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ESSENTIAL AMENABILITY OF DUAL BANACH ALGEBRAS

MOHSEN ZIAMANESH, BEHROUZ SHOJAEE and AMIN MAHMOODI[✉]

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Abstract

We show that an essentially amenable Banach algebra need not have an approximate identity. This answers a question posed by Ghahramani and Loy [‘Generalized notions of amenability’, *J. Funct. Anal.* **208** (2004), 229–260]. Essentially Connes-amenable dual Banach algebras are introduced and studied.

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1. Introduction

The notion of amenable Banach algebras was introduced by Johnson in [3]. Two modifications of this concept are essentially amenable Banach algebras [2] and Connes-amenable Banach algebras [7]; the latter had been considered previously under different names. We recall the definitions in Definitions 2.1 and 2.2 below. In this note, besides answering an open question concerning the existence of an approximate identity for essentially amenable Banach algebras, we introduce and investigate the notion of essential Connes-amenability for dual Banach algebras.

The organisation of the paper is as follows. Section 2 is devoted to recalling some background notions and definitions. In Section 3, we first answer an open question raised by Ghahramani and Loy [2, Section 9, Question 3]. They asked if an essentially amenable Banach algebra necessarily has an approximate identity. We present a class of algebras of the form $\mathcal{V}_\varphi$ to answer the question in the negative. Then we define a new notion of amenability of dual Banach algebras, namely essential Connes-amenability. Among other things, we give examples to show that essential Connes-amenability is weaker than Connes-amenability, but they are the same in the presence of an identity. In Section 4, we discuss essential Connes-amenability of algebras such as matrix algebras and Lau algebras as well as algebras over locally compact groups. In an Appendix (Appendix A) we give another approach to essential amenability of the Banach algebra $\mathcal{V}_\varphi$.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to \mathcal{X}$ is a *derivation* if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathcal{A}$. For every $x \in \mathcal{X}$, the map $ad_x : \mathcal{A} \to \mathcal{X}$, defined by $ad_x(a) = a \cdot x - x \cdot a$ for $a \in \mathcal{A}$, is a derivation; these are *inner* derivations. It is known that the dual Banach space $\mathcal{X}^*$ is also an $\mathcal{A}$-bimodule through

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle, \quad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle \quad (a \in \mathcal{A}, x \in \mathcal{X}, x^* \in \mathcal{X}^*).$$

A Banach algebra $\mathcal{A}$ is *amenable* if, for every Banach $\mathcal{A}$-bimdule $\mathcal{X}$, every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is inner. An $\mathcal{A}$-bimodule $\mathcal{X}$ is *neo-unital* if $\mathcal{X} = \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A}$ where $\mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A} = \{a \cdot x \cdot b : a, b \in \mathcal{A}, x \in \mathcal{X}\}$.

Definition 2.1 [2]. A Banach algebra $\mathcal{A}$ is *essentially amenable* if every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is inner for any neo-unital Banach $\mathcal{A}$-bimodule $\mathcal{X}$.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is *dual*, if there exists a closed submodule $\mathcal{X}_*$ of $\mathcal{X}^*$ such that $\mathcal{X} = (\mathcal{X}_*)^*$. We call $\mathcal{X}_*$ the *predual* of $\mathcal{X}$. Moreover, a Banach algebra $\mathcal{A}$ is a *dual Banach algebra* if it is dual as a Banach $\mathcal{A}$-bimodule. Let $\mathcal{A}$ be a dual Banach algebra. A dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is *normal* if the module actions of $\mathcal{A}$ on $\mathcal{X}$ are ω^*-continuous.

Definition 2.2 [7]. A dual Banach algebra $\mathcal{A}$ is *Connes-amenable*, if for every normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every ω^*-continuous derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is inner.

For a Banach algebra $\mathcal{A}$, we write $\sigma(\mathcal{A})$ for the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. Moreover, if $\mathcal{A}$ is a dual Banach algebra, then we write $\sigma_{\omega^*}(\mathcal{A})$ for the subset of all ω^*-continuous elements in $\sigma(\mathcal{A})$.

Let $\mathcal{V}$ be a Banach space, and let φ be a nonzero element of $\mathcal{V}^*$ with $\|\varphi\| \le 1$. Throughout, we denote by $\mathcal{V}_\varphi$ the Banach algebra $\mathcal{V}$ equipped with the multiplication $ab = \varphi(b)a$ for $a, b \in \mathcal{V}$. It is not hard to see that $\sigma(\mathcal{V}_\varphi) = \{\varphi\}$.

3. Essential [Connes-] amenability

We start by giving some essentially amenable Banach algebras without approximate identities.

Example 3.1. Let $\mathcal{V}$ be a Banach space of dimension (at least) 2. Then $\mathcal{V}_\varphi$ is essentially amenable but does not have an approximate identity. Suppose that $\mathcal{X}$ is a neo-unital Banach $\mathcal{V}_\varphi$-bimodule and that $D : \mathcal{V}_\varphi \longrightarrow \mathcal{X}^*$ is a derivation. Since $\mathcal{X}$ is neo-unital, $x \cdot a = \varphi(a)x$ for $x \in \mathcal{X}$, $a \in \mathcal{V}_\varphi$. Hence $a \cdot x^* = \varphi(a)x^*$ for $x^* \in \mathcal{X}^*$, $a \in \mathcal{V}_\varphi$. Choosing $a_0 \in \mathcal{V}_\varphi$ with $\varphi(a_0) = 1$, we see that

$$\begin{aligned} a \cdot D(a_0) = \varphi(a)D(a_0) &= D(\varphi(a)a_0) = D(a_0 a) \\ &= D(a_0) \cdot a + a_0 \cdot D(a) = D(a_0) \cdot a + D(a). \end{aligned}$$

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Therefore, $D(a) = a \cdot D(a_0) - D(a_0) \cdot a$ for all $a \in \mathcal{V}_\varphi$, so that D is inner. This shows that $\mathcal{V}_\varphi$ is essentially amenable. Next, assume that $(e_\alpha)_\alpha$ is a left approximate identity for $\mathcal{V}_\varphi$. Since $\mathcal{V}_\varphi$ is not one-dimensional, it is possible to choose a nonzero element $a_1 \in \ker\varphi$. Thus $a_1 = \lim_\alpha e_\alpha\ a_1 = 0$, a contradiction. It should be remarked that, because of the lack of an approximate identity, $\mathcal{V}_\varphi$ is not amenable.

Example 3.2. Any nonzero Banach space $\mathcal{A}$, equipped with the multiplication $ab = 0$ for $a, b \in \mathcal{A}$, is an essentially amenable Banach algebra [11, Example 4.2(a)]. Trivially, this algebra does not possess an approximate identity.

The following is the basic definition of the current paper.

Definition 3.3. A dual Banach algebra $\mathcal{A}$ is *essentially Connes-amenable* if for every normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X} = (\mathcal{X}_*)^*$ with neo-unital $\mathcal{X}_*$, every ω^*-continuous derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is inner.

Theorem 3.4. *Let $\mathcal{A}$ be a dual Banach algebra and let $\mathcal{B}$ be a dual Banach subalgebra of $\mathcal{A}$ containing $\mathcal{A}^2$. If $\mathcal{B}$ is essentially Connes-amenable, then so is $\mathcal{A}$.*

Proof. Suppose that $\mathcal{X} = (\mathcal{X}_*)^*$ is a normal dual Banach $\mathcal{A}$-bimodule with neo-unital $\mathcal{X}_*$ and that $D : \mathcal{A} \longrightarrow \mathcal{X}$ is an ω^*-continuous derivation. It follows from

$$\mathcal{X}_* = \mathcal{A} \cdot \mathcal{X}_* \cdot \mathcal{A} = \mathcal{A} \cdot (\mathcal{A} \cdot \mathcal{X}_* \cdot \mathcal{A}) \cdot \mathcal{A} \subseteq \mathcal{B} \cdot \mathcal{X}_* \cdot \mathcal{B} \subseteq \mathcal{X}_*$$

that $\mathcal{X}_* = \mathcal{B} \cdot \mathcal{X}_* \cdot \mathcal{B}$. Hence we may consider $\mathcal{X}$ as a normal dual Banach $\mathcal{B}$-bimodule with neo-unital $\mathcal{X}_*$, and then $D|_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{X}$ is an ω^*-continuous derivation. By the assumption, there exists $x \in \mathcal{X}$ such that $D(b) = ad_x(b)$ for each $b \in \mathcal{B}$. Setting $\widetilde{D} := D - ad_x$, we observe that $\widetilde{D}$ is an ω^*-continuous derivation from $\mathcal{A}$ into $\mathcal{X}$ and $\widetilde{D}|_\mathcal{B} = 0$. An argument similar to the proof of [11, Proposition 4.1] shows that $\widetilde{D} = 0$ and so $D = ad_x$, as required. □

It is immediate from the definition that Connes-amenable dual Banach algebras are essentially Connes-amenable. The converse however is not true, as the following example shows.

Example 3.5. Suppose that $\mathcal{A}$ is a dual Banach space equipped with the multiplication $ab = 0$ for all $a, b \in \mathcal{A}$. It is easy to verify that $\mathcal{A}$ is a dual Banach algebra. Obviously, $\mathcal{B} := \mathcal{A}^2 = \{0\}$ is essentially Connes-amenable. Then by Theorem 3.4, $\mathcal{A}$ is essentially Connes-amenable as well. Notice that because of the lack of an identity, $\mathcal{A}$ is not Connes-amenable.

Example 3.6. Suppose that $\mathcal{V}$ is a dual Banach space of dimension (at least) 2, and that φ be a nonzero, w^*-continuous functional on $\mathcal{V}$. It is known that $\mathcal{V}_\varphi$ is not a Connes-amenable dual Banach algebra (Behnamian and Mahmoodi, 'On pointwise amenability and biprojectivity of Banach algebras', submitted). As $\mathcal{V}_\varphi$ is essentially amenable, it is automatically essentially Connes-amenable.

Proposition 3.7. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{B}$ be a dual Banach algebra and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous epimorphism.*

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(i) *If $\mathcal{A}$ is essentially amenable, then $\mathcal{B}$ is essentially Connes-amenable.*

(ii) *If $\mathcal{A}$ is an essentially Connes-amenable dual Banach algebra and if θ is ω^*-continuous, then $\mathcal{B}$ is essentially Connes-amenable.*

Proof. We only give the proof of part (ii).

Suppose that $\mathcal{X} = (\mathcal{X}_*)^*$ is a normal dual Banach $\mathcal{B}$-bimodule with neo-unital $\mathcal{X}_*$. Then we may consider $\mathcal{X}$ as a normal dual Banach $\mathcal{A}$-bimodule with module actions $a \cdot x = \theta(a) \cdot x$ and $x \cdot a = x \cdot \theta(a)$ for $x \in \mathcal{X}$, $a \in \mathcal{A}$. It follows from

$$\mathcal{A} \cdot \mathcal{X}_* \cdot \mathcal{A} = \theta(\mathcal{A}) \cdot \mathcal{X}_* \cdot \theta(\mathcal{A}) = \mathcal{B} \cdot \mathcal{X}_* \cdot \mathcal{B} = \mathcal{X}_*$$

that $\mathcal{X}_*$ is neo-unital as an $\mathcal{A}$-bimodule. Now, suppose that $D : \mathcal{B} \longrightarrow \mathcal{X}$ is an ω^*-continuous derivation. It is easy to check that $D \circ \theta : \mathcal{A} \longrightarrow \mathcal{X}$ is also an ω^*-continuous derivation. Since $\mathcal{A}$ is essentially Connes-amenable, there exists $x \in \mathcal{X}$ such that $D \circ \theta(a) = \lim(a \cdot x - x \cdot a)$, $a \in \mathcal{A}$. By surjectivity of θ, it follows that $D(a) = \lim(a \cdot x - x \cdot a)$ for $a \in \mathcal{A}$. □

The concept of essential Connes-amenability is only interesting for nonunital algebras, because of the following proposition.

Proposition 3.8. *Let $\mathcal{A}$ be a dual Banach algebra. Then $\mathcal{A}$ is Connes-amenable if and only if $\mathcal{A}$ is unital and essentially Connes-amenable.*

Proof. Only the necessity part needs a proof. We follow the standard argument in [8, Proposition 2.1.5].

Suppose that $\mathcal{X} = (\mathcal{X}_*)^*$ is a normal dual Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}$ is an ω^*-continuous derivation. By Cohen's factorisation theorem, $\mathcal{X}_0 := \mathcal{A} \cdot \mathcal{X}_* \cdot \mathcal{A}$ is a closed $\mathcal{A}$-submodule of $\mathcal{X}_*$ which is also neo-unital. Let $\pi : \mathcal{X} \longrightarrow \mathcal{X}_0^*$ be the restriction map. It is easy to check that π is an ω^*-continuous module homomorphism. Thus $\mathcal{X}_0^*$ is a normal dual Banach $\mathcal{A}$-bimodule and $\pi \circ D : \mathcal{A} \longrightarrow \mathcal{X}_0^*$ is an ω^*-continuous derivation. Since $\mathcal{A}$ is essentially Connes-amenable, there exists $\lambda \in \mathcal{X}_0^*$ such that $\pi \circ D = ad_\lambda$. Take $x \in \mathcal{X}$ for which $x|_{\mathcal{X}_0} = \lambda$. It is routinely checked that $\widetilde{D} := D - ad_\lambda$ is an ω^*-continuous derivation from $\mathcal{A}$ into $\mathcal{X}_0^\perp \subseteq \mathcal{X}$. One may check that $\mathcal{X}_0^\perp \cong (\mathcal{X}_*/\mathcal{X}_0)^*$ is a normal dual Banach $\mathcal{A}$-bimodule. It follows from the definition of $\mathcal{X}_0$ that $\mathcal{A} \cdot (\mathcal{X}_*/\mathcal{X}_0) = \{0\}$. Now $\widetilde{D}$ is inner, by [8, Proposition 2.1.3]. Notice that here the existence of an identity is needed. Therefore, there exists $y \in \mathcal{X}_0^\perp$ such that $\widetilde{D} = ad_y$. Thus $D = ad_{x-y}$. □

Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \sigma(\mathcal{A})$. We write $\mathbb{C}_\varphi$ for the neo-unital Banach $\mathcal{A}$-bimodule $\mathbb{C}$ with operations $a \cdot z = z \cdot a = \varphi(a)z$ for $a \in \mathcal{A}$, $z \in \mathbb{C}$. Recall that a *point derivation d at φ* is a derivation $d : \mathcal{A} \to \mathbb{C}_\varphi$.

Proposition 3.9. *Let $\mathcal{A}$ be a Banach algebra.*

(i) *If $\mathcal{A}$ is essentially amenable, then there are no nonzero point derivations on $\mathcal{A}$.*

(ii) *If $\mathcal{A}$ is an essentially Connes-amenable dual Banach algebra, then there are no nonzero ω^*-continuous point derivations on $\mathcal{A}$.*

Proof. (i) Every point derivation $d : \mathcal{A} \to \mathbb{C}_\varphi$, $\varphi \in \sigma(\mathcal{A})$, may be regarded as a derivation into a dual bimodule with a neo-unital predual. By the assumption, d must be inner, but any inner derivation is zero.

(ii) As $\mathbb{C}_\varphi$ is a normal dual Banach $\mathcal{A}$-bimodule, the proof is similar to part (i). We just note that if $d : \mathcal{A} \to \mathbb{C}_\varphi$ is an ω^*-continuous point derivation for $\varphi \in \sigma(\mathcal{A})$, then φ is necessarily ω^*-continuous. To see this, suppose that d is ω^*-continuous and that $a_i \xrightarrow{\omega^*} 0$ in $\mathcal{A}$. Then $a_i b \xrightarrow{\omega^*} 0$ for any $b \in \mathcal{A}$. By the ω^*-continuity of d, both $(d(a_i))_i$ and $(d(a_i b))_i$ tend to zero. Hence from $d(a_i b) = \varphi(a_i)d(b) + \varphi(b)d(a_i)$, we conclude that $\varphi(a_i) \longrightarrow 0$, as required. □

Example 3.10. It is well-known that the discrete convolution algebra ℓ^1 consisting of all sequences $(a(n))_n$ with $\|a\| = \Sigma_{n=0}^\infty |a(n)| < \infty$ is a dual Banach algebra [1, Theorem 9.6]. Also by [4, Example 2.5], $\sigma(\ell^1) = \{\varphi_z : z \in \mathbb{C}, |z| \le 1\}$, where φ_z is the point evaluation at z, that is, $\varphi_z(a) = \Sigma_{n=0}^\infty a(n)z^n$ for all $a \in \ell^1$. Since $\varphi_z = (z, z^2, z^3, \cdots) \in c_0$, for each $z \in \mathbb{C}$ with $|z| < 1$, it follows that $\sigma_{\omega^*}(\ell^1) = \{\varphi_z : z \in \mathbb{C}, |z| < 1\}$. Consequently, the map $f \mapsto f'(z)$ is a nonzero ω^*-continuous point derivation at φ_z for each $z \in \mathbb{C}$ with $|z| < 1$ (see also [5, Example 2.8] or Shojaee ('A generalization on character Connes-amenability', submitted)). From Proposition 3.9, ℓ^1 is neither essentially amenable nor essentially Connes-amenable.

4. On essential [Connes-] amenability of some algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with $\sigma(\mathcal{B}) \neq \emptyset$, and let $\theta \in \sigma(\mathcal{B})$. From [12, Definition 2.1], we recall that the *θ-Lau product* $\mathcal{A} \times_\theta \mathcal{B}$ is the Cartesian product $\mathcal{A} \times \mathcal{B}$ equipped with the multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \theta(b_1)a_2 + \theta(b_2)a_1, b_1b_2),$$

and with the norm $\|(a_1, b_1)\| = \|a_1\| + \|b_1\|$, where $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. The space $\mathcal{A} \times_\theta \mathcal{B}$ is a Banach algebra. Further, if $\mathcal{A} = (\mathcal{A}_*)^*$ and $\mathcal{B} = (\mathcal{B}_*)^*$ are dual Banach algebras and if $\theta \in \sigma_{\omega^*}(\mathcal{B})$, then $\mathcal{A} \times_\theta \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \bigoplus_\infty \mathcal{B}_*$, where $\bigoplus_\infty$ denotes the l^∞-direct sum.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\theta \in \sigma(\mathcal{B})$.*

(i) *If $\mathcal{A} \times_\theta \mathcal{B}$ is essentially amenable, then $\mathcal{A}$ and $\mathcal{B}$ are essentially amenable.*

Further, let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and let $\theta \in \sigma_{\omega^*}(\mathcal{B})$.

(ii) *If $\mathcal{A} \times_\theta \mathcal{B}$ is essentially Connes-amenable, then $\mathcal{A}$ and $\mathcal{B}$ are essentially Connes-amenable.*

Proof. We prove part (ii). Suppose $\mathcal{A} \times_\theta \mathcal{B}$ is essentially Connes-amenable, $\mathcal{X} = (\mathcal{X}_*)^*$ is a normal dual Banach $\mathcal{A}$-bimodule with neo-unital $\mathcal{X}_*$ and $D : \mathcal{A} \to \mathcal{X}$ is an ω^*-continuous derivation. We may consider $\mathcal{X}$ as a Banach $\mathcal{A} \times_\theta \mathcal{B}$-bimodule with actions

$$(a, b) \cdot x = a \cdot x + \theta(b)x \quad \text{and} \quad x \cdot (a, b) = x \cdot a + \theta(b)x,$$

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where $a \in \mathcal{A}, b \in \mathcal{B}$ and $x \in \mathcal{X}$. Hence

$$(\mathcal{A} \times_\theta \mathcal{B}) \cdot \mathcal{X}_* \cdot (\mathcal{A} \times_\theta \mathcal{B}) = \mathcal{A} \cdot \mathcal{X}_* \cdot \mathcal{A} + \mathcal{X}_* = \mathcal{X}_* + \mathcal{X}_* = \mathcal{X}_*.$$

Therefore, $\mathcal{X}$ is a normal dual Banach $\mathcal{A} \times_\theta \mathcal{B}$ -bimodule with neo-unital $\mathcal{X}_*$. Define the map $\tilde{D} : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{X}$ with $\tilde{D}(a, b) = D(a)$. For each $(a_1, b_1), (a_2, b_2) \in \mathcal{A} \times_\theta \mathcal{B}$,

$$\begin{aligned}
\tilde{D}((a_1, b_1) \cdot (a_2, b_2)) &= \tilde{D}(a_1 a_2 + \theta(b_1)a_2 + \theta(b_2)a_1, b_1 b_2) \\
&= D(a_1 a_2 + \theta(b_1)a_2 + \theta(b_2)a_1) \\
&= D(a_1) \cdot a_2 + a_1 \cdot D(a_2) + \theta(b_2)D(a_1) + \theta(b_1)D(a_2) \\
&= D(a_1) \cdot (a_2, b_2) + (a_1, b_1) \cdot D(a_2) \\
&= \tilde{D}(a_1, b_1) \cdot (a_2, b_2) + (a_1, b_1) \cdot \tilde{D}(a_2, b_2).
\end{aligned}$$

Hence $\tilde{D}$ is an ω^*-continuous derivation. By the assumption, there is an $x \in \mathcal{X}$ such that $\tilde{D} = ad_x$. Then, for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$\begin{aligned}
D(a) = \tilde{D}(a, b) &= (a, b) \cdot x - x \cdot (a, b) \\
&= a \cdot x + \theta(b)x - (x \cdot a + \theta(b)x) \\
&= a \cdot x - x \cdot a.
\end{aligned}$$

Thus D is inner, so that $\mathcal{A}$ is essentially Connes-amenable.

Next, the map $\varphi : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{B}$ defined by $\varphi(a, b) = b$ is an ω^*-continuous epimorphism. So by Proposition 3.7(ii), $\mathcal{B}$ is essentially Connes-amenable. □

We write $\mathcal{A}^\sharp$ for the *unitisation* of a Banach algebra $\mathcal{A}$. It is known that $\mathcal{A}^\sharp = \mathcal{A} \times_i \mathbb{C}$ [12, Remarks 2.2], where i is the identity map on $\mathbb{C}$.

Remark 4.2. The converse of Theorem 4.1 is not true. To see this, suppose that $\mathcal{A}$ is an essentially [Connes-] amenable Banach algebra but not [Connes-] amenable. Both $\mathcal{A}$ and $\mathbb{C}$ are essentially [Connes-] amenable, but $\mathcal{A} \times_i \mathbb{C}$ is not essentially [Connes-] amenable. For if $\mathcal{A} \times_i \mathbb{C} = \mathcal{A}^\sharp$ were essentially [Connes-] amenable, then $\mathcal{A}^\sharp$ would necessarily be [Connes-] amenable because of the existence of an identity. Hence $\mathcal{A}$ must be [Connes-] amenable, which is not the case.

Corollary 4.3. *Let $\mathcal{A}$ be a [dual] Banach algebra. If $\mathcal{A}^\sharp$ is essentially [Connes-] amenable, then so is $\mathcal{A}$. The converse does not hold.*

Proof. This is a consequence of Theorem 4.1 and Remark 4.2. □

Let $\mathcal{X}$ be a Banach space and let $n \in \mathbb{N}$. The collection of all $n \times n$ matrices $(x_{i,j})$ with entries from $\mathcal{X}$ is denoted by $\mathbb{M}_n(\mathcal{X})$. In particular, $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ is a unital algebra with *matrix units* $\mathcal{E}_{i,j}$, so that $\mathcal{E}_{i,j}\mathcal{E}_{k,l} = \delta_{j,k}\mathcal{E}_{i,l}$. We regard $\mathbb{M}_n(\mathcal{X})$ as a Banach space with the norm

$$\|(x_{i,j})\| = \sum\{\|x_{i,j}\| : 1 \le i, j \le n\}, \quad ((x_{i,j}) \in \mathbb{M}_n(\mathcal{X})).$$

We identify $\mathbb{M}_n(\mathcal{X})^*$ with $\mathbb{M}_n(\mathcal{X}^*)$, using the duality

$$\langle x, \Lambda \rangle = \sum\{\langle x_{i,j}, \lambda_{i,j} \rangle : 1 \le i, j \le n\},$$

for $x = (x_{i,j}) \in \mathbb{M}_n(\mathcal{X})$ and $\Lambda = (\lambda_{i,j}) \in \mathbb{M}_n(\mathcal{X}^*)$.

Let $\mathcal{A}$ be an algebra. Then $\mathbb{M}_n(\mathcal{A})$ is also an algebra in the obvious way. The matrix $(a_{i,j})$ is identified with $\sum_{i,j=1}^{n} \mathcal{E}_{i,j} \otimes a_{i,j}$, so that $\mathbb{M}_n(\mathcal{A})$ is isomorphic to $\mathbb{M}_n \otimes \mathcal{A}$. In the case where $\mathcal{A}$ is a Banach algebra, the algebra $\mathbb{M}_n(\mathcal{A})$ is a Banach algebra with respect to the norm defined above.

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. We shall regard $\mathbb{M}_n(\mathcal{X})$ as a Banach $\mathbb{M}_n(\mathcal{A})$-bimodule through

$$(a \cdot x)_{i,j} := \sum_{k=1}^{n} a_{i,k} \cdot x_{k,j} = \sum_{k=1}^{n} \mathcal{E}_{i,j} \otimes a_{i,k} \cdot x_{k,j}$$

and

$$(x \cdot a)_{i,j} := \sum_{k=1}^{n} x_{i,k} \cdot a_{k,j} = \sum_{k=1}^{n} \mathcal{E}_{i,j} \otimes x_{i,k} \cdot a_{k,j}$$

for $a = (a_{i,j}) \in \mathbb{M}_n(\mathcal{A})$ and $x = (x_{i,j}) \in \mathbb{M}_n(\mathcal{X})$. In particular, $\mathbb{M}_n(\mathcal{X}^*)$ is a Banach $\mathbb{M}_n(\mathcal{A})$-bimodule. For $a = (a_{i,j}) \in \mathbb{M}_n(\mathcal{A})$ and $\Lambda = (\lambda_{i,j}) \in \mathbb{M}_n(\mathcal{X}^*)$,

$$(a \cdot \Lambda)_{i,j} = \sum_{k=1}^{n} a_{j,k} \cdot \lambda_{i,k} = \sum_{k=1}^{n} \mathcal{E}_{i,j} \otimes a_{j,k} \cdot \lambda_{i,k}$$

and

$$(\Lambda \cdot a)_{i,j} = \sum_{k=1}^{n} \lambda_{k,j} \cdot a_{k,i} = \sum_{k=1}^{n} \mathcal{E}_{i,j} \otimes \lambda_{k,j} \cdot a_{k,i}.$$

Now, suppose that $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra and that $\mathcal{X} = (\mathcal{X}_*)^*$ is a normal dual Banach $\mathcal{A}$-bimodule. It is easy to check that $M_n \otimes \mathcal{A} = (M_n \otimes \mathcal{A}_*)^*$ is a dual Banach algebra and $M_n \otimes \mathcal{X} = (M_n \otimes \mathcal{X}_*)^*$ is a normal dual Banach $M_n \otimes \mathcal{A}$-bimodule (see for instance [6]).

Theorem 4.4. *Let $\mathcal{A}$ be a Banach algebra and let $n \in \mathbb{N}$.*

(i) *If $M_n(\mathcal{A})$ is essentially amenable, then so is $\mathcal{A}$.*

Further, suppose that $\mathcal{A}$ is a dual Banach algebra.

(ii) *If $M_n(\mathcal{A})$ is essentially Connes-amenable, then so is $\mathcal{A}$.*

Proof. (i) Let $\mathcal{X}$ be a neo-unital Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ be a derivation. It is easily seen that the map

$$I \otimes D : M_n \otimes \mathcal{A} \longrightarrow M_n \otimes \mathcal{X}^*; \quad (\lambda_{ij}) \otimes a \longmapsto (\lambda_{ij}) \otimes Da$$

is a derivation, where I is the identity map on M_n. Moreover,

$$(M_n \otimes \mathcal{A}) \cdot (M_n \otimes \mathcal{X}) \cdot (M_n \otimes \mathcal{A}) = M_n \otimes \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A} = M_n \otimes \mathcal{X}$$

so that $M_n \otimes \mathcal{X}$ is neo-unital as an $M_n \otimes \mathcal{A}$-bimodule. Since $M_n \otimes \mathcal{A}$ is essentially amenable, there exists $(\lambda_{ij}) \otimes x^* \in M_n \otimes \mathcal{X}^*$ such that

$$(I \otimes D)((\gamma_{ij}) \otimes a) = (\gamma_{ij}) \otimes a \cdot (\lambda_{ij}) \otimes x^* - (\lambda_{ij}) \otimes x^* \cdot (\gamma_{ij}) \otimes a$$

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for each $(\gamma_{ij}) \in M_n$ and $a \in A$. Take the matrix (γ_{ij}) with 1 in the (1,1)-position and 0 elsewhere. Hence $Da = a \cdot (\lambda_{11} x^*) - (\lambda_{11} x^*) \cdot a$. It follows that D is inner and $\mathcal{A}$ is essentially amenable.

(ii) Let $\mathcal{X} = (\mathcal{X}_*)^*$ be a normal dual Banach $\mathcal{A}$-bimodule with neo-unital $\mathcal{X}_*$, and let $D : \mathcal{A} \longrightarrow \mathcal{X}$ be an ω^*-continuous derivation. An argument similar to part (i) shows that D is inner. Notice that the derivation $I \otimes D$ here is ω^*-continuous. □

We do not know whether or not the converse of Theorem 4.4 is true.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathrm{WAP}(\mathcal{A}^*)$ stands for the set of all *weakly almost periodic functionals* on $\mathcal{A}^*$. It is known that $\mathrm{WAP}(\mathcal{A}^*)^*$ is a dual Banach algebra with the following universal property: for every dual Banach algebra $\mathcal{B}$ and every homomorphism $\theta : \mathcal{A} \longrightarrow \mathcal{B}$, there is a unique ω^*-continuous homomorphism $\widetilde{\theta} : \mathrm{WAP}(\mathcal{A}^*)^* \longrightarrow \mathcal{B}$ for which $\widetilde{\theta} \circ \iota = \theta$, where $\iota : \mathcal{A} \longrightarrow \mathrm{WAP}(\mathcal{A}^*)^*$ is the canonical map [10, Theorem 4.10].

To our knowledge, it is unknown if essential amenability of a Banach algebra $\mathcal{A}$ implies essential Connes-amenability of $\mathrm{WAP}(\mathcal{A}^*)^*$. However, they are equivalent in the case of $\mathcal{A} = L^1(G)$ (see Proposition 4.9 below).

The following result is likely to be well-known, but since we could not locate a reference, we include a proof.

Theorem 4.5. *Let $\mathcal{A}$ be a dual Banach algebra. The following statements are equivalent:*

(i) *$\mathcal{A}$ is Connes-amenable;*
(ii) $\mathrm{WAP}(\mathcal{A}^*)^*$ *is Connes-amenable.*

Proof. It should be mentioned that if $\mathcal{A}$ is a dual Banach algebra, then the map ι is an isometry [10, page 141].

(i) ⇒ (ii) This is immediate by [7, Proposition 4.2(ii)], because $\mathcal{A}$ is ω^*-dense in $\mathrm{WAP}(\mathcal{A}^*)^*$.

(ii) ⇒ (i) Suppose that $\mathrm{WAP}(\mathcal{A}^*)^*$ is Connes-amenable. By the universal property of $\mathrm{WAP}(\mathcal{A}^*)^*$, there exists an ω^*-continuous homomorphism $\theta : \mathrm{WAP}(\mathcal{A}^*)^* \longrightarrow \mathcal{A}$ such that $\theta \circ \iota = id_{\mathcal{A}}$. Here θ is an epimorphism. Again by [7, Proposition 4.2(ii)], $\mathcal{A}$ is Connes-amenable. □

For the essential case, we have the following result.

Theorem 4.6. *Let $\mathcal{A}$ be a dual Banach algebra. If* $\mathrm{WAP}(\mathcal{A}^*)^*$ *is essentially Connes-amenable, then so is $\mathcal{A}$.*

Proof. The proof is similar to that of the implication (ii) ⇒ (i) of Theorem 4.5, with Proposition 3.7(ii) in place of [7, Proposition 4.2(ii)]. □

The converse of Theorem 4.6 is unresolved.

Example 4.7. From Example 3.10, ℓ^1 is not essentially Connes-amenable. Hence by Theorem 4.6, $\mathrm{WAP}(\ell^\infty)^*$ is not essentially Connes-amenable. As a consequence, $\mathrm{WAP}(\ell^\infty)^*$ is not essentially amenable.

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Corollary 4.8. *Let $\mathcal{A}$ be a dual Banach algebra, which is Arens regular. If $\mathcal{A}^{**}$ is essentially Connes-amenable, then so is $\mathcal{A}$.*

Proof. This is a special case of Theorem 4.6. We just note that $\mathrm{WAP}(\mathcal{A}^*)^* = \mathcal{A}^{**}$, whenever $\mathcal{A}$ is Arens regular. □

Let G be a locally compact group. It is a classical result of Johnson that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group [3]. We learn from [2] that for a Banach algebra with an approximate identity, amenability and essential amenability are the same notions. Next, it was shown by Runde that amenability of G is equivalent to Connes-amenability of both the measure algebra $M(G)$ and $\mathrm{WAP}(L^\infty(G))^*$ (see [9] and [10, Proposition 4.11]). These facts together with Proposition 3.8 lead us to the following result.

Proposition 4.9. *For a locally compact group G, the following are equivalent:*

(i) *G is amenable;*
(ii) *$L^1(G)$ is amenable;*
(iii) *$L^1(G)$ is essentially amenable;*
(iv) *$M(G)$ is Connes-amenable;*
(v) *$M(G)$ is essentially Connes-amenable;*
(vi) $\mathrm{WAP}(L^\infty(G))^*$ *is Connes-amenable;*
(vii) $\mathrm{WAP}(L^\infty(G))^*$ *is essentially Connes-amenable.*

Appendix A

We have seen in Example 3.1 that the Banach algebra $\mathcal{V}_\varphi$ is essentially amenable. In this appendix we explore a second approach to these algebras.

Let $\mathcal{A}$ be a Banach algebra, and let $\varphi \in \sigma(\mathcal{A})$. We write $\mathbb{M}_\varphi$ for the set of all Banach $\mathcal{A}$-bimodules $\mathcal{X}$ for which the right module action is given by $x \cdot a = \varphi(a)x$ for $a \in \mathcal{A}$, $x \in \mathcal{X}$.

Lemma A.1. *Let $\mathcal{V}$ be a Banach space and let φ be a nonzero element of $\mathcal{V}^*$ with $\|\varphi\| \le 1$. If $\mathcal{X}$ is a neo-unital Banach $\mathcal{V}_\varphi$-bimodule, then $\mathcal{X} \in \mathbb{M}_\varphi$.*

Proof. Take arbitrary elements $a \in \mathcal{V}_\varphi$ and $x \in \mathcal{X}$. There exist $b \in \mathcal{V}_\varphi$ and $y \in \mathcal{X}$ such that $x = y \cdot b$. Hence $x \cdot a = y \cdot ba = \varphi(a) y \cdot b = \varphi(a)x$, as required. □

Proposition A.2. *Let $\mathcal{V}$ be a Banach space of dimension at least 2 and let φ be a nonzero element of $\mathcal{V}^*$ with $\|\varphi\| \le 1$. Then $\mathcal{V}_\varphi$ is essentially amenable without an approximate identity.*

Proof. Let $\mathcal{X}$ be a neo-unital Banach $\mathcal{V}_\varphi$-bimodule, and let $D : \mathcal{V}_\varphi \longrightarrow \mathcal{X}^*$ be a derivation. By Lemma A.1, $\mathcal{X} \in \mathbb{M}_\varphi$. It was shown in Behnamian and Mahmoodi ('On pointwise amenability and biprojectivity of Banach algebras', submitted) that $\mathcal{V}_\varphi$ is φ-amenable, in the sense of [4]. Therefore, D is inner, by [4, Theorem 1.1]. □

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MOHSEN ZIAMANESH, Department of Mathematics,
South Tehran Branch, Islamic Azad University, Tehran, Iran
e-mail: st_m_ziamanesh@azad.ac.ir

BEHROUZ SHOJAEE, Department of Mathematics,
Karaj Branch, Islamic Azad University, Karaj, Iran
e-mail: shoujaei@kiau.ac.ir

AMIN MAHMOODI, Department of Mathematics,
Central Tehran Branch, Islamic Azad University, Tehran, Iran
e-mail: a_mahmoodi@iauctb.ac.ir