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Uniform approximation problems of expanding Markov maps

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Abstract. Let $T:[0,1]\to[0,1]$ be an expanding Markov map with a finite partition. Let μ_ϕ be the invariant Gibbs measure associated with a Hölder continuous potential ϕ. For $x\in[0,1]$ and $\kappa>0$, we investigate the size of the uniform approximation set

$$\mathcal{U}^\kappa(x):=\{y\in[0,1]:\ \text{for all } N\gg 1,\ \text{there exists } n\le N,\ \text{such that } |T^nx-y|<N^{-\kappa}\}.$$

The critical value of κ such that $\dim_{\mathrm{H}}\mathcal{U}^\kappa(x)=1$ for μ_ϕ-almost every (a.e.) x is proven to be $1/\alpha_{\max}$, where $\alpha_{\max}=-\int\phi\,d\mu_{\max}/\int\log|T'|\,d\mu_{\max}$ and $\mu_{\max}$ is the Gibbs measure associated with the potential $-\log|T'|$. Moreover, when $\kappa>1/\alpha_{\max}$, we show that for μ_ϕ-a.e. x, the Hausdorff dimension of $\mathcal{U}^\kappa(x)$ agrees with the multifractal spectrum of μ_ϕ.

Key words: uniform Diophantine approximation, expanding Markov map, Hausdorff dimension, Gibbs measure, multifractal spectrum
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1. *Introduction and motivation*

Denote by $\|\cdot\|$ the distance to the nearest integer. The famous Dirichlet theorem asserts that for any real number $x\in[0,1]$ and $N\ge 1$, there exists a positive integer n such that

$$\|nx\|<\frac{1}{N}\quad\text{and}\quad n\le N. \tag{1.1}$$

As a corollary, for any $x\in[0,1]$, there exist infinitely many positive integers n such that

$$\|nx\|\le\frac{1}{n}. \tag{1.2}$$

Let $(\{nx\})_{n\geq 0}$ be the orbit of 0 of the rotation by an irrational number x, where $\{nx\}$ is the fractional part of nx. From the dynamical perspective, the Dirichlet theorem and its corollary describe the rate at which 0 is approximated by the orbit $(\{nx\})_{n\geq 0}$ in a uniform and asymptotic way, respectively.

In general, one can study the Hausdorff dimension of the set of points which are approximated by the orbit $(\{nx\})_{n\geq 0}$ with a faster speed. For the asymptotic approximation, Bugeaud [8] and, independently, Schmeling and Troubetzkoy [28] proved that for any $x \in [0, 1] \setminus \mathbb{Q}$,

$$\dim_{\mathrm{H}}\{y \in [0, 1] : \|nx - y\| < n^{-\kappa} \text{ for infinitely many } n\} = \frac{1}{\kappa},$$

where $\dim_{\mathrm{H}}$ stands for the Hausdorff dimension. The corresponding uniform approximation problem was recently studied by Kim and Liao [18] who obtained the Hausdorff dimension of the set

$$\{y \in [0, 1] : \text{ for all } N \gg 1, \text{ there exists } n \leq N, \text{ such that } \|nx - y\| < N^{-\kappa}\}.$$

Naturally, one wonders about the analog results when the orbit $(\{nx\})_{n\geq 0}$ is replaced by an orbit $(T^n x)_{n\geq 0}$ of a general dynamical system $([0, 1], T)$. For any $\kappa > 0$, Fan, Schmeling, and Troubetzkoy [13] considered the set

$$\mathcal{L}^{\kappa}(x) := \{y \in [0, 1] : |T^n x - y| < n^{-\kappa} \text{ for infinitely many } n\}$$

of points that are asymptotically approximated by the orbit $(T^n x)_{n\geq 0}$ with a given speed $n^{-\kappa}$, where T is the doubling map. It seems difficult to investigate the size of $\mathcal{L}^{\kappa}(x)$ when x is not a dyadic rational, as the distribution of $(T^n x)_{n\geq 0}$ is not as well studied as that of $(\{nx\})_{n\geq 0}$, see for example [1] for more details about the distribution of $(\{nx\})_{n\geq 0}$. However, from the viewpoint of ergodic theory, Fan, Schmeling, and Troubetzkoy [13] obtained the Hausdorff dimension of $\mathcal{L}^{\kappa}(x)$ for μ_ϕ almost all points x, where μ_ϕ is the Gibbs measure associated with a Hölder continuous potential ϕ. They found that the size of $\mathcal{L}^{\kappa}(x)$ is closely related to the local dimension of μ_ϕ and to the first hitting time for shrinking targets.

In their paper [21], Liao and Seuret extended the results of [13] to expanding Markov maps. Later, Persson and Rams [24] considered more general piecewise expanding interval maps, and proved some similar results to those of [13, 21]. These studies are also closely related to the metric theory of a random covering set; see [3, 11, 12, 17, 29, 30, 32] and references therein.

As a counterpart of the dynamically defined asymptotic approximation set $\mathcal{L}^{\kappa}(x)$, we would like to study the corresponding uniform approximation set $\mathcal{U}^{\kappa}(x)$ defined as

$$\begin{aligned}\mathcal{U}^{\kappa}(x) :&= \{y \in [0, 1] : \text{ for all } N \gg 1, \text{ there exists } n \leq N, \text{ such that } |T^n x - y| < N^{-\kappa}\}\\ &= \bigcup_{i=1}^{\infty} \bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}),\end{aligned}$$

where $B(x, r)$ is the open ball with center x and radius r, and T is an expanding Markov map (see Definition 1.1).

As the studies on $\mathcal{L}^\kappa(x)$, we are interested in the sizes (Lebesgue measure and Hausdorff dimension) of $\mathcal{U}^\kappa(x)$. By a simple argument, one can check that $\mathcal{U}^\kappa(x) \setminus \{T^n x\}_{n\geq 0} \subset \mathcal{L}^\kappa(x)$. Thus trivially, one has $\lambda(\mathcal{U}^\kappa(x)) \leq \lambda(\mathcal{L}^\kappa(x))$ and $\dim_H \mathcal{U}^\kappa(x) \leq \dim_H \mathcal{L}^\kappa(x)$. Here, λ denotes the Lebesgue measure on [0, 1].

Our first result asserts that for any $\kappa > 0$, the Lebesgue measure and the Hausdorff dimension of $\mathcal{U}^\kappa(x)$ are constants almost surely with respect to a T-invariant ergodic measure.

THEOREM 1.1. *Let T be an expanding Markov map on* [0, 1] *and ν be a T-invariant ergodic measure. Then for any $\kappa > 0$, both $\lambda(\mathcal{U}^\kappa(x))$ and $\dim_H \mathcal{U}^\kappa(x)$ are constants almost surely.*

To further describe the size of $\mathcal{U}^\kappa(x)$ for almost all points, we impose a stronger condition, the same as that of Fan, Schmeling, and Troubetzkoy [13] and Liao and Seuret [21], that ν is a Gibbs measure. Precisely, let ϕ be a Hölder continuous potential and μ_ϕ be its associated Gibbs measure. Let $\mathcal{B}^\kappa(x) := [0, 1] \setminus \mathcal{U}^\kappa(x)$. We remark that the sets $\mathcal{U}^\kappa(x)$ are decreasing (the sets $\mathcal{B}^\kappa(x)$ are increasing) with respect to κ. Then, we want to ask, for μ_ϕ almost all points x, how the size of $\mathcal{U}^\kappa(x)$ (and $\mathcal{B}^\kappa(x)$) changes with respect to κ. We thus would like to answer the following questions.

(Q1) When $\mathcal{U}^\kappa(x) = [0, 1]$ for μ_ϕ-almost every (a.e.) x, what is the critical value

$$\kappa_\phi := \sup\{\kappa \geq 0 : \mathcal{U}^\kappa(x) = [0, 1] \text{ for } \mu_\phi\text{-a.e. } x\}?$$

(Q2) When $\lambda(\mathcal{U}^\kappa(x)) = 1$ for μ_ϕ-a.e. x, what is the critical value

$$\kappa_\phi^\lambda := \sup\{\kappa \geq 0 : \lambda(\mathcal{U}^\kappa(x)) = 1 \text{ for } \mu_\phi\text{-a.e. } x\}?$$

(Q3) What are the Hausdorff dimensions of $\mathcal{U}^\kappa(x)$ and $\mathcal{B}^\kappa(x)$ for μ_ϕ-a.e. x? What is the critical value

$$\kappa_\phi^H := \sup\{\kappa \geq 0 : \dim_H(\mathcal{U}^\kappa(x)) = 0 \text{ for } \mu_\phi\text{-a.e. } x\}?$$

In this paper, we answer these questions when T is an expanding Markov map of the interval [0, 1] with a finite partition—an expanding Markov map, for short. We stress that according to our definition, an expanding Markov map is mixing (see the following Theorem 4.2 whose proof can be found in [2, 22, 23, 27]).

Definition 1.1. (Expanding Markov map) A transformation $T : [0, 1] \to [0, 1]$ is an expanding Markov map with a finite partition provided that there is a partition of [0, 1] into subintervals $I(i) = (a_i, a_{i+1})$ for $i = 0, \ldots, Q-1$ with endpoints $0 = a_0 < a_1 < \cdots < a_Q = 1$ satisfying the following properties.

(1) There is an integer n_0 and a real number ρ such that $|(T^{n_0})'| \geq \rho > 1$.
(2) T is strictly monotonic and can be extended to a C^2 function on each $\overline{I(i)}$.
(3) If $I(j) \cap T(I(k)) \neq \emptyset$, then $I(j) \subset T(I(k))$.
(4) There is an integer R such that $I(j) \subset \bigcup_{n=1}^R T^n(I(k))$ for every k, j.
(5) For every $k \in \{0, 1, \ldots, Q-1\}$, $\sup_{(x,y,z)\in I(k)^3}(|T''(x)|/|T'(y)||T'(z)|) < \infty$.

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For a probability measure ν and for $y \in [0, 1]$, we set

$$\underline{d}_\nu(y) := \liminf_{r\to 0} \frac{\log \nu(B(y,r))}{\log r} \quad \text{and} \quad \overline{d}_\nu(y) := \limsup_{r\to 0} \frac{\log \nu(B(y,r))}{\log r},$$

which are called respectively the lower and upper local dimensions of ν at y. When $\underline{d}_\nu(y) = \overline{d}_\nu(y)$, their common value is denoted by $d_\nu(y)$, and is simply called the local dimension of ν at y. Let D_ν be the multifractal spectrum of ν defined by

$$D_\nu(s) := \dim_{\mathrm{H}}\{y \in [0, 1] : d_\nu(y) = s\} \quad \text{for all } s \in \mathbb{R}.$$

Our answers to questions (Q1)–(Q3) are stated in the following theorem. For an expanding Markov map T and a Hölder continuous potential ϕ, we first define

$$\alpha_- := \min_{\nu\in\mathcal{M}_{\mathrm{inv}}} \frac{-\int \phi \, d\nu}{\int \log |T'| \, d\nu}, \tag{1.3}$$

$$\alpha_{\max} := \frac{-\int \phi \, d\mu_{\max}}{\int \log |T'| \, d\mu_{\max}}, \tag{1.4}$$

$$\alpha_+ := \max_{\nu\in\mathcal{M}_{\mathrm{inv}}} \frac{-\int \phi \, d\nu}{\int \log |T'| \, d\nu}, \tag{1.5}$$

where $\mathcal{M}_{\mathrm{inv}}$ is the set of T-invariant probability measures on $[0, 1]$ and $\mu_{\max}$ is the Gibbs measure associated with the potential $-\log |T'|$. By definition, it holds that $\alpha_- \le \alpha_{\max} \le \alpha_+$. Indeed, the quantities α_-, $\alpha_{\max}$, and α_+ depend on T and ϕ. However, for simplicity, we leave out the dependence unless the context requires specification.

The following main theorem tells us that the three critical values demanded in questions (Q1)–(Q3) are $1/\alpha_+$, $1/\alpha_{\max}$, and $1/\alpha_-$, correspondingly.

THEOREM 1.2. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure.*

(1) *The critical value κ_ϕ is $1/\alpha_+$. Namely, for μ_ϕ-a.e.x, $\mathcal{U}^\kappa(x) = [0, 1]$ if $1/\kappa > \alpha_+$, and $\mathcal{U}^\kappa(x) \ne [0, 1]$ if $1/\kappa < \alpha_+$.*

(2) *The critical value κ_ϕ^λ is $1/\alpha_{\max}$. Moreover, for μ_ϕ-a.e.x,*

$$\lambda(\mathcal{U}^\kappa(x)) = 1 - \lambda(\mathcal{B}^\kappa(x)) = \begin{cases} 0 & \text{if } 1/\kappa \in (0, \alpha_{\max}), \\ 1 & \text{if } 1/\kappa \in (\alpha_{\max}, +\infty). \end{cases}$$

(3) *The critical value κ_ϕ^H is $1/\alpha_-$. Moreover, for μ_ϕ-a.e.x,*

$$\dim_{\mathrm{H}} \mathcal{U}^\kappa(x) = \begin{cases} D_{\mu_\phi}(1/\kappa) & \text{if } 1/\kappa \in (0, \alpha_{\max}] \setminus \{\alpha_-\}, \\ 1 & \text{if } 1/\kappa \in (\alpha_{\max}, +\infty). \end{cases}$$

(4) *For μ_ϕ-a.e.x,*

$$\dim_{\mathrm{H}} \mathcal{B}^\kappa(x) = \begin{cases} 1 & \text{if } 1/\kappa \in (0, \alpha_{\max}), \\ D_{\mu_\phi}(1/\kappa) & \text{if } 1/\kappa \in [\alpha_{\max}, +\infty) \setminus \{\alpha_+\}. \end{cases}$$

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Remark 1. Item (2) of Theorem 1.2 is valid in a wider setting (in particular, the Markov assumption can be dropped) and is a direct consequence of [**15**, Theorem 3.2] and [**19**, Proposition 1 and Theorem 5]. We will provide a self-contained proof for the reader's convenience.

Remark 2. It is worth noting that the multifractal spectrum $D_{\mu_\phi}(s)$ vanishes if $s \notin [\alpha_-, \alpha_+]$. So if $1/\kappa < \alpha_-$, then $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x) = 0$ for μ_ϕ-a.e. x.

Remark 3. The cases $1/\kappa = \alpha_-$ and α_+ are not covered by Theorem 1.2. However, if the multifractal spectrum D_{μ_ϕ} is continuous at α_- (respectively α_+), we get that $\dim_{\mathrm{H}} \mathcal{U}^{\alpha_-}(x) = 0$ (respectively $\dim_{\mathrm{H}} \mathcal{B}^{\alpha_+}(x) = 0$) for μ_ϕ-a.e. x. The situation becomes more subtle if $D_{\mu_\phi}(\cdot)$ is discontinuous at α_- (respectively α_+). Our methods do not work for obtaining the value of $\dim_{\mathrm{H}} \mathcal{U}^{\alpha_-}(x)$ (respectively $\dim_{\mathrm{H}} \mathcal{B}^{\alpha_+}(x) = 0$) for μ_ϕ-a.e. x.

Let $\dim_{\mathrm{H}} \nu$ be the dimension of the Borel probability measure ν defined by

$$\dim_{\mathrm{H}} \nu = \inf\{\dim_{\mathrm{H}} E : E \text{ is a Borel set of } [0, 1] \text{ and } \nu(E) > 0\}.$$

Remark 4. As already discussed above, $\mathcal{U}^\kappa(x) \setminus \{T^n x\}_{n\geq 0} \subset \mathcal{L}^\kappa(x)$, one may wonder whether the sets $\mathcal{U}^\kappa(x)$ and $\mathcal{L}^\kappa(x)$ are essentially different. More precisely, is it possible that $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$ is strictly less than $\dim_{\mathrm{H}} \mathcal{L}^\kappa(x)$? Theorem 1.2 affirmatively answers this question. Compared with the asymptotic approximation set $\mathcal{L}^\kappa(x)$, the structure of the uniform approximation set $\mathcal{U}^\kappa(x)$ does have a notable feature. When $1/\kappa \in (0, \dim_{\mathrm{H}} \mu_\phi) \setminus \{\alpha_-\}$, the map $1/\kappa \mapsto \dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$ agrees with the multifractal spectrum $D_{\mu_\phi}(1/\kappa)$, while the map $1/\kappa \mapsto \dim_{\mathrm{H}} \mathcal{L}^\kappa(x)$ is the linear function $f(1/\kappa) = 1/\kappa$ independent of the multifractal spectrum. Therefore, $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x) < \dim_{\mathrm{H}} \mathcal{L}^\kappa(x)$. See Figure 1 for an illustration.

Remark 5. For the asymptotic approximation set $\mathcal{L}^\kappa(x)$, the most difficult part lies in establishing the lower bound for $\dim_{\mathrm{H}} \mathcal{L}^\kappa(x)$ when $1/\kappa < \dim_{\mathrm{H}} \mu_\phi$, for which a multifractal mass transference principle for Gibbs measure is applied, see [**13**, §8], [**21**, §5.2], and [**24**, §6]. Specifically, since $\mu_\phi(\mathcal{L}^\delta(x)) = 1$ for all $1/\delta > \dim_{\mathrm{H}} \mu_\phi$, the multifractal mass transference principle guarantees the lower bound $\dim_{\mathrm{H}} \mathcal{L}^\kappa(x) \geq (\dim_{\mathrm{H}} \mu_\phi)\delta/\kappa$ for all $1/\kappa < \dim_{\mathrm{H}} \mu_\phi$. By letting $1/\delta$ monotonically decrease to $\dim_{\mathrm{H}} \mu_\phi$ along a sequence (δ_n), we get immediately the expected lower bound $\dim_{\mathrm{H}} \mathcal{L}^\kappa(x) \geq 1/\kappa$ for all $1/\kappa < \dim_{\mathrm{H}} \mu_\phi$. However, recent progresses in uniform approximation [**9**, **18**, **20**, **34**] indicate that there is no mass transference principle for uniform approximation set. Therefore, we cannot expect that $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$ decreases linearly with respect to $1/\kappa$ as $\dim_{\mathrm{H}} \mathcal{L}^\kappa(x)$ does. The main new ingredient of this paper is the difficult upper bound for $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$ when $1/\kappa < \dim_{\mathrm{H}} \mu_\phi$. To overcome the difficulty, we fully develop and combine the methods in [**13**, **20**].

To illustrate our main theorem, let us give some examples.

Example 1. Suppose that T is the doubling map and $\mu_\phi := \lambda$ is the Lebesgue measure. Applying Theorem 1.2, we have that for λ-a.e. x,

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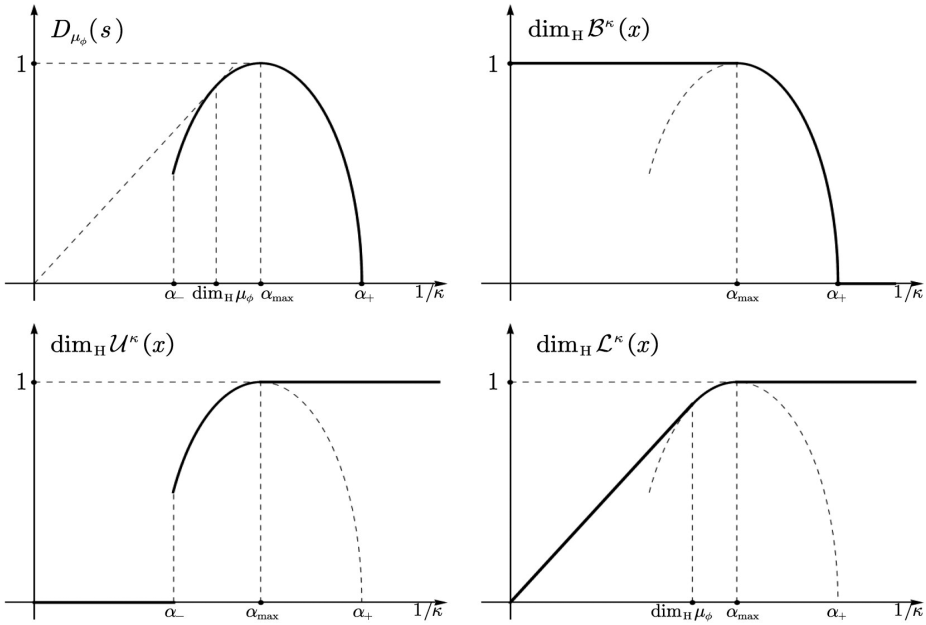

FIGURE 1. The multifractal spectrum of μ_ϕ and the maps $1/\kappa \mapsto \dim_{\rm H} \mathcal{B}^\kappa(x)$, $1/\kappa \mapsto \dim_{\rm H} \mathcal{U}^\kappa(x)$, and $1/\kappa \mapsto \dim_{\rm H} \mathcal{L}^\kappa(x)$.

$$\dim_{\rm H} \mathcal{U}^\kappa(x) = \begin{cases} 0 & \text{if } 1/\kappa \in (0,1), \\ 1 & \text{if } 1/\kappa \in (1,+\infty). \end{cases}$$

The Lebesgue measure is monofractal and hence the corresponding multifractal spectrum D_λ is discontinuous at 1. Theorem 1.2 fails to provide any metric statement for the set $\mathcal{U}^1(x)$ for λ-a.e. x. However, we can conclude that $\mathcal{U}^1(x)$ is a Lebesgue null set for λ-a.e. x from the Fubini theorem and a zero-one law established in [15, Theorem 2.1]. Further, by Theorem 1.1, $\dim_{\rm H} \mathcal{U}^1(x)$ is Lebesgue almost surely a constant.

Some sets similar to $\mathcal{U}^1(x)$ in Example 1 have recently been studied by Koivusalo, Liao, and Persson [20]. In their paper, instead of the orbit $(T^n x)_{n\ge 0}$, they investigated the sets of points uniformly approximated by an independent and identically distributed sequence $(\omega_n)_{n\ge 1}$. Specifically, they showed that with probability one, the lower bound of the Hausdorff dimension of the set

$$\{y \in [0,1] : \text{ for all } N \gg 1, \text{ there exists } n \le N, \text{ such that } |\omega_n - y| < 1/N\}$$

is larger than 0.217 744 429 848 5995 [20, Theorem 5].

Example 2. Let $p \in (1/2, 1)$. Suppose that T is the doubling map on [0, 1] and μ_p is the $(p, 1-p)$ Bernoulli measure. It is known that the multifractal spectrum D_{μ_p} is continuous on $(0, +\infty)$ and attains its unique maximal value 1 at $-\log_2(p(1-p))/2$. Theorem 1.2 then gives that for μ_p-a.e. x,

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$$\dim_{\mathrm{H}} \mathcal{U}^\kappa(x) = \begin{cases} D_{\mu_p}(1/\kappa) & \text{if } 1/\kappa \in \left(0, \frac{-\log_2(p(1-p))}{2}\right), \\ 1 & \text{if } 1/\kappa \in \left[\frac{-\log_2(p(1-p))}{2}, +\infty\right). \end{cases}$$

Our paper is organized as follows. We start in §2 with some preparations on an expanding Markov map, and then use ergodic theory to give a proof of Theorem 1.1. Section 3 contains some recollections on multifractal analysis and a variational principle which are essential in the proof of Theorem 1.2. Section 4 describes some relations among hitting time, approximating rate, and local dimension of μ_ϕ. From these relations, we then derive items (1), (2), and (4) of Theorem 1.2 in §5.1, as well as the lower bound of $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$ in §5.2. In the same §5.2, we establish the upper bound of $\dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$, which is arguably the most substantial part.

2. *Basic definitions and the proof of Theorem 1.1*

2.1. *Covering of* $[0, 1]$ *by basic intervals.* Let T be an expanding Markov map as defined in Definition 1.1. For each $(i_1 i_2 \cdots i_n) \in \{0, 1, \ldots, Q-1\}^n$, we call

$$I(i_1 i_2 \cdots i_n) := I(i_1) \cap T^{-1}(I(i_2)) \cap \cdots \cap T^{-n+1}(I(i_n))$$

a basic interval of generation n. It is possible that $I(i_1 i_2 \cdots i_n)$ is empty for some $(i_1 i_2 \cdots i_n) \in \{0, 1, \ldots, Q-1\}^n$. The collection of non-empty basic intervals of a given generation n will be denoted by Σ_n. Let $\mathcal{E}$ denote the set of endpoints of basic intervals. The set $\mathcal{E}$ is a countable set, so $\dim_{\mathrm{H}} \mathcal{E} = 0$. For any $x \in [0, 1] \setminus \mathcal{E}$, we denote $I_n(x)$ the unique basic interval $I \in \Sigma_n$ containing x.

By the definition of an expanding Markov map, we obtain the following bounded distortion property on basic intervals: there is a constant $L > 1$ such that for any $x \in [0, 1] \setminus \mathcal{E}$,

$$\text{for any } n \geq 1, \quad L^{-1}|(T^n)'(x)|^{-1} \leq |I_n(x)| \leq L|(T^n)'(x)|^{-1}, \tag{2.1}$$

where $|I|$ is the length of the interval I. Consequently, we can find two constants $1 < L_1 < L_2$ such that

$$\text{for every } I \in \Sigma_n, \quad L_2^{-n} \leq |I| \leq L_1^{-n}. \tag{2.2}$$

2.2. *Proof of Theorem 1.1.* Let us start with a simple but crucial observation.

LEMMA 2.1. *Let T be an expanding Markov map. For any $x \in [0, 1]$ and $\kappa > 0$, we have $\mathcal{U}^\kappa(x) \setminus \{Tx\} \subset \mathcal{U}^\kappa(Tx)$.*

Proof. Let $y \in \mathcal{U}^\kappa(x) \setminus \{Tx\}$ and let i be the smallest integer satisfying $y \notin B(Tx, i^{-\kappa})$. By the definition of $\mathcal{U}^\kappa(x)$, for any integer $N > i$ large enough, there exists $1 \leq n \leq N$ such that $y \in B(T^n x, N^{-\kappa})$. Moreover, the condition $N > i$ implies that $n \neq 1$, and hence $y \in B(T^{n-1}(Tx), N^{-\kappa})$. This gives that $y \in \mathcal{U}^\kappa(Tx)$; therefore, $\mathcal{U}^\kappa(x) \setminus \{Tx\} \subset \mathcal{U}^\kappa(Tx)$. □

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Recall that a T-invariant measure ν is ergordic if and only if any T-invariant function is constant almost surely. The proof of Theorem 1.1 falls naturally into two parts. We first deal with the Lebesgue measure part.

Proof of Theorem 1.1: Lebesgue measure part. For any $\kappa > 0$, define the function $g_\kappa : x \mapsto \lambda(\mathcal{U}^\kappa(x))$. We claim that g_κ is measurable. In fact, it suffices to observe that

$$g_\kappa(x) = \lim_{i\to\infty} \lim_{m\to\infty} \lambda\bigg(\bigcap_{N=i}^{m} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg)$$

and that, by the piecewise continuity of T^n ($n \geq 1$), the set

$$\bigg\{ x \in [0, 1] : \lambda\bigg(\bigcap_{N=i}^{m} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) > t \bigg\}$$

is measurable for any $t \in \mathbb{R}$.

By Lemma 2.1, we see that $\lambda(\mathcal{U}^\kappa(Tx)) \geq \lambda(\mathcal{U}^\kappa(x))$, or equivalently, $g_\kappa(Tx) \geq g_\kappa(x)$. Since g_κ is measurable, by the fact $0 \leq g_\kappa \leq 1$ and the invariance of ν, we have that g_κ is invariant with respect to ν, that is, $g_\kappa(Tx) = g_\kappa(x)$ for ν-a.e. x. In the presence of ergodicity of ν, g_κ is constant almost surely. □

Proof of Theorem 1.1: Hausdorff dimension part. Fix $\kappa > 0$ and define the function $f_\kappa : x \mapsto \dim_{\mathrm{H}} \mathcal{U}^\kappa(x)$. Again by Lemma 2.1, we see that $f_\kappa(Tx) \geq f_\kappa(x)$. Proceeding in the same way as the Lebesgue measure part, we get that f_κ is constant almost surely provided that f_κ is measurable.

To show that f_κ is measurable, it suffices to prove that for any $t > 0$, the set

$$A(t) := \{x \in [0, 1] : f_\kappa(x) < t\} = \bigg\{ x \in [0, 1] : \dim_{\mathrm{H}}\bigg(\bigcup_{i=1}^{\infty} \bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < t \bigg\}$$

is measurable. Throughout the proof of this part, we will assume that the ball $B(T^n x, N^{-\kappa})$ is closed. This makes the proof achievable and it does not change the Hausdorff dimension of $\mathcal{U}^\kappa(x)$.

By the definition of Hausdorff dimension, a point $x \in A(t)$ if and only if there exists $h \in \mathbb{N}$ such that

$$\mathcal{H}^{t-1/h}\bigg(\bigcup_{i=1}^{\infty} \bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) = 0,$$

or equivalently for all $i \geq 1$,

$$\mathcal{H}^{t-1/h}\bigg(\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) = 0. \tag{2.3}$$

By the definition of Hausdorff measure, equation (2.3) holds if and only if for any $j, k \in \mathbb{N}$,

$$\mathcal{H}_{1/j}^{t-1/h}\bigg(\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < \frac{1}{k}.$$

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Hence, we see that

$$\begin{aligned} A(t) &= \bigcup_{h=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \bigg\{ x \in [0,1] : \mathcal{H}^{t-1/h}_{1/j} \bigg(\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < \frac{1}{k} \bigg\} \\ &=: \bigcup_{h=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} B_{h,i,j,k}. \end{aligned}$$

If $x \in B_{h,i,j,k}$, then there is a countable open cover $\{U_p\}_{p\geq 1}$ with $0 < |U_p| < 1/j$ satisfying

$$\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \subset \bigcup_{p=1}^{\infty} U_p \quad \text{and} \quad \sum_{p=1}^{\infty} |U_p|^{t-1/h} < \frac{1}{k}. \tag{2.4}$$

The set $\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa})$ can be viewed as the intersection of a family of decreasing compact sets $\{\bigcap_{N=i}^{l} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa})\}_{l\geq i}$, and hence there exists $l_0 \geq i$ satisfying

$$\bigcap_{N=i}^{l_0} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \subset \bigcup_{p=1}^{\infty} U_p,$$

which implies

$$\mathcal{H}^{t-1/h}_{1/j} \bigg(\bigcap_{N=i}^{l_0} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < \frac{1}{k}.$$

We then deduce that

$$B_{h,i,j,k} \subset \bigcup_{l=i}^{\infty} \bigg\{ x \in [0,1] : \mathcal{H}^{t-1/h}_{1/j} \bigg(\bigcap_{N=i}^{l} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < \frac{1}{k} \bigg\} =: \bigcup_{l=i}^{\infty} C_{h,i,j,k,l}. \tag{2.5}$$

If $x \in C_{h,i,j,k,l}$ for some $l \geq 1$, then

$$\mathcal{H}^{t-1/h}_{1/j} \bigg(\bigcap_{N=i}^{\infty} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) \leq \mathcal{H}^{t-1/h}_{1/j} \bigg(\bigcap_{N=i}^{l} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa}) \bigg) < \frac{1}{k}.$$

Hence, $x \in B_{i,j,k,l}$ and the reverse inclusion of equation (2.5) is proved.

Notice that $T, T^2, \ldots, T^l$ are continuous on every basic interval of generation greater than l. For any $x \in C_{h,i,j,k,l}$, denote $\mathcal{S}(x) := \bigcap_{N=i}^{l} \bigcup_{n=1}^{N} B(T^n x, N^{-\kappa})$. There is an open cover $(V_p)_{p\geq 1}$ of $\mathcal{S}(x)$ with $0 < |V_p| < 1/j$ and $\sum_p |V_p|^{t-1/h} < 1/k$. Since $\mathcal{S}(x)$ is compact, we see that the distance δ between $\mathcal{S}(x)$ and the complement of $\bigcup_{p\geq 1} V_p$ is positive. By the continuities of $T, T^2, \ldots, T^l$, there is some $l_1 := l_1(\delta) > l$ for which if $y \in I_{l_1}(x)$, then $\mathcal{S}(y)$ is contained in the $\delta/2$-neighborhood of $\mathcal{S}(x)$. Thus, $\mathcal{S}(y)$ can also be covered by $\bigcup_{p\geq 1} V_p$ and $I_{l_1}(x) \subset C_{h,i,j,k,l}$. Finally, $C_{h,i,j,k,l}$ is a union of some basic intervals, which is measurable.

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Now, combining the equalities obtained above, we have

$$A(t)=\bigcup_{h=1}^{\infty}\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{l=i}^{\infty}C_{h,i,j,k,l},$$

which is a Borel measurable set. □

3. *Multifractal properties of Gibbs measures*

In this section, we review some of the standard facts on multifractal properties of Gibbs measures.

Definition 3.1. A Gibbs measure μ_ϕ associated with a potential ϕ is a probability measure satisfying the following: there exists a constant $\gamma>0$ such that

$$\text{for any basic interval } I\in\Sigma_n,\quad \gamma^{-1}\le\frac{\mu_\phi(I)}{e^{S_n\phi(x)-nP(\phi)}}\le\gamma\quad\text{for every } x\in I,$$

where $S_n\phi(x)=\phi(x)+\cdots+\phi(T^{n-1}x)$ is the nth Birkhoff sum of ϕ at x, and $P(\phi)$ is the topological pressure of ϕ defined by

$$P(\phi)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{I\in\Sigma_n}\sup_{x\in I}e^{S_n\phi(x)}.$$

The following theorem ensures the existence and uniqueness of invariant Gibbs measure.

THEOREM 3.1. [6, 33] *Let $T:[0,1]\to[0,1]$ be an expanding Markov map. Then for any Hölder continuous function ϕ, there exists a unique T-invariant Gibbs measure μ_ϕ associated with ϕ. Further, μ_ϕ is ergodic.*

The Gibbs measure μ_ϕ also satisfies the quasi-Bernoulli property (see [21, Lemma 4.1]), that is, for any $n>k\ge1$, for any basic interval $I(i_1\cdots i_n)\in\Sigma_n$, we have

$$\gamma^{-3}\mu_\phi(I')\mu_\phi(I'')\le\mu_\phi(I)=\mu_\phi(I'\cap T^{-k}I'')\le\gamma^3\mu_\phi(I')\mu_\phi(I''),\tag{3.1}$$

where $I'=I(i_1\cdots i_k)\in\Sigma_k$ and $I''=I(i_{k+1}\cdots i_n)\in\Sigma_{n-k}$. It follows immediately that

$$\text{for any } m\ge k,\quad \mu_\phi(I'\cap T^{-m}U)\le\gamma^3\mu_\phi(I')\mu_\phi(U),\tag{3.2}$$

where U is an open set in $[0,1]$.

We adopt the convention that ϕ is normalized, that is, $P(\phi)=0$. If it is not the case, we can replace ϕ by $\phi-P(\phi)$.

Now, let us recall some standard facts on multifractal analysis which aims at studying the multifractal spectrum D_{μ_ϕ}. Some multifractal analysis results were summarized in [21] and we present them as follows. The proofs can also be found in Refs. [4, 7, 10, 25, 26, 31].

THEOREM 3.2. [21, Theorem 2.5] *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure. Then, the following hold.*

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(1) *The function* D_{μ_ϕ} *of* μ_ϕ *is a concave real-analytic map on the interval* (α_-, α_+), *where* α_- *and* α_+ *are defined in equations (1.3) and (1.5), respectively.*
(2) *The spectrum* D_{μ_ϕ} *reaches its maximum value* 1 *at* $\alpha_{\max}$ *defined in equation (1.4).*
(3) *The graph of* D_{μ_ϕ} *and the line with equation* $y = x$ *intersect at a unique point which is* $(\dim_H \mu_\phi, \dim_H \mu_\phi)$. *Moreover,* $\dim_H \mu_\phi$ *satisfies*

$$\dim_H \mu_\phi = \frac{-\int \phi \, d\mu_\phi}{\int \log |T'| \, d\mu_\phi}.$$

PROPOSITION 3.3. [**21**, §2.3] *For every* $q \in \mathbb{R}$, *there is a unique real number* $\eta_\phi(q)$ *such that the topological pressure* $P(-\eta_\phi(q) \log |T'| + q\phi)$ *is equal to* 0. *Further,* $\eta_\phi(q)$ *is real-analytic and concave.*

Remark 6. For simplicity, denote by μ_q the T-invariant Gibbs measure associated with the potential $-\eta_\phi(q) \log |T'| + q\phi$. Certainly, $\eta_\phi(0) = 1$ and the corresponding measure μ_0 is associated with the potential $-\log |T'|$. By the bounded distortion property in equation (2.1), the Gibbs measure μ_0, coinciding with $\mu_{\max}$, is strongly equivalent to the Lebesgue measure λ, which means that there exists a constant $c \geq 1$ such that for every measurable set E, we have

$$c^{-1}\mu_0(E) \leq \lambda(E) \leq c\mu_0(E).$$

For every $q \in \mathbb{R}$, we introduce the exponent

$$\alpha(q) = \frac{-\int \phi \, d\mu_q}{\int \log |T'| \, d\mu_q}. \tag{3.3}$$

PROPOSITION 3.4. [**21**, §2.3] *Let* μ_q *and* $\alpha(q)$ *be as above. The following statements hold.*
(1) *The Gibbs measure* μ_q *is supported by the level set* $\{y : d_{\mu_\phi}(y) = \alpha(q)\}$ *and* $D_{\mu_\phi}(\alpha(q)) = \dim_H \mu_q = \eta_\phi(q) + q\alpha(q)$.
(2) *The map* $\alpha(q)$ *is decreasing, and*

$$\lim_{q \to +\infty} \alpha(q) = \alpha_-, \qquad \lim_{q \to -\infty} \alpha(q) = \alpha_+,$$
$$\alpha(1) = \dim_H \mu_\phi, \qquad \alpha(0) = \alpha_{\max}.$$

(3) *The inverse of* $\alpha(q)$ *exists and is denoted by* $q(\alpha)$. *Moreover,* $q(\alpha) < 0$ *if* $\alpha \in (\alpha_{\max}, \alpha_+)$, *and* $q(\alpha) \geq 0$ *if* $\alpha \in (\alpha_-, \alpha_{\max}]$.

Recall that $\mathcal{E}$ is the set of endpoints of basic intervals which has Hausdorff dimension 0. For a measure ν and a point $y \in [0, 1] \setminus \mathcal{E}$, define the lower and upper Markov pointwise dimensions respectively by

$$\underline{M}_\nu(y) := \liminf_{n\to\infty} \frac{\log \nu(I_n(y))}{\log |I_n(y)|}, \qquad \overline{M}_\nu(y) := \limsup_{n\to\infty} \frac{\log \nu(I_n(y))}{\log |I_n(y)|}.$$

When $\underline{M}_\nu(y) = \overline{M}_\nu(y)$, their common value is denoted by $M_\nu(y)$. By equations (2.1) and (2.2), we have

$$\overline{d}_\nu(y) \leq \overline{M}_\nu(y),$$

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which implies the inclusions

$$\{y : d_\nu(y) = s\} \setminus \mathcal{E} \subset \{y : \underline{d}_\nu(y) \geq s\} \setminus \mathcal{E} \subset \{y : \overline{d}_\nu(y) \geq s\} \setminus \mathcal{E} \subset \{y : \overline{M}_\nu(y) \geq s\}. \tag{3.4}$$

By the Gibbs property of μ_ϕ and the bounded distortion property on basic intervals in equation (2.1), the definitions of Markov pointwise dimensions can be reformulated as

$$\overline{M}_{\mu_\phi}(y) = \limsup_{n\to\infty} \frac{S_n\phi(y)}{S_n(-\log T')(y)} \quad \text{and} \quad M_{\mu_\phi}(y) = \lim_{n\to\infty} \frac{S_n\phi(y)}{S_n(-\log T')(y)}. \tag{3.5}$$

This allows us to derive the following lemma, which is an alternative version of a proposition due to Jenkinson [16, Proposition 2.1]. We omit its proof since the argument is similar.

LEMMA 3.5. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure. Then,*

$$\sup_{y\in[0,1]} \overline{M}_{\mu_\phi}(y) = \sup_{y:\, M_{\mu_\phi}(y) \text{ exists}} M_{\mu_\phi}(y) = \max_{\nu\in\mathcal{M}_{\text{inv}}} \frac{-\int \phi\, d\nu}{\int \log|T'|\, d\nu} = \alpha_+.$$

In particular, for any $s > \alpha_+$,

$$\{y : d_{\mu_\phi}(y) = s\} = \{y : \overline{d}_{\mu_\phi}(y) \geq s\} = \emptyset.$$

We finish the section with a variational principle.

LEMMA 3.6. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure.*

(1) *For every $s < \alpha_{\max}$*, $\dim_H\{y : \underline{d}_{\mu_\phi}(y) \leq s\} = \dim_H\{y : \overline{d}_{\mu_\phi}(y) \leq s\} = D_{\mu_\phi}(s)$.

(2) *For every $s \in (\alpha_{\max}, +\infty) \setminus \alpha_+$*, $\dim_H\{y : \underline{d}_{\mu_\phi}(y) \geq s\} = \dim_H\{y : \overline{d}_{\mu_\phi}(y) \geq s\} = D_{\mu_\phi}(s)$.

Proof. (1) Note that

$$\{y : d_{\mu_\phi}(y) = s\} \subset \{y : \overline{d}_{\mu_\phi}(y) \leq s\} \subset \{y : \underline{d}_{\mu_\phi}(y) \leq s\}.$$

In [21, Proposition 2.8], the leftmost set and the rightmost set were shown to have the same Hausdorff dimension. This together with the above inclusions completes the proof of the first point of the lemma.

(2) When T is the doubling map, the statement was formulated by Fan, Schmeling, and Trobetzkoy [13, Theorem 3.3]. Our proof follows their idea closely, we include it for completeness.

By Lemma 3.5, we can assume without loss of generality that $s < \alpha_+$. The inclusions in equation (3.4) imply the following inequalities:

$$\dim_H\{y : d_{\mu_\phi}(y) = s\} \leq \dim_H\{y : \overline{d}_{\mu_\phi}(y) \geq s\} \leq \dim_H\{y : \overline{M}_{\mu_\phi}(y) \geq s\}.$$

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We turn to prove the reverse inequalites. By Proposition 3.4 and the condition $s > \alpha_{\max}$, there exists a real number $q_s := q(s) < 0$ such that

$$s = \frac{-\int \phi \, d\mu_{q_s}}{\int \log |T'| \, d\mu_{q_s}} \quad \text{and} \quad D_{\mu_\phi}(s) = \dim_{\mathrm{H}} \mu_{q_s} = \eta_\phi(q_s) + q_s s,$$

where μ_{q_s} is the Gibbs measure associated with the potential $-\eta_\phi(q_s) \log |T'| + q_s\phi$. Now let y be any point such that $\overline{M}_{\mu_\phi}(y) \geq s$. By Proposition 3.3, the topological pressure $P(-\eta_\phi(q_s) \log |T'| + q_s\phi)$ is 0. Then, we can apply the Gibbs property of μ_{q_s} and equation (2.1) to yield

$$\begin{aligned}
\underline{M}_{\mu_{q_s}}(y) &= \liminf_{n\to\infty} \frac{\log e^{S_n(-\eta_\phi(q_s) \log |T'| + q_s\phi)(y)}}{\log |I_n(y)|} \\
&= \liminf_{n\to\infty} \bigg(\frac{-\eta_\phi(q_s) \log |(T^n)'(y)|}{\log |I_n(y)|} + q_s \cdot \frac{\log e^{S_n\phi(y)}}{\log |I_n(y)|} \bigg) \\
&= \eta_\phi(q_s) + q_s \cdot \limsup_{n\to\infty} \frac{\log \mu_\phi(I_n(y))}{\log |I_n(y)|} \\
&= \eta_\phi(q_s) + q_s \overline{M}_{\mu_\phi}(y) \\
&\leq \eta_\phi(q_s) + q_s s = D_{\mu_\phi}(s),
\end{aligned}$$

where the inequality holds because $q_s < 0$.

Finally, Billingsley's lemma [5, Lemma 1.4.1] gives

$$\begin{aligned}
\dim_{\mathrm{H}}\{y : \overline{M}_{\mu_\phi}(y) \geq s\} &\leq \dim_{\mathrm{H}}\{y : \underline{M}_{\mu_{q_s}}(y) \leq D_{\mu_\phi}(s)\} \leq D_{\mu_\phi}(s) \\
&= \dim_{\mathrm{H}}\{y : d_{\mu_\phi}(y) = s\}.
\end{aligned}$$

□

4. *Covering questions related to hitting time and local dimension*

In §4.1, we reformulate the uniform approximation set $\mathcal{U}^\kappa(x)$ in terms of hitting time. Thereafter, we relate the first hitting time for shrinking balls to local dimension in §4.2.

4.1. *Covering questions and hitting time.*

Denote $\mathcal{O}^+(x) := \{T^n x : n \geq 1\}$.

Definition 4.1. For every $x, y \in [0, 1]$ and $r > 0$, define the first hitting time of the orbit of x into the ball $B(y, r)$ by

$$\tau_r(x, y) := \inf\{n \geq 1 : T^n x \in B(y, r)\}.$$

Set

$$\underline{R}(x, y) := \liminf_{r\to 0} \frac{\log \tau_r(x, y)}{-\log r} \quad \text{and} \quad \overline{R}(x, y) := \limsup_{r\to 0} \frac{\log \tau_r(x, y)}{-\log r}.$$

For convenience, when $\mathcal{O}^+(x) \cap B(y, r) = \emptyset$, we set $\tau_r(x, y) = \infty$ and $\underline{R}(x, y) = \overline{R}(x, y) = \infty$. If $\underline{R}(x, y) = \overline{R}(x, y)$, we denote the common value by $R(x, y)$.

For any ball $B \subset [0, 1]$, define the first hitting time $\tau(x, B)$ by

$$\tau(x, B) := \inf\{n \geq 1 : T^n x \in B\}.$$

Similarly, we set $\tau(x, B) = \infty$ when $\mathcal{O}^+(x) \cap B = \emptyset$.

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The following lemma exhibits a relation between $\mathcal{U}^\kappa(x)$ and hitting time.

LEMMA 4.1. *For any $\kappa > 0$, we have*

$$\left\{y \in [0,1] : \overline{R}(x,y) > \frac{1}{\kappa}\right\} \subset \mathcal{B}^\kappa(x) \subset \left\{y \in [0,1] : \overline{R}(x,y) \geq \frac{1}{\kappa}\right\},$$
$$\left\{y \in [0,1] : \overline{R}(x,y) < \frac{1}{\kappa}\right\} \subset \mathcal{U}^\kappa(x) \subset \left\{y \in [0,1] : \overline{R}(x,y) \leq \frac{1}{\kappa}\right\}.$$

Proof. The top left and bottom right inclusions imply one another. Let us prove the bottom right inclusion. Suppose that $y \in \mathcal{U}^\kappa(x)$. Then for all large enough N, there is an $n \leq N$ such that $T^n x \in B(y, N^{-\kappa})$. Thus, $\tau_{N^{-\kappa}}(x,y) \leq N$ for all N large enough, which implies $\overline{R}(x,y) \leq 1/\kappa$.

The top right and bottom left inclusions imply one another. So, it remains to prove the bottom left inclusion. Consider y such that $\overline{R}(x,y) < 1/\kappa$. If $y \in \mathcal{O}^+(x)$ with $y = T^{n_0}x$ for some $n_0 \geq 1$, then the system

$$|T^n x - y| = |T^n x - T^{n_0}x| < N^{-\kappa} \quad \text{and} \quad 1 \leq n \leq N$$

always has a trivial solution $n = n_0$ for all $N \geq n_0$. Therefore, $y \in \mathcal{U}^\kappa(x)$. Now assume that $y \notin \mathcal{O}^+(x)$. By the definition of $\overline{R}(x,y)$, there is a positive real number $r_0 < 1$ such that

$$\tau_r(x,y) < r^{-1/\kappa} \quad \text{for all } 0 < r < r_0.$$

Denote $n_r := \tau_r(x,y)$ for all $0 < r < r_0$. Since $y \notin \mathcal{O}^+(x)$, the family of positive integers $\{n_r : 0 < r < r_0\}$ is unbounded. For each $N > r_0^{-1/\kappa}$, denote $t := N^{-\kappa}$. The definition of n_t implies that

$$T^{n_t}x \in B(y,t) = B(y, N^{-\kappa}).$$

We conclude $y \in \mathcal{U}^\kappa(x)$ by noting that $n_t < t^{-1/\kappa} = N$. □

4.2. *Relation between hitting time and local dimension.* As Lemma 4.1 shows, we need to study the hitting time $\overline{R}(x,y)$ of the Gibbs measure μ_ϕ. We will prove that the hitting time is related to local dimension when the measure is exponential mixing.

Definition 4.2. A T-invariant measure ν is exponential mixing if there exist two constants $C > 0$ and $0 < \beta < 1$ such that for any ball A and any Borel measurable set B,

$$|\nu(A \cap T^{-n}B) - \nu(A)\nu(B)| \leq C\beta^n \nu(B). \tag{4.1}$$

THEOREM 4.2. [2, 22, 23, 27] *The T-invariant Gibbs measure μ_ϕ associated with a Hölder continuous potential ϕ of an expanding Markov map T is exponential mixing.*

The exponential mixing property allows us to apply the following theorem which describes a relation between hitting time and local dimension of invariant measure.

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THEOREM 4.3. [14] *Let (X, T, ν) be a measure-theoretic dynamical system. If ν is superpolynomial mixing and if $d_\nu(y)$ exists, then for ν-a.e.x, we have*

$$R(x, y) = d_\nu(y).$$

It should be noticed that the superpolynomial mixing property is much weaker than the exponential mixing property.

Now, we turn to the study of the expanding Markov map T on the interval [0, 1]. An application of Fubini's theorem yields the following corollary.

COROLLARY 4.4. [21, Corollary 3.8] *Let T be an expanding Markov map. Let μ_ϕ and μ_ψ be two T-invariant Gibbs probability measures on* [0, 1] *associated with Hölder potentials ϕ and ψ, respectively. Then,*

$$\text{for } \mu_\phi \times \mu_\psi\text{-a.e. } (x, y), \quad R(x, y) = d_{\mu_\phi}(y) = \frac{-\int \phi \, d\mu_\psi}{\int \log |T'| \, d\mu_\psi}.$$

5. *The studies of $\mathcal{B}^\kappa(x)$ and $\mathcal{U}^\kappa(x)$*

5.1. *The study of $\mathcal{B}^\kappa(x)$.* In this subsection, we are going to prove Theorem 1.2 except for item (3). Let us start with the lower bound for $\dim_H \mathcal{B}^\kappa(x)$.

LEMMA 5.1. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure. For any $\kappa > 0$, the following hold.*

(a) *If $1/\kappa \in (0, \alpha_{\max})$, then $\lambda(\mathcal{B}^\kappa(x)) = 1$ for μ_ϕ-a.e.x.*

(b) *If $1/\kappa \in [\alpha_{\max}, +\infty) \setminus \{\alpha_+\}$, then $\dim_H \mathcal{B}^\kappa(x) \geq D_{\mu_\phi}(1/\kappa)$ for μ_ϕ-a.e.x.*

Proof. (a) Let $1/\kappa \in (0, \alpha_{\max})$. As already observed in §3, the Gibbs measure μ_0 associated with $\log |T'|$ is strongly equivalent to the Lebesgue measure λ. Thus, a set F has full μ_0-measure if and only if F has full λ-measure. Corollary 4.4 implies that

$$\text{for } \mu_\phi \times \mu_0\text{-a.e. } (x, y), \quad R(x, y) = d_{\mu_\phi}(y) = \frac{-\int \phi \, d\mu_0}{\int \log |T'| \, d\mu_0} = \alpha_{\max}.$$

By Fubini's theorem, for μ_ϕ-a.e. x, the set $\{y : R(x, y) = d_{\mu_\phi}(y) = \alpha_{\max}\}$ has full μ_0-measure. Then for μ_ϕ-a.e. x, we have

$$\mu_0(\{y : \overline{R}(x, y) > 1/\kappa\}) \geq \mu_0(\{y : R(x, y) = d_{\mu_\phi}(y) = \alpha_{\max}\}) = 1.$$

By Lemma 4.1, we arrive at the conclusion.

(b) By Lemma 3.5, the level set $\{y : d_{\mu_\phi}(y) = 1/\kappa\}$ is empty if $1/\kappa > \alpha_+$. Thus, $D_{\mu_\phi}(1/\kappa) = 0$ and therefore $\dim_H \mathcal{B}^\kappa(x) \geq 0 = D_{\mu_\phi}(1/\kappa)$ trivially holds for all $1/\kappa > \alpha_+$.

Let $1/\kappa \in [\alpha_{\max}, \alpha_+)$. We can suppose that $\alpha_{\max} \neq \alpha_+$, since otherwise $[\alpha_{\max}, \alpha_+) = \emptyset$ and there is nothing to prove. For any $s \in (1/\kappa, \alpha_+)$, by Proposition 3.4, there exists a real number $q_s := q(s)$ such that

$$s = \frac{-\int \phi \, d\mu_{q_s}}{\int \log |T'| \, d\mu_{q_s}} \quad \text{and} \quad \mu_{q_s}(\{y : d_{\mu_\phi}(y) = s\}) = 1.$$

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Applying Corollary 4.4, we obtain

$$\text{for } \mu_\phi \times \mu_{q_s}\text{-a.e. } (x, y), \quad R(x, y) = d_{\mu_\phi}(y) = \frac{-\int \phi \, d\mu_{q_s}}{\int \log |T'| \, d\mu_{q_s}} = s.$$

It follows from Fubini's theorem that, for μ_ϕ-a.e. x, the set $\{y : R(x, y) = d_{\mu_\phi}(y) = s\}$ has full μ_{q_s}-measure. Consequently, for μ_ϕ-a.e. x,

$$\begin{aligned} \dim_{\mathrm{H}}\{y : \overline{R}(x, y) > 1/\kappa\} &\geq \dim_{\mathrm{H}}\{y : R(x, y) = d_{\mu_\phi}(y) = s\} \\ &\geq \dim_{\mathrm{H}} \mu_{q_s} = D_{\mu_\phi}(s). \end{aligned}$$

We conclude by noting that $s \in (1/\kappa, \alpha_+)$ is arbitrary and D_{μ_ϕ} is continuous on $[\alpha_{\max}, \alpha_+)$. □

We are left to determine the upper bound of $\dim_{\mathrm{H}} \mathcal{B}^\kappa(x)$. The following four lemmas were initially proved by Fan, Schmeling, and Troubetzkoy [13] for the doubling map, and later by Liao and Sereut [21] in the context of expanding Markov maps. We follow their ideas and demonstrate more general results. In Lemmas 5.2–5.4, we will not assume that T is an expanding Markov map.

LEMMA 5.2. *Let T be a map on* $[0, 1]$ *and ν be a T-invariant exponential mixing measure. Let $A_1, A_2, \ldots, A_k$ be k subsets of* $[0, 1]$ *such that each A_i is a union of at most m disjoint balls. Then,*

$$\prod_{i=1}^{k} \bigg(1 - \frac{mC\beta^d}{\nu(A_i)}\bigg) \leq \frac{\nu(A_1 \cap T^{-d} A_2 \cap \cdots \cap T^{-d(k-1)} A_k)}{\nu(A_1)\nu(A_2) \cdots \nu(A_k)} \leq \prod_{i=1}^{k} \bigg(1 + \frac{mC\beta^d}{\nu(A_i)}\bigg),$$

where β is the constant appearing in equation (4.1).

Proof. Since each A_i is a union of at most m disjoint balls, the exponential mixing property of ν gives that, for every $d \geq 1$,

$$|\nu(A_i \cap T^{-d} B) - \nu(A_i)\nu(B)| \leq mC\beta^d \nu(B), \tag{5.1}$$

where B is a Borel measurable set. In particular, defining

$$B_i = A_i \cap T^{-d} A_{i+1} \cap \cdots \cap T^{-d(k-i)} A_k,$$

we get, for any $i < k$,

$$|\nu(A_i \cap T^{-d}(B_{i+1})) - \nu(A_i)\nu(B_{i+1})| \leq mC\beta^d \nu(B_{i+1}).$$

The above inequality can be written as

$$1 - \frac{mC\beta^d}{\nu(A_i)} \leq \frac{\nu(A_i \cap T^{-d} B_{i+1})}{\nu(A_i)\nu(B_{i+1})} \leq 1 + \frac{mC\beta^d}{\nu(A_i)}.$$

Multiplying over all $i \leq k$ and using the identity

$$B_{i+1} = A_{i+1} \cap T^{-d} B_{i+2},$$

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we have

$$\prod_{i=1}^{k}\left(1-\frac{mC\beta^d}{\nu(A_i)}\right)\le\frac{\nu(A_1\cap T^{-d}A_2\cap\cdots\cap T^{-d(k-1)}A_k)}{\nu(A_1)\nu(A_2)\cdots\nu(A_k)}\le\prod_{i=1}^{k}\left(1+\frac{mC\beta^d}{\nu(A_i)}\right). \qquad \square$$

The following lemma illustrates that balls with small local dimension for exponential mixing measure are hit with big probability.

LEMMA 5.3. *Let T be a map on $[0, 1]$ and ν be a T-invariant exponential mixing measure. Let h and ϵ be two positive real numbers. For each $n \in \mathbb{N}$, consider $N \le 2^n$ distinct balls $B_1, \ldots, B_N$ satisfying $|B_i| = 2^{-n}$ and $\nu(B_i) \ge 2^{-n(h-\epsilon)}$ for all $1 \le i \le N$. Set*

$$\mathcal{C}_{n,N,h} = \{x \in [0, 1] : \textit{ there exists } 1 \le i \le N \textit{ such that } \tau(x, B_i) \ge 2^{nh}\}.$$

Then there exists an integer $n_h \in \mathbb{N}$ independent of N such that

$$\textit{for every } n \ge n_h, \quad \nu(\mathcal{C}_{n,N,h}) \le 2^{-n}.$$

Proof. For each $i \le N$, let

$$\Delta_i := \{x \in [0, 1] : \quad \text{for all } k \le 2^{nh}, T^k x \notin B_i\}.$$

Obviously, we have $\mathcal{C}_{n,N,h} = \bigcup_{i=1}^{N} \Delta_i$, so it suffices to bound from above each $\nu(\Delta_i)$. Pick an integer ω such that $\omega > \log_{\beta^{-1}} 2^h$. Let $k = [2^{nh}/(\omega n)]$ be the integer part of $2^{nh}/(\omega n)$. Then,

$$\Delta_i \subset \bigcap_{j=1}^{k}\{x \in [0, 1] : T^{j\omega n}x \notin B_i\} = \bigcap_{j=1}^{k} T^{-j\omega n}B_i^c.$$

Since $\omega > \log_{\beta^{-1}} 2^h$, there is an n_h large enough such that for any $n \ge n_h$,

$$2C\beta^{\omega n} < 2^{-nh-1} \le \nu(B_i)/2 \tag{5.2}$$

and

$$2^{n+1}\exp\left(\frac{-2^{n\epsilon}}{2\omega n}\right) \le 2^{-n}. \tag{5.3}$$

Now applying Lemma 5.2 to $A_l = B_i$ for all $l \le N$ and to $m = 2$, we conclude from equation (5.2) that

$$\begin{aligned}\nu(\Delta_i) \le \nu\Big(\bigcap_{j=1}^{k} T^{-j\omega n}B_i^c\Big) &\le (\nu(B_i^c) + 2C\beta^{\omega n})^k\\ &\le (1-\nu(B_i)/2)^k\\ &\le (1-2^{-n(h-\epsilon)-1})^{2^{nh}/(\omega n)-1}\\ &= (1-2^{-n(h-\epsilon)-1})^{-1}\exp\left(\frac{2^{nh}\log(1-2^{-n(h-\epsilon)-1})}{\omega n}\right)\\ &\le 2\exp\left(\frac{-2^{n\epsilon}}{2\omega n}\right).\end{aligned}$$

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By equation (5.3),

$$\nu(C_{n,N,h}) \le \sum_{i=1}^{N} \nu(B_i) \le 2^{n+1} \exp\left(\frac{-2^{n\epsilon}}{2\omega n}\right) \le 2^{-n}. \qquad \square$$

Let us recall that $\{y : \overline{R}(x, y) \ge s\}$ is a random set depending on the random element x, but $\{y : \overline{d}_{\mu_\phi}(y) \ge s\}$ is independent of x. The following lemma reveals a connection between the random set $\{y : \overline{R}(x, y) \ge s\}$ and the deterministic set $\{y : \overline{d}_{\mu_\phi}(y) \ge s\}$.

LEMMA 5.4. *Let T be a map on $[0, 1]$ and ν be a T-invariant exponential mixing measure. Let $s \ge 0$. Then for ν-a.e.x,*

$$\{y : \overline{R}(x, y) \ge s\} \subset \{y : \overline{d}_\nu(y) \ge s\}.$$

Proof. The case $s = 0$ is obvious. We therefore assume $s > 0$. For any integer $n \ge 1$, let

$$\mathcal{R}_{n,s,\epsilon}(x) = \{y : \tau(x, B(y, 2^{-n+1})) \ge 2^{n(s-\epsilon)}\},$$
$$\mathcal{E}_{n,s,\epsilon} = \{y : \nu(B(y, 2^{-n})) \le 2^{-n(s-2\epsilon)}\}.$$

By definition, $y \in \{y : \overline{R}(x, y) \ge s\}$ if and only if for any $\epsilon > 0$, there exist infinitely many integers n such that

$$\frac{\log \tau(x, B(y, 2^{-n+1}))}{\log 2^n} \ge s - \epsilon.$$

Hence, we have

$$\{y : \overline{R}(x, y) \ge s\} = \bigcap_{\epsilon>0} \limsup_{n\to\infty} \mathcal{R}_{n,s,\epsilon}(x). \tag{5.4}$$

Similarly,

$$\{y : \overline{d}_{\mu_\phi}(y) \ge s\} = \bigcap_{\epsilon>0} \limsup_{n\to\infty} \mathcal{E}_{n,s,\epsilon}.$$

Thus, it is sufficient to prove that, for ν-a.e. x, there exists some integer $n(x)$ such that

$$\text{for all } n \ge n(x), \quad \mathcal{R}_{n,s,\epsilon}(x) \subset \mathcal{E}_{n,s,\epsilon}, \tag{5.5}$$

or equivalently,

$$\text{for all } n \ge n(x), \quad \mathcal{E}^c_{n,s,\epsilon}(x) \subset \mathcal{R}^c_{n,s,\epsilon}. \tag{5.6}$$

Notice that $\mathcal{E}^c_{n,s,\epsilon}$ can be covered by $N \le 2^n$ balls with center in $\mathcal{E}^c_{n,s,\epsilon}$ and radius 2^{-n}. Let $\mathcal{F}_{n,s,\epsilon} := \{B_1, B_2, \ldots, B_N\}$ be the collection of these balls. By definition, we have $\nu(B_i) \le 2^{-n(s-2\epsilon)}$. Applying Lemma 5.3 to the collection $\mathcal{F}_{n,s,\epsilon}$ of balls and to $h = s - \epsilon$, we see that

$$\sum_{n\ge n_h} \nu(\{x : \text{ there exists } B \in \mathcal{F}_{n,s,\epsilon} \text{ such that } \tau(x, B) \ge 2^{n(s-\epsilon)}\}) \le \sum_{n\ge n_h} 2^{-n} < \infty.$$

By the Borel–Cantelli lemma, for ν-a.e. x, there exists an integer $n(x)$ such that

$$\text{for all } n \ge n(x), \ \text{ for all } B \in \mathcal{F}_{n,s,\epsilon}, \quad \tau(x, B) < 2^{n(s-\epsilon)}.$$

If $y \in B$ for some $B \in \mathcal{F}_{n,s,\epsilon}$ and $n \geq n(x)$, then $B \subset B(y, 2^{-n+1})$, which implies that $\tau(x, B(y, 2^{-n+1})) < \tau(x, B)$. We then deduce that B is included in $\mathcal{R}^c_{n,s,\epsilon}$. This yields $\mathcal{E}^c_{n,s,\epsilon} \subset \mathcal{R}^c_{n,s,\epsilon}$, which is what we want. □

Remark 7. With the notation in Lemma 5.4, proceeding with the same argument as equation (5.4), we have

$$\{y : \underline{R}(x, y) \geq s\} = \bigcap_{\epsilon>0} \liminf_{n\to\infty} \mathcal{R}_{n,s,\epsilon}(x) \quad \text{and} \quad \{y : \underline{d}_{\mu_\phi}(y) \geq s\} = \bigcap_{\epsilon>0} \liminf_{n\to\infty} \mathcal{E}_{n,s,\epsilon}.$$

It then follows from equation (5.5) that for ν-a.e. x,

$$\{y : \underline{R}(x, y) \geq s\} \subset \{y : \underline{d}_\nu(y) \geq s\}.$$

Applying Lemma 5.4 to the Gibbs measure μ_ϕ, we get the following upper bound.

LEMMA 5.5. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure. Suppose $1/\kappa \geq \alpha_{\max}$. Then for μ_ϕ-a.e.x,*

$$\dim_H \mathcal{B}^\kappa(x) \leq D_{\mu_\phi}(1/\kappa).$$

Moreover, if $1/\kappa > \alpha_+$, then for μ_ϕ-a.e.x,

$$\mathcal{B}^\kappa(x) = \emptyset.$$

Proof. Recall that Lemma 4.1 asserts that

$$\mathcal{B}^\kappa(x) \subset \{y : \overline{R}(x, y) \geq 1/\kappa\} \cup \mathcal{O}^+(x).$$

A direct application of Proposition 3.6 and Lemma 5.4 yields the first conclusion.

The second conclusion follows from Lemmas 3.5 and 4.1. □

Collecting the results obtained in this subsection, we can prove Theorem 1.2 except for item (3).

Proof of the items (1), (2), and (4) of Theorem 1.2. Combining with Lemmas 5.1 and 5.5, we get the desired result. □

5.2. *The study of $\mathcal{U}^\kappa(x)$.* In this subsection, we prove the remaining part of Theorem 1.2, that is, item (3). We begin by showing the lower bound of $\dim_H \mathcal{U}^\kappa(x)$, which may be proved in much the same way as Lemma 5.1.

LEMMA 5.6. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure.*

(a) *If $1/\kappa \in (0, \alpha_{\max}] \setminus \{\alpha_-\}$, then $\dim_H \mathcal{U}^\kappa(x) \geq D_{\mu_\phi}(1/\kappa)$ for μ_ϕ-a.e.x.*

(b) *If $1/\kappa \in (\alpha_{\max}, +\infty)$, then $\dim_H \mathcal{U}^\kappa(x) = 1$ for μ_ϕ-a.e.x.*

Proof. (a) By Lemma 3.6, the Hausdorff dimension of the level set $\{y : d_{\mu_\phi}(y) = 1/\kappa\}$ is zero if $1/\kappa < \alpha_-$. Therefore, $\dim_H \mathcal{U}^\kappa(x) \geq 0 = D_{\mu_\phi}(1/\kappa)$.

The remaining case $1/\kappa \in (\alpha_-, \alpha_{\max}]$ holds by the same reasoning as proving Lemma 5.1(b).

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(b) Observe that the full Lebesgue measure statement implies the full Hausdorff dimension statement. It follows from item (2) of Theorem 1.2 that $\dim_{\mathrm{H}} \mathcal{U}^{\kappa}(x) = 1$ when $1/\kappa \in (\alpha_{\max}, +\infty)$. □

It is left to show the upper bound of $\dim_{\mathrm{H}} \mathcal{U}^{\kappa}(x)$ when $1/\kappa \leq \alpha_{\max}$. The proof combines the methods developed in [13, §7] and [20, Theorem 8]. Heuristically, the larger the local dimension of a point is, the less likely it is to be hit.

LEMMA 5.7. *Let T be an expanding Markov map. Let ϕ be a Hölder continuous potential and μ_ϕ be the corresponding Gibbs measure. Let $1/\kappa \leq \alpha_{\max}$, then for μ_ϕ-a.e. x,*

$$\dim_{\mathrm{H}} \mathcal{U}^{\kappa}(x) \leq D_{\mu_\phi}(1/\kappa).$$

Proof. The proof will be divided into two steps.

Step 1. Given any $a > 1/\kappa$, we are going to prove that

$$\dim_{\mathrm{H}}(\mathcal{U}^{\kappa}(x) \cap \{y : \underline{d}_{\mu_\phi}(y) > a\}) = 0 \quad \text{for } \mu_\phi\text{-a.e. } x. \tag{5.7}$$

Suppose now that equation (5.7) is established. Let $(a_m)_{m \geq 1}$ be a monotonically decreasing sequence of real numbers converging to $1/\kappa$. Applying equation (5.7) to each a_m yields a full μ_ϕ-measure set corresponding to a_m. Then, by taking the intersection of these countable full μ_ϕ-measure sets, we conclude from the countable stability of Hausdorff dimension that

$$\dim_{\mathrm{H}}(\mathcal{U}^{\kappa}(x) \cap \{y : \underline{d}_{\mu_\phi}(y) > 1/\kappa\}) = 0 \quad \text{for } \mu_\phi\text{-a.e. } x.$$

As a result, by Lemma 3.6, for μ_ϕ-a.e. x,

$$\begin{aligned}
\dim_{\mathrm{H}} \mathcal{U}^{\kappa}(x) &= \dim_{\mathrm{H}}(\mathcal{U}^{\kappa}(x) \cap (\{y : \underline{d}_{\mu_\phi}(y) \leq 1/\kappa\} \cup \{y : \underline{d}_{\mu_\phi}(y) > 1/\kappa\})) \\
&= \dim_{\mathrm{H}}(\mathcal{U}^{\kappa}(x) \cap \{y : \underline{d}_{\mu_\phi}(y) \leq 1/\kappa\}) \\
&\leq \dim_{\mathrm{H}}\{y : \underline{d}_{\mu_\phi}(y) \leq 1/\kappa\} = D_{\mu_\phi}(1/\kappa).
\end{aligned}$$

This clearly yields the lemma.

Choose $b \in (1/\kappa, a)$. Put $A_n := \{y : \mu_\phi(B(y, r)) < r^b \text{ for all } r < 2^{-n}\}$. By the definition of $\underline{d}_{\mu_\phi}(y)$, we have

$$\{y : \underline{d}_{\mu_\phi}(y) > a\} \subset \bigcup_{n=1}^{\infty} A_n.$$

Thus, equation (5.7) is reduced to showing that for any $n \geq 1$,

$$\dim_{\mathrm{H}}(\mathcal{U}^{\kappa}(x) \cap A_n) = 0 \quad \text{for } \mu_\phi\text{-a.e. } x. \tag{5.8}$$

Step 2. The next objective is to prove equation (5.8).

Fix $n \geq 1$. Let $\epsilon > 0$ be arbitrary. Choose a large integer $l \geq n$ with

$$12 \times 2^{-\kappa l} < 2^{-n} \quad \text{and} \quad \gamma^3 12^b 2^{(1-b\kappa)l} < \epsilon, \tag{5.9}$$

where the constant γ is defined in equation (3.1). Let $\theta_j = [\kappa j \log_{L_1} 2] + 1$, where L_1 is given in equation (2.2). Then, by equation (2.2), the length of each basic interval of

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generation θ_j is smaller than $2^{-\kappa j}$. Recall that $\mathcal{E}$ is the set of endpoints of basic intervals which is countable. For any $x \in [0, 1] \setminus \mathcal{E}$, define

$$\mathcal{I}_j(x) := \bigcup_{J \in \Sigma_{\theta_j} : d(I_{\theta_j}(x), J) < 2^{-\kappa j}} J,$$

where $d(\cdot, \cdot)$ is the Euclidean metric. Clearly, $\mathcal{I}_j(x)$ covers the ball $B(x, 2^{-\kappa j})$ and is contained in $B(x, 3 \times 2^{-\kappa j})$. Moreover, if $I_{\theta_j}(x) = I_{\theta_j}(y)$, then $\mathcal{I}_j(x) = \mathcal{I}_j(y)$. With the notation $\mathcal{I}_j(x)$, we consider the set

$$G_{l,i}(x) = A_n \cap \bigg(\bigcap_{j=l}^{i} \bigcup_{k=1}^{2^j} \mathcal{I}_j(T^k x) \bigg).$$

The advantage of using $\mathcal{I}_j(x)$ rather than $B(x, 2^{-\kappa j})$ is that the map $x \mapsto G_{l,i}(x)$ is constant on each basic interval of generation $2^i + \theta_i$. We are going to construct inductively a cover of $G_{l,i}(x)$ by the family $\{B(T^k x, 3 \times 2^{-\kappa i}) : k \in S_i(x)\}$ of balls, where $S_i(x) \subset \{1, 2, \ldots, 2^i\}$.

For $i = l$, we let $S_l(x) \subset \{1, 2, \ldots, 2^l\}$ consist of those $k \le 2^l$ such that $\mathcal{I}_l(T^k x)$ intersects A_n. Suppose now that $S_i(x)$ has been defined. We define $S_{i+1}(x)$ to consist of those $k \le 2^{i+1}$ such that $\mathcal{I}_{i+1}(T^k x)$ intersects $G_{l,i}(x)$. Then the family $\{B(T^k x, 3 \times 2^{-\kappa(i+1)}) : k \in S_{i+1}(x)\}$ of balls forms a cover of $G_{l,i+1}(x)$, and the construction is completed.

With the aid of the notation $\mathcal{I}_{i+1}(T^k x)$, one can verify that $x \mapsto S_{i+1}(x)$ is constant on each basic interval of generation $2^{i+1} + \theta_{i+1}$. Let $N_{i+1}(x) := \sharp S_{i+1}(x)$, then $x \mapsto N_{i+1}(x)$ is also constant on each basic interval of generation $2^{i+1} + \theta_{i+1}$.

To establish equation (5.8), we need to estimate $N_{i+1}(x)$. For those $k \in S_{i+1}(x) \cap \{1, 2, \ldots, 2^i\}$, since $\mathcal{I}_{i+1}(T^k x) \subset \mathcal{I}_i(T^k x)$ and $G_{l,i}(x) \subset G_{l,i-1}(x)$, we must have that $\mathcal{I}_i(T^k x)$ intersects $G_{l,i-1}(x)$, and hence $k \in S_i(x)$. However, since $\mathcal{I}_{i+1}(T^k x)$ is contained in $B(T^k x, 3 \times 2^{-\kappa(i+1)})$, if $\mathcal{I}_{i+1}(T^k x)$ has non-empty intersection with $G_{l,i}(x)$, then the distance between $T^k x$ and $G_{l,i}(x)$ is less than $3 \times 2^{-\kappa(i+1)}$. In particular,

$$T^k x \in \{y : d(y, G_{l,i}(x)) < 3 \times 2^{-\kappa(i+1)}\} \subset \bigcup_{J \in \Sigma_{\theta_i} : d(J, G_{l,i}(x)) < 3 \times 2^{-\kappa(i+1)}} J. \tag{5.10}$$

Denote the right-hand side union as $\hat{G}_{l,i}(x)$. The set $\hat{G}_{l,i}(x)$ is nothing but the union of cylinders of level θ_i whose distance from $G_{l,i}(x)$ is less than $3 \times 2^{-\kappa(i+1)}$. Thus, by the fact that $x \mapsto G_{l,i}(x)$ is constant on each basic interval of generation $2^i + \theta_i$, we have

$$\hat{G}_{l,i}(x) = \hat{G}_{l,i}(y), \quad \text{whenever } I_{2^i + \theta_i}(x) = I_{2^i + \theta_i}(y). \tag{5.11}$$

According to the above discussion, it holds that

$$N_{i+1}(x) \le N_i(x) + M_{i+1}(x),$$

where $M_{i+1}(x)$ is the number of $2^i < k \le 2^{i+1}$ for which $T^k x$ intersects $\hat{G}_{l,i}(x)$.

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The function $M_{i+1}(x)$ can further be written as

$$M_{i+1}(x)=\sum_{k=2^i+1}^{2^{i+1}}\chi_{\hat{G}_{l,i}(x)}(T^kx)=\sum_{k=2^i+1}^{2^i+\theta_i}\chi_{\hat{G}_{l,i}(x)}(T^kx)+\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\chi_{\hat{G}_{l,i}(x)}(T^kx).$$

Since the countable set $\mathcal{E}$ has zero μ_ϕ-measure, all the functions given above are well defined for μ_ϕ-a.e. x. It follows from the locally constant property in equation (5.11) of $\hat{G}_{l,i(x)}$ that

$$\begin{aligned}\int M_{i+1}(x)d\mu_\phi(x)&=\sum_{k=2^i+1}^{2^i+\theta_i}\int\chi_{\hat{G}_{l,i}(x)}(T^kx)d\mu_\phi(x)+\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\int\chi_{\hat{G}_{l,i}(x)}(T^kx)d\mu_\phi(x)\\&\le\theta_i+\sum_{J\in\Sigma_{2^i+\theta_i}}\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\int\chi_J(x)\chi_{\hat{G}_{l,i}(x)}(T^kx)d\mu_\phi(x)\\&=\theta_i+\sum_{J\in\Sigma_{2^i+\theta_i}}\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\int\chi_J(x)\chi_{\hat{G}_{l,i}(x_J)}(T^kx)d\mu_\phi(x),\end{aligned}\tag{5.12}$$

where x_J is any fixed point of J. Now the task is to deal with the right-hand side summation. We deduce from the quasi-Bernoulli property in equation (3.2) of μ_ϕ that for each $k>2^i+\theta_i$,

$$\int\chi_J(x)\chi_{\hat{G}_{l,i}(x_J)}(T^kx)d\mu_\phi(x)=\mu_\phi(J\cap T^{-k}(\hat{G}_{l,i}(x_J)))\le\gamma^3\mu_\phi(J)\mu_\phi(\hat{G}_{l,i}(x_J)).\tag{5.13}$$

Since $G_{l,i}(x_J)$ can be covered by the family $\{B(T^kx_J,3\times2^{-\kappa i}):k\in S_i(x_J)\}$ of balls, then by equation (5.10), the family $\mathcal{F}_i(x_J):=\{B(T^kx_J,6\times2^{-\kappa i}):k\in S_i(x_J)\}$ of enlarged balls forms a cover of $\hat{G}_{l,i}(x_J)$. Observe that each enlarged ball $B\in\mathcal{F}_i(x_J)$ intersects A_n, and thus $B\subset B(y,12\times2^{-\kappa i})$ for some $y\in A_n$. Then, by the definition of A_n and equation (5.9),

$$\mu_\phi(B)\le\mu_\phi(B(y,12\times2^{-\kappa i}))\le12^b2^{-b\kappa i}.$$

Accordingly,

$$\mu_\phi(\hat{G}_{l,i}(x_J))\le12^b2^{-b\kappa i}N_i(x_J).\tag{5.14}$$

Recall that $x\mapsto N_i(x)$ is constant on each basic interval of generation $2^i+\theta_i$. Applying the upper bound of equation (5.14) on $\mu_\phi(\hat{G}_{l,i}(x_J))$ to equation (5.13), and then substituting equation (5.13) into equation (5.12), we have

$$\begin{aligned}\int M_{i+1}(x)\,d\mu_\phi(x)&\le\theta_i+\sum_{J\in\Sigma_{2^i+\theta_i}}\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\gamma^3\mu_\phi(\hat{G}_{l,i}(x_J))\mu_\phi(J)\\&\le\theta_i+\sum_{J\in\Sigma_{2^i+\theta_i}}\sum_{k=2^i+\theta_i+1}^{2^{i+1}}\gamma^312^b2^{-b\kappa i}N_i(x_J)\mu_\phi(J)\end{aligned}$$

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$$= \theta_i + \gamma^3 12^b 2^{-b\kappa i}(2^i - \theta_i) \int N_i(x)\, d\mu_\phi(x)$$
$$\leq \theta_i + \epsilon \int N_i(x)\, d\mu_\phi(x),$$

where the last inequality follows from equation (5.9). Since $N_{i+1}(x) \leq N_i(x) + M_{i+1}(x)$, we have

$$\int N_{i+1}(x)\, d\mu_\phi(x) \leq \theta_i + (1+\epsilon) \int N_i(x)\, d\mu_\phi(x). \tag{5.15}$$

Note that equation (5.15) holds for all $i \geq l$ and $N_l(x) \leq 2^l$. Then,

$$\int N_{i+1}(x)\, d\mu_\phi(x) \leq \sum_{k=l}^{i} (1+\epsilon)^{i-k}\theta_k + (1+\epsilon)^{i-l+1} \int N_l(x)\, d\mu_\phi(x)$$
$$< (1+\epsilon)^i (i\theta_i + 2^l).$$

By Markov's inequality,

$$\mu_\phi(\{x : N_{i+1}(x) \geq (1+\epsilon)^{2i}(i\theta_i + 2^l)\})$$
$$\leq \mu_\phi\bigg(\bigg\{x : N_{i+1}(x) \geq (1+\epsilon)^i \int N_{i+1}(x)\, d\mu_\phi(x)\bigg\}\bigg)$$
$$\leq (1+\epsilon)^{-i},$$

which is summable over i. Hence, for μ_ϕ-a.e. x, there is an $i_0(x)$ such that

$$N_{i+1}(x) \leq (1+\epsilon)^{2i}(i\theta_i + 2^l) \tag{5.16}$$

holds for all $i \geq i_0(x)$.

Denote by $F_{l,\epsilon}$ the full measure set on which equation (5.16) holds. Let $x \in F_{l,\epsilon}$. Then, such an $i_0(x)$ exists. With $i \geq i_0(x)$, we may cover the set

$$G_l(x) = A_n \cap \bigg(\bigcap_{j=l}^{\infty} \bigcup_{k=1}^{2^j} \mathcal{I}_j(T^k x) \bigg)$$

by $N_i(x)$ balls of radius $3 \times 2^{-\kappa i}$. Since $\theta_i = [\kappa i \log_{L_1} 2] + 1 \leq ci$ for some $c > 0$, we have

$$\dim_{\mathrm{H}} G_l(x) \leq \limsup_{i\to\infty} \frac{\log N_i(x)}{\log 2^{\kappa i}} \leq \limsup_{i\to\infty} \frac{\log((1+\epsilon)^{2i}(i\theta_i + 2^l))}{\log 2^{\kappa i}} = \frac{2\log(1+\epsilon)}{\kappa \log 2}. \tag{5.17}$$

Let $(\epsilon_m)_{m\geq 1}$ be a monotonically decreasing sequence of real numbers converging to 0. For each ϵ_m, choose an integer l_m satisfying equation (5.9), but with ϵ replaced by ϵ_m. For every $m \geq 1$, by the same reason as equation (5.17), there exists a full μ_ϕ-measure set F_{l_m,ϵ_m} such that

$$\text{for all } x \in F_{l_m,\epsilon_m}, \quad \dim_{\mathrm{H}} G_{l_m}(x) \leq \frac{2\log(1+\epsilon_m)}{\kappa \log 2}.$$

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By taking the intersection of the countable full μ_ϕ-measure sets $(F_{l_m,\epsilon_m})_{m\geq 1}$, and using the fact that $G_l(x)$ is increasing in l, we obtain that for μ_ϕ-a.e. x,

$$\text{for any } l \geq 1, \quad \dim_{\mathrm{H}} G_l(x) \leq \lim_{m\to\infty} \dim_{\mathrm{H}} G_{l_m}(x) = 0.$$

We conclude equation (5.8) by noting that

$$\mathcal{U}^\kappa(x) \cap A_n \subset \bigcup_{l\geq 1} G_l(x). \qquad \square$$

Remark 8. Recall that Lemma 4.1 exhibits a relation between $\mathcal{U}^\kappa(x)$ and hitting time:

$$\{y : \overline{R}(x, y) < 1/\kappa\} \subset \mathcal{U}^\kappa(x) \subset \{y : \overline{R}(x, y) \leq 1/\kappa\}.$$

With this relation in mind, it is natural to investigate the size of the level sets

$$\{y : R(x, y) = 1/\kappa\}, \quad \kappa \in (0, \infty).$$

Lemmas 5.4 and 5.7 together with the inclusions

$$\{y : R(x, y) = 1/\kappa\} \subset \{y : \overline{R}(x, y) \leq 1/\kappa\} \quad \text{and}$$
$$\{y : R(x, y) = 1/\kappa\} \subset \{y : \overline{R}(x, y) \geq 1/\kappa\}$$

give the upper bound for Hausdorff dimension:

$$\dim_{\mathrm{H}}\{y : R(x, y) = 1/\kappa\} \leq D_{\mu_\phi}(1/\kappa) \quad \text{for } \mu_\phi\text{-a.e. } x.$$

The lower bound, coinciding with the upper bound, can be proved by the same argument as that of Lemma 5.1. Thus for μ_ϕ-a.e. x,

$$\dim_{\mathrm{H}}\{y : R(x, y) = 1/\kappa\} = D_{\mu_\phi}(1/\kappa) \quad \text{if } 1/\kappa \notin \{\alpha_-, \alpha_+\}.$$

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