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ON THE ORGANIZATION OF ROUND ROBIN TOURNAMENTS WITH CONSTRAINTS

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Abstract

A sporting league places every team into one of several divisions of equal size, and runs a round robin tournament for each division. Some teams are paired with another team, not necessarily in the same division, to share facilities. It is shown that however many teams are paired and whatever the pairings, it is always possible to schedule the fixtures in the minimum time, so that no two paired teams have home matches simultaneously.

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1. Introduction

Many sporting leagues use a round robin tournament format, where each team plays each other team in the first half of the season, and the same fixtures are repeated with home advantage reversed in the second half of the season. As a result, a team's facilities lie idle half the time and an obvious economy would be for teams to pair up and share facilities. The tournament fixtures would, however, then need to be arranged, so that paired teams never both had home matches simultaneously. It is not difficult to find such an arrangement. This paper is concerned with the (frequently occurring, but more complicated) situation where a sporting league has several divisions, each team plays only the other teams in its division and each division has its own round robin tournament. It is shown that it is always possible to schedule the fixtures so that no paired teams have home matches simultaneously, even if teams in different divisions may be paired and however many teams are paired.

This result does not appear to be in the literature, although the arrangement of round robin tournaments with other constraints is a well-studied subject [2, 3]. The result will be proved by describing how to construct an appropriate fixture list, with references to the theorems that guarantee and underlie the various steps.

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2. Preliminaries and plan of proof

Let a sporting league have r divisions, and suppose that each division plays its own round robin tournament, as described in the introduction. Suppose further that no team plays more than one match each week and the tournaments are to be played in the minimum time. If there are $2n$ teams in each division, the half-season must last at least $2n - 1$ weeks and this will be the minimum time. If there are $2n - 1$ teams in each division, the half-season must still last at least $2n - 1$ weeks, as each week in each division at least one team must have no match. It is standard procedure to introduce a dummy team in each division and use a schedule for $2n$ teams, interpreting a match against the dummy team as a free week. Thus, it suffices to consider the case of r divisions, where each division has $2n$ teams.

Now suppose that some teams are paired up to share facilities, where paired teams may play in the same or different divisions.

Theorem 2.1. *However many teams are paired up, and whatever divisions these teams are in, it is possible to arrange the fixture schedules for the round robin tournaments so that no two paired teams ever have a home match simultaneously.*

Without loss of generality, assume that every team is paired with another since additional pairings can only make the task of arranging the schedule more difficult.

The first step in the proof of Theorem 2.1 is to label the $2n$ teams in each division $A_1, B_1, \ldots, A_n, B_n$ in such a manner that two teams which are paired are labelled A_i and B_i for the same value of i, regardless of which divisions they are in. It requires proof that such a labelling always exists.

The following example demonstrates that assigning the labels can be a delicate matter.

Example 2.2. Let $r = 3$ and $n = 3$. Suppose that four teams in Division 1 are paired with teams in Division 2, the remaining two teams in Division 1 are paired with two teams in Division 3, the remaining two teams in Division 2 are paired with teams in Division 3, and finally, the remaining two teams in Division 3 are paired with each other. We could not just label the first four pairs (A_1^1, B_1^2), (B_1^1, A_1^2), (A_2^1, B_2^2), (B_2^1, A_2^2), where the superscript indicates the division of the team, because the next two pairings would have to be (A_3^1, B_3^3), (B_3^1, A_3^3), and we would be unable to label the next two pairings. However in this example, there is a labelling that does work, namely, (A_1^1, B_1^2), (B_1^1, A_1^2), (A_2^1, B_2^2), (A_3^1, B_3^2) for the first four pairings, (B_3^1, A_3^3), (B_2^1, A_2^3) for the next two pairings, (A_2^2, B_2^3), (A_3^2, B_3^3) for the next two pairings, and (A_1^3, B_1^3) for the final pairing.

The second step in the proof of Theorem 2.1 is to exhibit a fixture list in the case when $r = 1$, that is for a round robin tournament with $2n$ teams $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that, for each i, teams A_i and B_i are never assigned to be the home team in the same week. This same fixture list can be used in each division using the labelling found in the first step. Proofs of the first and second steps are outlined in Sections 3 and 4, respectively.

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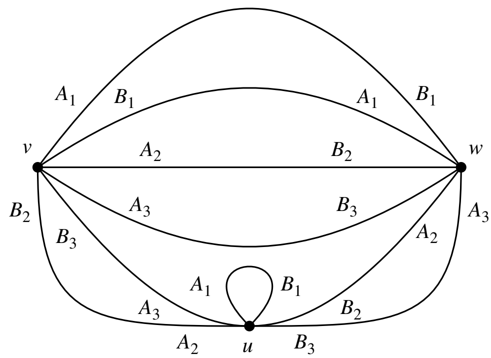

FIGURE 1. Graph G.

3. How to label the teams

Construct a graph G where each division is represented by a vertex, and each pairing of a team in Division v with a team in Division w is represented by an edge between vertex v and vertex w. Keep track of which teams are involved in each pairing by writing the name of the team in Division v on the edge next to vertex v and the name of the team in Division w at the other end of the edge next to vertex w. Here v and w may or may not be the same. Since every team has been assumed to be paired off with another team, each vertex of the graph G has degree $2n$, that is, there are $2n$ lines incident to v, each one associated to one of the teams in Division v. The task is to label each team with one of the labels $A_1, \ldots, A_n, B_1, \ldots, B_n$, so that each of these labels occurs exactly once next to each vertex (as these are the teams of a division) and each edge is labelled A_iB_i for some i (indicating a pairing).

For Example 2.2, the graph G in Figure 1 is the underlying graph below (without the A, B labelling of the edges), where Divisions 1, 2, 3 are represented by the vertices v, w, u, respectively. The A, B labelling of the edges is the consistent labelling described in this example.

In general, first insert all the labels A_n, B_n onto graph G as follows. Select a collection of edges of graph G with the property that together these edges form disjoint cycles such that each vertex in graph G appears exactly once on just one of these cycles. That this can be done is guaranteed by Petersen's theorem that every $2n$-regular graph has a 2-factor [1, Corollary 2.1.5]. More instructively, here is a way of constructing such cycles. For ease of exposition, the following account applies when graph G is connected; if graph G is not connected, apply the method to each of its components. Let the vertices of graph G be denoted by $v_1, \ldots, v_r$. By Euler's theorem [1, Theorem 1.8.1] graph G has an Euler tour, that is, a closed path which contains every edge of G precisely once (and, therefore, passes through every vertex exactly n times). The standard proof of Euler's theorem explains how to construct such a tour. Arbitrarily assign a direction to the Euler tour. Construct a new graph H with $2r$ vertices $x_1, \ldots, x_r, y_1, \ldots, y_r$ where there is an edge between x_i and y_j if and only if

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the Euler circuit contains an edge starting at v_i and ending at v_j. Every vertex of graph H has degree n. Given any subset S of the vertices x_i, let T be the set of all vertices y_j such that there is an edge in graph H between y_j and at least one of the vertices of S. If S consists of s vertices, then T must consist of at least s vertices, since each of the ns edges that emerge from S must end somewhere, and not more than n of them can end at the same vertex of T. By Hall's marriage theorem, there is a pairing between the set $\{x_1, \ldots, x_r\}$ and the set $\{y_1, \ldots, y_r\}$, such that paired vertices are joined by an edge of graph H. For every pair (x_i, y_j) in this pairing, label the edge in graph G from v_i to v_j as A_nB_n. Each vertex v of graph G will occur twice, once as a v_i and once as a v_j. Note that the above procedure is constructive if one invokes a constructive proof of Hall's marriage theorem.

This first step labels certain edges of graph G, such that at every vertex there is exactly one incident edge labelled A_n and exactly one incident edge labelled B_n. Delete these edges from graph G to obtain a $(2n-2)$-regular subgraph G' of G with the same vertices as G. Similar to the first step, select a collection of edges of graph G' with the property that their union forms disjoint cycles such that each vertex of G' appears just once on these cycles. Label the edges of these cycles $A_{n-1}B_{n-1}$. Continue in this way. After n steps, every edge of graph G will be labelled A_iB_i for some i and the labelling task is completed.

4. A fixture list for a single division

To complete the proof of Theorem 2.1, it remains to show that a round robin tournament between teams $A_1, \ldots, A_n, B_1, \ldots, B_n$ can be arranged so that, for each i, the teams A_i and B_i are never assigned home advantage simultaneously. Indeed, the fixtures obtained by Kirkman's circle method can be used for an appropriate labelling of the teams, which is now described. The fixture schedule is most easily envisaged geometrically. Let $P_0, P_1, \ldots, P_{2n-2}$ be equally spaced adjacent points on a circle with centre C, and let D denote the diameter through P_0. For the first round place team B_n at C, A_n at P_0, and $B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-1}$ at $P_1, \ldots, P_{2n-2}$, respectively. Let B_n play A_n, and the other $n-1$ matches be between teams that occupy positions that reflect to each other in D, that is, B_i plays A_{n-i} for each $i \leq n$. For the second round, rotate the teams around the circle by one position. The team A_{n-1} is now at P_0 and plays B_n; the other matches are between the teams that occupy positions which reflect to each other in D. For the third round rotate the teams further one position and use reflection in D to determine the matches. Continue in this way. In every round, let the teams at $P_1, \ldots, P_{n-1}$ be the home teams, and let B_n be a home team if and only if A_n is not in one of the positions $P_1, \ldots, P_{n-1}$. It is geometrically clear that A_i and B_i never have home advantage simultaneously.

It is of course impossible to arrange a round robin tournament so that every team has alternating home and away matches since two teams that have home matches in the first week, cannot both have home matches only in odd-numbered weeks, as they have to play each other at some stage. The match schedule just described has the undesirable property that for each team the runs of consecutive home matches and consecutive away matches are long. This shortcoming can be overcome by reversing the home advantage in all even-numbered weeks.

It is perhaps worth mentioning that the fixture list obtained by the circle method is not the only arrangement of a round robin tournament satisfying the conditions of the first sentence of this section. When $n = 4$, one can write down a fixture list which cannot be transformed into Kirkman's using the operations of relabelling a pair (A_i, B_i) as (B_i, A_i), permuting the subscripts $1, 2, 3, 4$ and reordering the weeks. Since this nonuniqueness is hardly surprising, the proof is omitted. The nonuniqueness widens the possibility of finding a fixture schedule that simultaneously has other desirable features [2, 3].

5. Concluding remarks

This study was motivated by a proposal which a player put to the Annual General Meeting of the Edinburgh Table Tennis League in August 2011. His suggestion was to have smaller divisions with fewer teams in each division, instead of having four large divisions. He felt that it would be more motivating if teams were promoted or relegated twice a year instead of once. The committee was concerned that it would be too time-consuming and complicated, if not impossible, to rearrange the fixture schedule if there were more divisions and if teams, which were paired by clubs to share home tables, could move more freely between divisions. Fortunately, their concerns were unfounded.

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