 0$ such that

$$N(n \leqq x : c_n \not\equiv 0 \pmod{m}) = O(x(\log x)^{-\alpha}) \tag{1}$$

as $x \to \infty$, where throughout this note $N(n \leqq x : P)$ denotes the number of positive integers $n \leqq x$ with the property P.

We shall refer below to three special cases of (1):

(i) $c_n = \tau(n)$ (Ramanujan's function),

the coefficient in the expansion of

$$\Delta = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24};$$

(ii) $c_n = \sigma_{2k-1}(n) = \sum_{d \mid n} d^{2k-1}$,

the coefficient in the Eisenstein series of weight $2k \geqq 4$;

(iii) $c_n = r_{2k}(n)$ (= the number of representations of n as a sum of $2k$ integer squares) the coefficient obtained from the $2k$-th power of the well-known theta function and thus from a modular form of weight k.

For (ii) above, a more precise conclusion than (1) is a corollary of a general result in our earlier paper **[8]**. The proofs in **[8]** and related papers **[3]**, **[4]** by W. Narkiewicz depend largely on elementary and classical analytic number theoretic arguments (including an application of a tauberian theorem of Delange **[1]**), in contrast to the deep algebraical arguments employed by Deligne and Serre in their far-ranging papers. The overlap in some instances of the results in **[10]** and **[8]** provides one of the motives for writing this note, the main purpose of which is to describe the result obtained by the method of **[8]** in the case $k = 1$ of (iii) above. The problem of characterizing in some way the divisibility properties of such a well-known function as $r_2(n)$ is of intrinsic interest, and in the theorem below we give asymptotic formulae for the properties $d \parallel r_2(n)$ and $d \nmid r_2(n)$ (where $d \parallel m$ means that $d \mid m$ but $(d, m/d) = 1$).

https://doi.org/10.1017/S0017089500003128 Published online by Cambridge University Press

First we recall the origins of the problem. In [12], G. N. Watson proved that for each positive integer d and odd v,

$$N(n \leqq x : d \nmid \sigma_v(n)) = O(x(\log x)^{-1/\phi(d)}) \tag{2}$$

as $x \to \infty$ (thus establishing (1) in case (ii)). Hence, using the congruence

$$\tau(n) \equiv \sigma_{11}(n) \qquad (\text{mod } 691),$$

he deduced that $\tau(n)$ is almost always divisible by 691, as had been conjectured by Ramanujan. Improvements in (2), leading to an asymptotic formula for both odd and even v, were obtained by R. A. Rankin in [5] for d prime and certain other d, and by this author in [6] and [8] for all the remaining d. From these asymptotic formulae, one could deduce results about the divisibility of $\tau(n)$ or $r_{2k}(n)$ in the few cases when these are related to some divisor function $\sigma_v(n)$ by a congruence or a simple expression.

The main results in [3], [4], [8] concern elements of a general class $\mathscr{C}$ of integer-valued multiplicative functions that consists of those functions f for which there exists a polynomial W with integer coefficients such that

$$f(p) = W(p) \qquad \text{for all primes } p; \tag{3}$$

f is then said to be *polynomial-like*. Euler's function ϕ and the divisor functions

$$\sigma_v \qquad (v = 0, 1, 2, \ldots)$$

belong to $\mathscr{C}$. W. Narkiewicz [3], [4] and this author [6], [7], [8] obtained asymptotic formulae for the divisibility properties $d \,\|\, f(n)$ and $d \nmid f(n)$ for $f \in \mathscr{C}$. Also in [8; Theorem 4], we showed that $d \mid f(n)$ for almost all n for any $f \in \mathscr{C}$ for which, to each prime $p \mid d$, there exists x with $p \nmid x$, $p \mid W(x)$. When this latter condition fails to hold, the density of the set $\{n : d \mid f(n)\}$ (when it is non-empty) lies strictly between 0 and 1.

In [7] and [8; §8], the coefficients of W in (3) were allowed to depend on the value of $p\,(\text{mod } Q)$ for some fixed integer $Q > 1$; thus W was really replaced by Q polynomials, one for each residue class $(\text{mod } Q)$. In particular we considered the divisibility of

$$\sigma_v(n, \chi) = \sum_{k \mid n} \chi(k) k^v, \tag{4}$$

where χ is a real non-principal character $(\text{mod } Q)$ and v is a positive integer, and from [8; Theorem 5] it follows that for most χ, $d \mid \sigma_v(n, \chi)$ for almost all n.

The case $v = 0$ in (4) gives rise to an important application, namely $k = 1$ in (iii) above, which was not discussed in [7] or [8]; for if χ denotes the real non-principal character (mod 4),

$$r_2(n) = 4 \sum_{k \mid n} \chi(k) = 4\sigma_0(n, \chi).$$

The following results hold:

THEOREM. (i) *As* $x \to \infty$,

$$N(n \leqq x : d \,\|\, r_2(n)) \sim \begin{cases} Bx(\log x)^{-\frac{1}{2}} & \text{if } d \text{ is odd} \\ Bx(\log\log x)^{a-3}(\log x)^{-1} & \text{if } 2^a \,\|\, d,\ a \geqq 3, \end{cases}$$

where B denotes a positive constant depending on d.

(ii) *If d has an odd prime divisor, then as* $x \to \infty$,

$$N(n \leqq x : d \nmid r_2(n)) \sim Cx(\log x)^{-\frac{1}{2}},$$

and

$$0 < \lim_{x \to \infty} \{N(n \leqq x : d \nmid r_2(n))/N(n \leqq x : r_2(n) \neq 0)\} < 1.$$

If $a \geqq 4$, *then as* $x \to \infty$,

$$N(n \leqq x : 2^a \nmid r_2(n)) \sim Cx(\log\log x)^{a-4}(\log x)^{-1},$$

and

$$N(n \leqq x : 8 \nmid r_2(n)) = O(x^{\frac{1}{2}}).$$

C denotes a positive constant depending on d or 2^a.

Part (ii) gives a result that is more precise than is (1) in the case $c_n = r_2(n)$. Part (i) of this Theorem comes from the generalization of Narkiewicz's result [**4**; Theorem 1], or see [**8**; Theorem 2], described in [**8**; §8], and part (ii) is deduced from this by applying the method of [**8**], in particular that of §5 and §8; the detailed argument is routine, and so we shall not give it here.

Finally we remark that the corresponding results for the function $\sigma_0(n, \chi)$ for any real non-principal character $\chi \pmod Q$ and any $Q > 1$ are simply those obtained by replacing $r_2(n)$ by $4\sigma_0(n, \chi)$ in the statement of the Theorem, but the constants B, C will then depend on χ also.

REFERENCES

1. H. Delange, Généralisation du théorème de Ikehara, *Ann. Sci. École Norm. Sup.* (3) **71** (1954), 213–242.

2. P. Deligne and J.-P. Serre, Formes modulaires de poids 1, *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 507–530.

3. W. Narkiewicz, Divisibility properties of a class of multiplicative functions, *Colloq. Math.* **18** (1967), 219–232.

4. W. Narkiewicz, Divisibility properties of some multiplicative functions, *Number Theory* (*Colloq. Math. Soc. János Bolyai* 2, Debrecen, 1968) (North-Holland, 1970), 147–159.

5. R. A. Rankin, The divisibility of divisor functions, *Proc. Glasgow Math. Assoc.* **5** (1961), 35–40.

6. E. J. Scourfield, On the divisibility of $\sigma_v(n)$, *Acta Arith.* **10** (1964), 245–285.

7. E. J. Scourfield, On the divisibility of a modified divisor function, *Proc. London Math. Soc.* (3) **21** (1970), 145–159.

8. E. J. Scourfield, Non-divisibility of some multiplicative functions, *Acta Arith.* **22** (1973), 287–314.

9. J.-P. Serre, Congruences et formes modulaires, *Séminaire Bourbaki* (1971/72) exposé 416, Lecture Notes in Mathematics 317 (Springer-Verlag, 1973), 319–338.

10. J.-P. Serre, Divisibilité des coefficients des formes modulaires de poids entier, *C.R. Acad. Sc. Paris Sér. A* **279** (1974), 679–682.

11. J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*, Séminaire Delange–Pisot–Poitou (Théorie des nombres), 16e année (1974/75) exposé 20, 28p.

12. G. N. Watson, Über Ramanujansche Kongruenzeigenschaften der Zerfällungsanzahlen (I), *Math. Z.* **39** (1935), 712–731.

DEPARTMENT OF MATHEMATICS
WESTFIELD COLLEGE
LONDON NW3 7ST

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
PRINTERS TO THE UNIVERSITY OF GLASGOW