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FINITE p-SOLUBLE GROUPS WITH IRREDUCIBLE MODULAR REPRESENTATIONS OF GIVEN DEGREES

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Let G be a finite group, p be a prime and K be a field of characteristic p. Let

$$K(G) = B_1 \oplus \cdots \oplus B_t$$

be a decomposition of the group ring of G over K as a sum of indecomposable two-sided ideals. An irreducible $K(G)$-module is said to be in the block B_i if it occurs as a composition factor of B_i. The block containing the trivial $K(G)$-module is called the principal block of G.

Let I be a subset of the positive integers with $1 \in I$. We denote by $\mathfrak{X}(I)$ (or by $\mathfrak{X}$ when I is fixed) the class of finite p-soluble groups G such that the dimension of every irreducible $K(G)$-module is in I, and by $\mathfrak{X}_1(I)$ (or by $\mathfrak{X}_1$) the class of finite p-soluble groups G such that the dimension of each irreducible $K(G)$-module in the principal block of G is in I. The object of this note is to investigate the relationship between $\mathfrak{X}(I)$ and $\mathfrak{X}_1(I)$, and to calculate these classes in some simple cases. Denoting by $O_P(G)$ the largest normal p-subgroup of a group G, and by $F_p(G)$ the largest normal subgroup of G which has a normal p-complement, we can now state our main result.

Theorem 1. *Let I be a subset of the positive integers with $1 \in I$. Let $\mathfrak{X}(I)$ and $\mathfrak{X}_1(I)$ be as defined above. Then*

(i) $\mathfrak{X}(I) = \{G \mid G/O_p(G) \cong H/F_p(H)$ *for some* $H \in \mathfrak{X}_1(I)\}$ *and*

(ii) $\mathfrak{X}_1(I) = \{G \mid G/F_p(G) \in \mathfrak{X}(I)\}$.

Proof. (i) Let T be an irreducible $K(G)$-module and suppose $G/O_p(G) \cong H/F_p(H)$ with $H \in \mathfrak{X}_1$. By Lemma 1.2 of (2), the kernel of T contains $O_p(G)$ and so we may regard T as a $K(H)$-module whose kernel contains $F_p(H)$. Lemma 2.3 of (2) now gives that T is in the principal block of $K(H)$. (In fact Lemma 2.3 of (2) is proved under the assumption that K is algebraically closed, but the techniques of (2) yield the general result quite easily). Thus dim $T \in I$ and so $G \in \mathfrak{X}$.

Conversely, let $G \in \mathfrak{X}$ and define $M = G/O_p(G)$. Since every irreducible $K(M)$-module may be regarded as an irreducible $K(G)$-module, $M \in \mathfrak{X}$. By construction $O_p(M) = 1$, so by Satz VI.7.20 of (4) there is a faithful, completely reducible $K(M)$-module B, say, over the field with p

elements. Let H be the semi-direct product MB. A standard argument as used in the proof of (**4**: VI.7.24), for example, shows that $F_p(H)=B$.

We have thus constructed a group H with

$$H/F_p(H)\cong MB/B\cong M=G/O_p(G).$$

By Lemma 2.3 of (**2**), each irreducible $K(H)$-module in the principal blocks of H has $F_p(H)$ in its kernel, and so may be regarded as an irreducible $K(G)$-module. Since $G\in\mathfrak{X}$, we have that $H\in\mathfrak{X}_1$.

(ii) By definition, a p-soluble group G is in $\mathfrak{X}_1$ if and only if the dimension of each irreducible $K(G)$-module in the principal block of G is in I. In view of Lemma 2.3 of (**2**), this is equivalent to saying that the dimension of each irreducible $K(G)$-module whose kernel contains $F_p(G)$ is in I. Thus $G\in\mathfrak{X}_1$ if and only if the dimension of each irreducible $K(G/F_p(G))$-module is in I, that is if and only if $G/F_p(G)$ is in $\mathfrak{X}$.

Corollary 1. *With the notation of the theorem, $\mathfrak{X}(I)$ is a formation if and only if $\mathfrak{X}_1(I)$ is a saturated formation.*

Proof. If $\mathfrak{X}_1(I)$ is a saturated formation, Theorem 1(i) together with Hilfssatz VI.7.24 of (**4**) give that $\mathfrak{X}(I)$ is a formation.

Conversely, suppose that $\mathfrak{X}(I)$ is a formation. Theorem 1 (ii) implies that a group G is in $\mathfrak{X}_1(I)$ if and only if $G/F_p(G)\in\mathfrak{X}(I)$ and $G/F_q(G)$ is a p-soluble group for $q\neq p$. Thus $\mathfrak{X}(I)$ is a locally defined formation and so is saturated by the main theorem of Gaschütz (see (**4**; VI.7.5)).

Corollary 2. *With the notation of the theorem,*

(i) *for $G\in\mathfrak{X}_1(I)$ and N any normal subgroup of G, $G/N\in\mathfrak{X}_1(I)$,*

(ii) *$G\in\mathfrak{X}_1(I)$ if and only if $G/\phi(G)\in\mathfrak{X}_1(I)$.*

Proof. Let $H\in\mathfrak{X}(I)$ and L be a normal subgroup of H. Since each irreducible $K(H/L)$-module is an irreducible $K(H)$-module, $H/L\in\mathfrak{X}(I)$. Thus $\mathfrak{X}(I)$ is closed under epimorphic images. The corollary can now be deduced from a proof of the theorem of Gaschütz referred to above.

In view of Corollaries 1 and 2, it is natural to ask if, given I, every finite p-soluble group has $\mathfrak{X}_1(I)$-projectors. This is not the case, as is shown by the following.

Example 1. Let N_1 be elementary of order 9, and Q_1 be a subgroup of $GL(2,3)$ isomorphic to the quaternion group of order 8. Let G_1 be the semi-direct product of N_1 by Q_1 with the natural action of Q_1 on N_1. Let $G_2\cong G_1$ be the semi-direct product of N_2 ($\cong N_1$) by Q_2 ($\cong Q_1$) and let $G=G_1\times G_2$.

Let K be an algebraically closed field of characteristic 3 and $I=\{1,2\}$. Then $G_i\in\mathfrak{X}(I)$ for $i=1$, 2 but $G\notin\mathfrak{X}(I)$ since there is a 4 dimensional irreducible $K(G)$-module. Also G has just one block (since $O_{3'}(G)=1$), so

$G \notin \mathfrak{X}_1(I)$. However $G/N_i \in \mathfrak{X}_1(I)$ $(i = 1, 2)$ since

$$G/N_1 \cong G_2 \times Q_1 \cong G/N_2.$$

Suppose that G has an $\mathfrak{X}_1(I)$-projector F. Then $G = FN_1$ and so $F \cap N_1$ is a normal subgroup of G, giving that $F \cap N_1 = 1$ by the minimality of N_1. Hence a Sylow 3-subgroup N of F has order 9 and N is normalized by F and by N_1 since $N_1 \times N_2$, the unique Sylow 3-subgroup of G, is abelian. Hence N is a normal subgroup of G. Since the non-identity elements of N_i are permuted transitively by the elements of Q_i $(i = 1, 2)$, it follows easily that N_1 and N_2 are the only two minimal normal subgroups of G. Thus $N = N_2$ and $FN_2 = F \neq G$. This shows that G has no $\mathfrak{X}_1(I)$-projectors.

As our first application of Theorem 1, we consider the situation where k is the field with p elements, n is a positive integer and I_n is the set of positive integers which divide n.

Theorem 2. *Let n be a positive integer and G be a group of order coprime to n. Then $G \in \mathfrak{X}(I_n)$ if and only if $G/O_p(G)$ is abelian of exponent dividing $p^n - 1$.*

Proof. Suppose $G \in \mathfrak{X}(I_n)$ and V is an irreducible $k(G)$-module with $N = \ker V$. By (**4**: VI.8.1), G/N is cyclic of order dividing $p^n - 1$. Since the intersection of the kernels of the irreducible $k(G)$-modules is $O_p(G)$, we have that $G/O_p(G)$ is abelian of exponent dividing $p^n - 1$.

Conversely, suppose that $G/O_p(G)$ is abelian of exponent dividing $p^n - 1$ and that V is an irreducible $k(G)$-module. It follows by (**4**: II.3.10) that the dimension of V divides n.

Recall that for any p-soluble group G, the arithmetic p-rank of G is defined to be the lowest common multiple of the k-dimensions of the p-chief factors of G.

Corollary 3. *Let n be a positive integer and G be a group of order coprime to n. Then G has arithmetic p-rank dividing n if and only if $G \in \mathfrak{X}_1(I_n)$.*

Proof. By Theorem 1, Theorem 2 and (**4**: VI.8.3).

For our second application, denote by $\bar{Q}$ the algebraic closure of the field Q of rational numbers. Choose a fixed extension ν_p of the p-adic exponential valuation of Q to $\bar{Q}$. Let R denote the local ring of ν_p in $\bar{Q}$, P denote the corresponding prime ideal, and $K = R/P$. Let $I_{p'}$ be the set of positive integers coprime to p. We then have

Lemma 2. *Each factor group and each normal subgroup of a group in $\mathfrak{X}(I_{p'})$ is also in $\mathfrak{X}(I_{p'})$.*

Proof. Let $G \in \mathfrak{X}(I_{p'})$ and suppose N is a normal subgroup of G. By Mackey's subgroup theorem (**4**: V.16.9), an irreducible $K(N)$-module T occurs as a component of $(T^G)_N$. However each composition factor of T^G

has dimension coprime to p and so by Clifford's theorem (**4**: V.17.3), the irreducible components of $(T^G)_N$ have dimension coprime to p.

The result for factor groups is trivial.

Theorem 3. *A p-soluble group G is in $\mathfrak{X}(I_{p'})$ if and only if G has a normal Sylow p-subgroup.*

Proof. Suppose G has a normal Sylow p-subgroup N. Then each irreducible $K(G)$-module may be regarded as an irreducible $K(G/N)$-module and so has dimension coprime to p.

We prove the converse implication by induction on $|G|$. Let N be a maximal normal subgroup of G, so that $M \in \mathfrak{X}$ by Lemma 2. By induction, M has a normal Sylow p-subgroup N. If p is coprime to $|G:M|$, N is a normal Sylow p-subgroup of G. Thus by the maximality of N, we may suppose that $|G:M| = p$. If $N \neq 1$, induction applied to G/N gives the result, so we may suppose that $|M|$ is coprime to p.

Since G has a normal p-complement and abelian Sylow p-subgroups, we can apply a result of Richen (**5**), to deduce that the restriction of the character of every irreducible $\bar{Q}(G)$-module to the p-regular conjugacy classes of G is a Brauer irreducible character of G. Thus each irreducible $\bar{Q}(G)$-module has dimension coprime to p and so, by a result of Fong (**1**: 3D), G has a normal Sylow p-subgroup as required.

Corollary 4. *A p-soluble G is in $\mathfrak{X}_1(I_{p'})$ if and only if G has p-length* 1.

Proof. By Theorem 1 and Theorem 3.

Remark. Theorem 3 is a modular analogue of Lemma 3D of (**1**) and Corollary 4 is an analogue of Theorem 3E of (**1**).

As a final application, we consider the case where I is a finite set.

Theorem 4. *There exists an integer valued function $f(p, m, n)$ so that for any finite p-soluble group G whose Sylow p-subgroups have order at most p^m with the property that each irreducible $K(G)$-module in the principal block of G has dimension at most n, there exists a normal subgroup N of G of index at most $f(p, m, n)$ such that N' has a normal p-complement.*

Proof. By Theorem 1 and the theorem of (**3**).

Example 2. Let $G = S_5$, the symmetric group on 5 symbols, so that $F_3(G) = 1$. Let K be algebraically closed of characteristic 3 and I be the set of integers coprime to 3. Since G has two irreducible $K(G)$-modules in its principal block, of degrees 1 and 4, $G \in \mathfrak{X}_1$. However $G \notin \mathfrak{X}$ since there is an irreducible $K(G)$-module of dimension 6. This example shows that neither (i) nor (ii) of Theorem 1 holds for arbitrary finite groups.

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