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TWO COMMUTATIVE AMENABLE NON-SYMMETRIC BANACH ALGEBRAS

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Abstract

We show that a commutative amenable Banach algebra need not be symmetric by constructing suitable examples.

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1. Introduction

The standard examples of commutative amenable Banach algebras, the group algebra of a locally compact abelian group and the algebra of continuous functions on a compact space, are extremely well behaved. This raises the question of whether various types of 'well behavedness' are actually implied by the assumption of amenability. In this paper we show that a commutative amenable Banach algebra can be non-symmetric.

DEFINITION 1.1. Let $M > 0$. A Banach algebra $\mathfrak{U}$ is *M-amenable* if, for all $\epsilon > 0$ and all finite subsets $\{a_1, \dots, a_n\}$ of $\mathfrak{U}$ there is $d \in \mathfrak{U}\hat{\otimes}\mathfrak{U}$ with $\|d\| \le M$, $\|a_i d - d a_i\| < \epsilon$ and $\|a_i \pi(d) - a_i\| < \epsilon$ $(i = 1, \dots, n)$.

In this π is the product map $\pi(a \otimes b) = ab$ and $\mathfrak{U}\hat{\otimes}\mathfrak{U}$ has the module structure $a(b \otimes c) = ab \otimes c$, $(b \otimes c)a = b \otimes ca$. Note that slight variations in the above give equivalent definitions. In particular, we can replace $d \in \mathfrak{U}\hat{\otimes}\mathfrak{U}$ by $d \in \mathfrak{U} \otimes \mathfrak{U}$ and we can replace $\{a_1, \dots, a_n\}$ by finite subsets of E where E is a subset of $\mathfrak{U}$ such that $\mathfrak{U}$ is the smallest closed subalgebra of $\mathfrak{U}$ containing E or even, although we shall not use this, the smallest closed, adverse-closed subalgebra containing E. A Banach algebra

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is amenable in the usual sense [1, Section 1] if and only if it is M-amenable for some M.

DEFINITION 1.2. Let $\mathfrak{U}$ be a Banach algebra. An element d of $\mathfrak{U}\hat{\otimes}\mathfrak{U}$ is a *diagonal* if $ad = da$ and $a = \pi(d)a$ $(a \in \mathfrak{U})$.

Clearly if d is a diagonal then $\mathfrak{U}$ is M amenable for any $M \geq \|d\|$. Conversely for finite dimensional algebras, M-amenability implies the existence of a diagonal d with $\|d\| \leq M$.

DEFINITION 1.3. A commutative Banach algebra $\mathfrak{U}$ is *symmetric* if for each a in $\mathfrak{U}$ there is b in $\mathfrak{U}$ with

$$\phi(b) = \phi(a)^- \qquad (\phi \in \hat{\mathfrak{U}})$$

where $\hat{\mathfrak{U}}$ is the space of non-zero multiplicative linear functionals on $\mathfrak{U}$ (the 'maximal ideal space').

An equivalent definition is that the set of Gelfand transforms of elements of $\mathfrak{U}$ is closed under the involution $f^*(\phi) = f(\phi)^-$ on $C_0(\hat{\mathfrak{U}})$.

If $\mathfrak{U}$ is a commutative Banach * algebra with hermitian involution then it is symmetric in this sense (see [3, p. 189]). Conversely, if $\mathfrak{U}$ is a semisimple symmetric commutative Banach algebra, then b in Definition 1.3 is uniquely determined by a and, denoting it by a^*, we have an involution on $\mathfrak{U}$ which is automatically continuous and hermitian. Thus, although a commutative semisimple Banach algebra may have many involutions, it has at most one hermitian involution and it is symmetric if and only if it has a hermitian involution.

2. The norm $\| \|_x$ on $\mathbb{C}^2$

With pointwise multiplication, $\mathbb{C}^2$ is a commutative algebra. We describe the harmonic analysis on $\mathbb{Z}_2 = \{e, g\}$, the group of order two. The group algebra $\ell^1(\mathbb{Z}_2)$ is the set of expressions $\alpha + \beta g (\alpha, \beta \in \mathbb{C})$ with $\|\alpha + \beta g\|_1 = |\alpha| + |\beta|$ and product $(\alpha + \beta g)(\alpha' + \beta' g) = (\alpha\alpha' + \beta\beta') + (\alpha\beta' + \alpha'\beta)g$. The characters on the group algebra are $\alpha + \beta g \mapsto \alpha + \beta$ and $\alpha + \beta g \mapsto \alpha - \beta$ so the Fourier-Gelfand transform is $(\alpha + \beta g)\hat{} = (\alpha + \beta, \alpha - \beta) \in \mathbb{C}^2$. The inverse of this is $(a, b) \mapsto (a + b)/2 + (a - b)/2$. The norm $\| \|_A$ on $\mathbb{C}^2$ corresponding under the Fourier transform to $\| \|_1$ is $\|(a, b)\|_A = (|a + b| + |a - b|)/2$. On $\mathbb{C}^2$ we also have the spectral radius norm $\|(a, b)\|_\infty = \max\{|a|, |b|\}$. If $a, b \in \mathbb{R}$ then $\|(a, b)\|_A = (|a + b| + |a - b|)/2 = \max\{|a|, |b|\} = \|(a, b)\|_\infty$ but this is not of course true throughout $\mathbb{C}^2$.

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Suppose now that $x \in \mathbb{C}^2$ with $\|x\|_\infty = 1$ and $\|x^3\|_A = 1$. We put $Z = \ell^1(\mathbb{Z}_2) \oplus \ell^1(\mathbb{Z}_2) \oplus \ell^1(\mathbb{Z}_2)$ (ℓ^1 direct sum) and define $T : Z \to \mathbb{C}^2$ by

$$T(p \oplus q \oplus r) = \hat{p} + \hat{q}x + \hat{r}x^2.$$

Put $Y = Z/\ker T$ with the quotient norm, let $T = T'q$ be the canonical factorisation of T, where q is the quotient map $Z \to Y$, and define $\|\,\|_x$ on $\mathbb{C}^2 = \operatorname{Im} T$ by $\|T'y\|_x = \|y\|_Y (y \in Y)$.

PROPOSITION 2.1. (a) *The closed unit ball B in $(\mathbb{C}^2, \|\,\|_x)$ is the absolutely convex hull C of $C_0 = \{\mathbf{1}, x, x^2, v, vx, vx^2\}$ where $\mathbf{1} = (1, 1)$, and $v = (1, -1)$.*

(b) *For all $c \in \mathbb{C}^2$, $\|c\|_\infty \le \|c\|_x \le \|c\|_A$.*

(c) *$\|\,\|_x$ is an algebra norm on $\mathbb{C}^2$.*

PROOF. B is the image under T of the unit ball in Z which is the closed absolutely convex cover of the six elements $p \oplus q \oplus r$ where one of $p, q, r \in \{e, g\}$ and the other two are 0. This proves (a).

For (b) note that if $c \in C_0$ then $\|c\|_\infty \le 1$ so if $c \in B$ then $\|c\|_\infty \le 1$ by (a). Thus $\|c\|_\infty \le \|c\|_x$ $(c \in \mathbb{C}^2)$. The $\|\,\|_A$ unit ball is the absolutely convex hull of $\{1, v\}$, because the corresponding statement is true in $\ell^1(\mathbb{Z}_2)$, and so contains C_0 and hence C. Thus $\|c\|_x \le \|c\|_A$ $(c \in \mathbb{C}^2)$.

To show (c) it is enough to show that B is closed under multiplication and this follows if we can prove that the product of two elements of C_0 is in C. Any such product is of the form $c = v^r x^s$ with $r \in \{0, 1\}$ and $s \in \{0, 1, 2, 3, 4\}$. This is in C_0 if $s \in \{0, 1, 2\}$. If $s = 3$, then $\|c\|_A \le \|v^r\|_A \|x^3\|_A \le 1$ so by (b), $\|c\|_x \le 1$. If $s = 4$, put $c' = v^r x^3$ so $\|c'\|_A \le 1$ as before, that is there are $\lambda, \mu \in \mathbb{C}$ with $c' = \lambda 1 + \mu v$ and $|\lambda| + |\mu| \le 1$. Then $c = \lambda x + \mu v x \in C$.

We now put $x = (1, \zeta)$ where $\zeta = \frac{1}{2}\exp(\frac{i\pi}{3})$. Then $\|x\|_\infty = 1$ and $\|x^3\|_A = \|x^3\|_\infty = \|\frac{1}{8}(1, -1)\|_\infty = 1$ because $\|\,\|_A = \|\,\|_\infty$ on $\mathbb{R}^2$.

PROPOSITION 2.2. *With respect to the norm $\|\,\|_x$ on $\mathbb{C}^2$ the map $(a, b) \mapsto (a, b)^* = (\bar{a}, \bar{b})$ has norm $\kappa > 1$.*

PROOF. We show $\|x^*\|_x > 1$ and the result follows because $\|x\|_x \le 1$. Suppose $\|x^*\| \le 1$. By Proposition 2.2 (a) there are $\lambda_1, \dots, \lambda_6 \in \mathbb{C}$ with $\Sigma|\lambda_i| \le 1$ and

$$\begin{aligned} x^* &= (1, \bar{\zeta}) \\ &= \lambda_1(1, 1) + \lambda_2(1, \zeta) + \lambda_3(1, \zeta^2) + \lambda_4(1, -1) + \lambda_5(1, -\zeta) + \lambda_6(1, -\zeta^2). \end{aligned}$$

Put $\lambda_i = |\lambda_i|\omega_i$ where $|\omega_i| = 1$ and $\omega_i = 1$ if $\lambda_i = 0$. Then $1 = \Sigma|\lambda_i|\omega_i$, so, as 1 is an extreme point of the unit disc in $\mathbb{C}$ and $\Sigma|\lambda_i| \le 1$, we have $\omega_i = 1$

for all i and $\lambda_i \geq 0$. Thus $\bar{\zeta}$ is in the convex hull of $\pm 1, \pm\zeta, \pm\zeta^2$. Consider the real-linear functional θ on $\mathbb{C}$ given by $\theta(z) = 4\mathrm{Re}\, z - 20\mathrm{Im}\, z/\sqrt{3}$. We have $\theta(\pm 1) = \pm 4, \theta(\pm\zeta) = \mp 4, \theta(\pm\zeta^2) = \mp 3$ and $\theta(\bar{\zeta}) = 6$ so $\bar{\zeta}$ is not in the convex hull of $\pm 1, \pm\zeta, \pm\zeta^2$ and hence x^* is not in the unit ball of $(\mathbb{C}^2, \|\,\|_x)$.

For each positive integer n we consider $\mathbb{C}^{2^n}$ as the tensor product of n copies of $\mathbb{C}^2$ and denote the greatest cross norm on $\mathbb{C}^{2^n}$ induced by $\|\,\|_A$ or $\|\,\|_x$ on all the factors by the same symbol. Thus $(\mathbb{C}^{2^n}, \|\,\|_x)$ and $(\mathbb{C}^{2^n}, \|\,\|_A)$ are both Banach algebras and the latter is isometrically isomorphic with $\ell^1((\mathbb{Z}_2)^n)$. For any finite group G, $d = |G|^{-1}\sum_{g\in G} g^{-1}\otimes g$ is a diagonal in $\ell^1(G)$ of norm 1. Thus $(\mathbb{C}^{2^n}, \|\,\|_A)$ is 1 amenable. More generally, the standard proof [1, Theorem 2.5] that the group algebra of an amenable group is amenable in fact shows that it is 1-amenable.

COROLLARY 2.3. *$(\mathbb{C}^{2^n}, \|\,\|_x)$ is a commutative symmetric semisimple 1-amenable Banach algebra with involution of norm κ^n.*

PROOF. $(\mathbb{C}^{2^n}, \|\,\|_x)$ is clearly a commutative symmetric semisimple Banach algebra. By Proposition 2.1(b) we have $\|t\|_x \leq \|t\|_A$ for all t in $\mathbb{C}^{2^n}$ so if d is the norm 1 diagonal for $(\mathbb{C}^{2^n}, \|\,\|_A)$ then it is a diagonal for $(\mathbb{C}^{2^n}, \|\,\|_x)$ with norm ≤ 1 and $(\mathbb{C}^{2^n}, \|\,\|_x)$ is 1 amenable.

The statement about the norm of the involution follows from the general identity $\|S\otimes T\| = \|S\|\|T\|$ for the tensor products of operators.

3. Direct sums of M-amenable Banach algebras

Throughout this section we will use $\oplus_\infty$ to indicate the direct sum with the supremum norm. Thus if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are Banach spaces, then $\mathfrak{X}_1 \oplus_\infty \mathfrak{X}_2$ is the direct sum with $\|x_1 \oplus x_2\| = \max\{\|x_1\|, \|x_2\|\}$ and if $\mathfrak{X}_1, \mathfrak{X}_2, \dots$ are Banach spaces then $(\oplus_\infty)_{n=1}^\infty \mathfrak{X}_n$ is the space of sequences x with $x_n \in \mathfrak{X}_n$, $n = 1, 2, \dots$ and $\|x_n\| \to 0$ as $n \to \infty$; the norm is given by $\|x\| = \max_n\{\|x_n\|\}$.

By considering elements of the form $d+d'$ in $(\mathfrak{A}\oplus_\infty\mathfrak{B})\hat{\otimes}(\mathfrak{A}\oplus_\infty\mathfrak{B})$ where $d \in \mathfrak{A}\hat{\otimes}\mathfrak{A}$ and $d' \in \mathfrak{B}\hat{\otimes}\mathfrak{B}$, it is easy to see that if $\mathfrak{A}$ and $\mathfrak{B}$ are M-amenable Banach algebras, then $\mathfrak{A}\oplus_\infty\mathfrak{B}$ is $2M$-amenable. We shall show that in fact it is M-amenable.

LEMMA 3.1. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $t \in \mathfrak{X}\otimes\mathfrak{Y}$ with $\|t\|_\cdot < M$ then there is k_0 such that for every integer $k > k_0$, t has an expression $t = \sum_{i=1}^k x_i \otimes y_i$ where $\|x_i\| < \delta$, $\|y_i\| < \delta$ and $\delta = (M/k)^{1/2}$.*

PROOF. We have $t = \sum_{j=1}^{\ell} r_j \otimes s_j$ for some $\ell \in \mathbb{N}$, $r_j \in \mathfrak{X}$, $s_j \in \mathfrak{Y}$ $(j = 1, \dots, \ell)$ and $\Sigma\|r_j\|\|s_j\| < M$. We may, of course, assume that none of the r_j or s_j is zero.

Choose $k_0 \in \mathbb{N}$ with $\ell M k_0^{-1} < M - \Sigma \|r_j\| \|s_j\|$, let $k > k_0$ and put $\delta = (M/k)^{1/2}$. Replacing each r_j by $\delta \|r_j\|^{-1} r_j$ and each s_j by $\delta^{-1} \|r_j\| s_j$ we may assume that $\|r_j\| = \delta$, $j = 1, \ldots, \ell$. If p_j is the integer part of $\|s_j\| \delta^{-1}$ then s_j is the sum of $1 + p_j$ terms of norm less than δ (we could take each term to be $(1 + p_j)^{-1} s_j$, for example). Decomposing the s_j in this way we get $t = \sum_{i=1}^{m} x_i' \otimes y_i'$ where $\|x_i'\| = \delta$ and $\|y_i'\| < \delta$. Because $p_j \delta \leq \|s_j\| < (1 + p_j)\delta$, the number of terms $m = \sum_{j=1}^{\ell} (1 + p_j)$ satisfies $(m - \ell)\delta^2 \leq \Sigma \|r_i\| \|s_i\| \leq m\delta^2$. Thus, $m\delta^2 \leq \Sigma \|r_i\| \|s_i\| + \ell\delta^2 \leq \Sigma \|r_i\| \|s_i\| + \ell M k_0^{-1} < M = k\delta^2$. This shows that $m < k$. To get the required expression we put $x_i = x_i' (\|y_i'\|/\delta)^{1/2}$, $y_i = y_i' (\|y_i'\|/\delta)^{-1/2}$, $(i = 1, \ldots, m)$ and $x_i = y_i = 0$ $(i = m + 1, \ldots, k)$.

LEMMA 3.2. *If $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}$ are Banach spaces and $t \in \mathfrak{X} \otimes \mathfrak{Y}$, $t' \in \mathfrak{Z} \otimes \mathfrak{W}$ with $\|t\|_\wedge < M$ and $\|t'\|_\wedge < M$, then $t + t' \in (\mathfrak{X} \oplus_\infty \mathfrak{Z}) \hat{\otimes} (\mathfrak{Y} \oplus_\infty \mathfrak{W})$ has $\|t + t'\|_\wedge < M$.*

PROOF. Apply Lemma 3.1 to t and t' getting integers k_0 and k_0'. Take $k > \max\{k_0, k_0'\}$, $\delta = (M/k)^{1/2}$ and put $t = \sum_{i=1}^{k} x_i \otimes y_i$, $t' = \sum_{i=1}^{k} z_i \otimes w_i$ where $\|x_i\| < \delta$, $\|y_i\| < \delta$, $\|z_i\| < \delta$ and $\|w_i\| < \delta$ $(1 = 1, \ldots, k)$. The expression

$$t + t' = \frac{1}{2} \sum_{r=0}^{1} \sum_{i=1}^{k} (x_i \oplus (-1)^r z_i) \otimes (y_i \oplus (-1)^r w_i)$$

shows that $\|t + t'\| < 1 \cdot 2 \cdot k \cdot \delta^2/2 = M$.

Applying Lemma 3.2 for every $M' > M$ we see that if $\|t\|_\wedge \leq M$ and $\|t'\|_\wedge \leq M$ then $\|t + t'\|_\wedge \leq M$.

PROPOSITION 3.3. (a) *If $\mathfrak{U}_1, \ldots, \mathfrak{U}_n$ are M-amenable Banach algebras then so is $(\oplus_\infty)_{i=1}^{n} \mathfrak{U}_i$.*

(b) *If $\mathfrak{U}_1, \mathfrak{U}_2, \ldots$ are M-amenable Banach algebras then so is $(\oplus_\infty)_{i=1}^{\infty} \mathfrak{U}_i$.*

PROOF. (a) follows easily from Lemma 3.2 by taking the elements d_i in $\mathfrak{U}_i \hat{\otimes} \mathfrak{U}_i$ from Definition 1.1 and forming $d = \sum d_i$.

To prove (b) we consider $\mathfrak{B}_n = \oplus_{i=1}^{n} \mathfrak{U}_i$ as embedded in $\mathfrak{B} = \oplus_{i=1}^{\infty} \mathfrak{U}_i$ in the obvious way. If $\mathfrak{B}_{00} = \cup_{n=1}^{\infty} \mathfrak{B}_n$ then any finite subset of $\mathfrak{B}_{00}$ lies in some $\mathfrak{B}_m$ so the conditions of Definition 1.1 are satisfied for finite subsets of $\mathfrak{B}_{00}$ and the result follows because $\mathfrak{B}_{00}$ is dense in $\mathfrak{B}$.

We are now in a position to give a counterexample.

THEOREM 3.4. *$\mathfrak{U} = (\oplus_\infty)_{n=1}^{\infty} (\mathbb{C}^{2^n}, \|\,\|_x)$ is a commutative semisimple 1-amenable Banach algebra which is not symmetric.*

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PROOF. It is easy to see that $\mathfrak{U}$ is a commutative semisimple Banach algebra because the $(\mathbb{C}^{2^n}, \| \|_x)$ are. It is 1-amenable by Proposition 3.3 and Corollary 2.3.

The maximal ideal space $\hat{\mathfrak{U}}$ of $\mathfrak{U}$ is the disjoint union of the maximal ideal spaces of the $\mathbb{C}^{2^n}$ and $C_0(\hat{\mathfrak{U}}) = (\oplus_\infty)_{n=1}^\infty(\mathbb{C}^{2^n}, \| \|_\infty)$. The hermitian involution of $C_0(\hat{\mathfrak{U}})$ is $\{a_i\}_{i=1}^\infty \mapsto \{a_i^*\}_{i=1}^\infty$. As the norm of this involution on $(\mathbb{C}^{2^n}, \| \|_x)$ is $\kappa^n \to \infty$ as $n \to \infty$, there is $a \in \mathfrak{U}$ with $\|a_n^*\|_x \to \infty$ as $n \to \infty$ so $a^* \notin \mathfrak{U}$ and $\mathfrak{U}$ is not symmetric.

4. Another example

We give a slightly different example which avoids the results in Section 3. For any group H we have the map $H \to H \times \mathbb{Z}_2$ given by $h \mapsto (h, e)$. Iterating this we get an inductive system

$$\mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \cdots$$

with limit which we denote by $\mathbb{Z}_{2,\infty}$. There is a corresponding sequence of injective isometric isomorphisms of group algebras

$$\ell^1(\mathbb{Z}_2) \to \ell^1(\mathbb{Z}_2 \times \mathbb{Z}_2) \to \cdots$$

with limit $\ell^1(\mathbb{Z}_{2,\infty})$. Denoting the group dual to $\mathbb{Z}_2$ by $\{1, -1\}$ there is a corresponding projective system

$$\{1, -1\} \leftarrow \{1, -1\}^2 \leftarrow \ldots$$

with limit the compact group $\{1, -1\}^\infty$. Here the linking maps are $(\epsilon_1, \ldots, \epsilon_n, \epsilon_{n+1}) \mapsto (\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i = \pm 1$. For the continuous function spaces we again have an injective system dual to this which, writing $\mathbb{C}^{2^n}$ for $C(\{1, -1\}^n)$ is

$$\mathbb{C}^2 \to \mathbb{C}^{2^2} \to \cdots$$

where the connecting maps are $f \mapsto f \otimes 1$. We have seen that, if we take the norms $\| \|_A$ on $\mathbb{C}^{2^n}$, then the limit is $A(\{1, -1\}^\infty)$. If we take the supremum norms on $\mathbb{C}^{2^n}$, the limit is $C(\{1, -1\}^\infty)$. The map $f \mapsto f \otimes 1$ is also an isometry in the $\| \|_x$ norms; we denote the inductive limit by $\mathfrak{U}$ and its norm by $\| \|_{x,\infty}$. The contractive isomorphisms $(\mathbb{C}^{2^n}, \| \|_A) \to (\mathbb{C}^{2^n}, \| \|_x) \to (\mathbb{C}^{2^n}, \| \|_\infty)$ given by the identity map give contractive homomorphisms $\Phi : A(\{1, -1\}^\infty) \to \mathfrak{U}$ and $\Psi : \mathfrak{U} \to C(\{1, -1\}^\infty)$ with dense ranges where $\Psi\Phi$ is the identity injection $A(\{1, -1\}^\infty) \to C(\{1, -1\}^\infty)$.

We will show that $\mathfrak{U}$ is semisimple so that Φ and Ψ are injective. Suppose that $r \in \operatorname{Rad}\mathfrak{U}$ and $c \in \mathbb{C}^{2^n}$ for some n. Taking the tensor product Θ_{np} of the identity map on $\mathbb{C}^{2^n}$ with p copies of the functional $(a, b) \mapsto a$ on $\mathbb{C}^2$ we get a coherent system of contractive maps from $(\mathbb{C}^{2^{n+p}}, \| \|_x)$ to $(\mathbb{C}^{2^n}, \| \|_x)$ and hence a contraction Θ from $\mathfrak{U}$

into $\mathbb{C}^{2^n}$ which is the identity on $\mathbb{C}^{2^n}$. In the same way we get a contraction $\tilde{\Theta}$ from $C(\{1,-1\}^\infty)$ onto $\mathbb{C}^{2^n}$. We have $\Psi\Theta = \tilde{\Theta}\Psi$ because this holds on the $\mathbb{C}^{2^{n+p}}$ spaces. Because $C(\{1,-1\}^\infty)$ is semisimple we have $\Psi r = 0$ and hence $\Psi\Theta r = \tilde{\Theta}\Psi r = 0$. However, Ψ is injective on $\mathbb{C}^{2^n}$ so $\Theta r = 0$. Thus $\|c - r\|_x \geq \|\Theta_c - \Theta_r\|_x = \|c\|_x$. However, if $\|r\|_x > 0$ then there is $n \in \mathbb{N}$ and $c \in \mathbb{C}^{2^n}$ with $\|c - r\|_x < \frac{1}{2}\|r\|_x$. We then have $\|r\|_x - \|c\|_x < \frac{1}{2}\|r\|_x$ so $\frac{1}{2}\|r\|_x < \|c\|_x$ and $\|c - r\|_x < \|c\|_x$. This contradiction shows that $\operatorname{Rad}\mathfrak{U} = 0$ and Φ and Ψ are injective.

If $\mathfrak{U}$ were symmetric then the involution on $C(\{1,-1\}^\infty)$ would restrict to a necessarily continuous involution on $\mathfrak{U}$. However, by Corollary 2.3, the restriction to $\mathbb{C}^{2^n}$ has norm κ^n where $\kappa > 1$. $\mathfrak{U}$ is 1 amenable because $A(\{1,-1\}^\infty) = \ell^1(\mathbb{Z}_{2,\infty})$ is and Φ is a contractive homomorphism into $\mathfrak{U}$ with dense range so the conditions of Definition 1.1 are satisfied with $a_1, \ldots, a_n \in \operatorname{Im}\Phi$ and $d = \Phi \otimes \Phi d'$ for a suitably chosen element d' of $A(\{1,-1\}^\infty) \otimes A(\{1,-1\}^\infty)$.

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