ON THE EXTENSION OF MEASURE BY THE METHOD OF BOREL

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Introduction. This paper concerns the problem of extending a given measure defined on a Boolean ring to a measure on the generated σ-ring. Two general methods are familiar to the literature, that of Lebesgue (outer measure) and a method proposed by Borel using transfinite induction (**4**, 49–134; **2**, 228–238). The problem of Borel, to extend a finite measure on an algebra of sets to a measure on the generated σ-algebra by way of transfinite induction on appropriate intermediate classes, has been solved by several authors (**5**; **1**). In the present paper we propose to develop the method of Borel in its full generality, that is, to extend, by transfinite induction, an arbitrary measure, not necessarily finite, defined on a Boolean ring of sets, to a measure on the generated σ-ring.

1. Boolean relations. The terminology and notation of Halmos (**3**) will be used without comment. A sequence of sets $\{E_n\}$ will be called "ascending" (descending) if

$$E_n \leqslant E_{n+1} (E_n \geqslant E_{n+1}), \qquad n = 1, 2, \ldots .$$

To indicate that a given sequence of sets $\{E_n\}$ is disjoint, the symbol $\sum$ or $+$ will be used instead of $\bigcup$.

1.1 A sequence of sets $\{E_n\}$ is said to converge if

$$\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i,$$

in which case, it converges to the limit

$$\lim E_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i.$$

If the sequences $\{E_n\}$, $\{F_n\}$ both converge, then also $\{E_n \cup F_n\}$ converges and

$$\lim (E_n \cup F_n) = \lim E_n \cup \lim F_n;$$

similarly for the intersection, difference and complement.

1.2 Let $\mathbf{C}$ denote a given class of subsets of the space: $\mathbf{C}^-$ ($\mathbf{C}^+$) denotes the class of limits of descending (ascending) sequences of sets of $\mathbf{C}$; $\mathbf{C}^0$ denotes the class of limits of convergent sequences of sets of $\mathbf{C}$. The letter $\mathbf{R}$ always denotes a ring of sets. $\mathbf{R}^-$ ($\mathbf{R}^+$) is a lattice closed under countable intersections (unions), and $\mathbf{R}^0$ is a ring; we have the following inclusions:

$$\mathbf{R} \leqslant \mathbf{R}^- \leqslant \mathbf{R}^{-+},\ \mathbf{R} \leqslant \mathbf{R}^+ \leqslant \mathbf{R}^{+-},\ \mathbf{R}^0 \leqslant \mathbf{R}^{-+},\ \mathbf{R}^0 \leqslant \mathbf{R}^{+-}.$$

Received January 5, 1956.

1.3 With a given ring $\mathbf{R}$ is associated a transfinite sequence of rings:

$$\mathbf{R}_0 \leqslant \mathbf{R}_1 \leqslant \mathbf{R}_2 \leqslant \ldots \leqslant \mathbf{R}_\alpha \leqslant \ldots,$$

where $\mathbf{R}_0 = \mathbf{R}$, $\mathbf{R}_\alpha = \mathbf{R}^0{}_{\alpha-1}$ if α is an ordinal number of the first kind, and

$$\mathbf{R}_\alpha = \bigcup_{\beta<\alpha} \mathbf{R}_\beta$$

if α is of the second kind. If ω_1 denotes the first non-countable ordinal, $\mathbf{R}\omega_1 = \mathbf{S}(\mathbf{R})$, the σ-ring generated by $\mathbf{R}$.

1.4 We say that a set F "covers" the set E if $F \geqslant E$. The class of sets covered by some set of $\mathbf{R}^+$ is a σ-ring, consequently every set of $\mathbf{S}(\mathbf{R})$ is covered by some set of $\mathbf{R}^+$.

1.5 By a "measure on a lattice" we mean a real function on the sets of the lattice satisfying the defining conditions corresponding to these of a (countably additive) measure on a ring, plus the monotone property: $\mu(E) \leqslant \mu(F)$ if $E \leqslant F$. Since a lattice of sets contains the null set (by definition), a measure on a lattice is additive in the finite sense, and subtractive.

2. Induction of the measure. The immediate object is the extension of a measure μ on $\mathbf{R}$ to a measure μ^0 on $\mathbf{R}^0$. The first step will be to extend μ to a measure μ^+ on the lattice $\mathbf{R}^+$, using the fact that μ is continuous from below on $\mathbf{R}$. The second step will be to extend μ^+ to μ^0 on $\mathbf{R}^0$, using the fact that every set of $\mathbf{R}^0$ is covered by a set of $\mathbf{R}^+$.

THEOREM 2.1. *A measure μ on a ring $\mathbf{R}$ admits an extension to a measure μ^+ on the lattice $\mathbf{R}^+$.*

Proof. If $\{E_m\}$, $\{F_n\}$ $(E_m, E_n \in \mathbf{R})$ are ascending sequences coverging to $E \in \mathbf{R}^+$ it is easily verified that

$$\lim_m \lim_n \mu(E_m \cap F_n) = \lim_n \lim_m \mu(E_m \cap F_n).$$

Since μ is continuous from below on $\mathbf{R}$,

$$\lim_m \mu(E_m) = \lim_m \mu(E_m \cap E) = \lim_m \mu(\lim_n(E_m \cap F_n)) = \lim_m \lim_n \mu(E_m \cap F_n).$$

Similarly,

$$\lim_n \mu(F_n) = \lim_n \lim_m \mu(E_m \cap F_n),$$

so that

$$\lim_m \mu(E_m) = \lim_n \mu(F_n).$$

We may therefore define, without ambiguity,

$$\mu^+(E) = \lim_m \mu(E_m),$$

https://doi.org/10.4153/CJM-1956-058-3 Published online by Cambridge University Press

where $\{E_m\}$ is any ascending sequence of sets of $\mathbf{R}$ which converges to E; furthermore μ^+ on $\mathbf{R}^+$ is an extension of the function μ on $\mathbf{R}$. Let F, E be sets of $\mathbf{R}^+$ such that $F \leqslant E$. Let $\{F_n\}$, $\{E_n\}$ be ascending sequences of sets of $\mathbf{R}$ converging to F, E respectively. $\{E_n \cap F_n\}$ is an ascending sequence of sets of $\mathbf{R}$ converging to F. Since $\mu(E_n \cap F_n) \leqslant \mu(E_n)$, we have in the limit, $\mu^+(F) \leqslant \mu^+(E)$. Suppose that

$$E = \sum_{n=1}^{\infty} E_n (E_n \in \mathbf{R});$$

then $E \in \mathbf{R}^+$, and

$$\mu^+(E) = \lim_n \mu\left(\sum_1^n E_i\right) = \lim_n \sum_1^n \mu(E_i) = \sum_1^{\infty} \mu(E_n).$$

Suppose that

$$E = \sum_1^{\infty} E_n (E_n \in \mathbf{R}^+);$$

then $E \in \mathbf{R}^+$ and the E_n may be decomposed:

$$E_n = \sum_{m=1}^{\infty} E_{nm} (E_{nm} \in \mathbf{R}),$$

so that

$$\mu^+(E) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(E_{nm}) = \sum_1^{\infty} \mu^+(E_n).$$

Thus μ^+ on $\mathbf{R}^+$ satisfies the conditions of 1.5. In what follows, μ^+ denotes the extension to R^+ of μ on $\mathbf{R}$, according to this theorem.

THEOREM 2.2. *The measure μ^+ on $\mathbf{R}^+$ enjoys the following properties:*

(1) *If E, $F \in \mathbf{R}^+$ have finite measures,*

$$\mu^+(E \cup F) = \mu^+(E) + \mu^+(F) - \mu^+(E \cap F).$$

(2) $$\mu^+(E \cup F) \leqslant \mu^+(E) + \mu^+(F) \qquad (E, F \in \mathbf{R}^+).$$

(3) $$\mu^+\left(\bigcup_1^{\infty} E_n\right) \leqslant \sum_1^{\infty} \mu^+(E_n) \qquad (E_n \in \mathbf{R}^+).$$

(4) *μ^+ is continuous from above on $\mathbf{R}^+$.*

Proof. (1): It suffices to consider, in the limit, the same relation for $\mathbf{R}$.
(2) A consequence of (1).
(3) If $E_n \in \mathbf{R} (n = 1, 2, \ldots)$ then

$$\mu\left(\bigcup_1^n E_i\right) \leqslant \sum_1^n \mu(E_i),$$

and therefore

$$\mu^+\left(\bigcup_1^{\infty} E_n\right) = \lim_n \mu\left(\bigcup_1^n E_i\right) \leqslant \lim_n \sum_1^n \mu(E_i) = \sum_1^{\infty} \mu(E_n).$$

https://doi.org/10.4153/CJM-1956-058-3 Published online by Cambridge University Press

If $E_n \in \mathbf{R}^+ (n = 1, 2, \ldots)$ the E_n may be decomposed

$$E_n = \sum_{m=1}^{\infty} E_{nm} (E_{nm} \in \mathbf{R}),$$

so that $\mu^+(E_n) = \sum \mu(E_{nm})$, and

$$\sum_1^{\infty} \mu^+(E_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(E_{nm}) \geqslant \mu^+\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{nm}\right) = \mu^+\left(\bigcup_{n=1}^{\infty} E_n\right).$$

(4) It must be shown that if $\{E_n\}$ $(E_n \in \mathbf{R}^+)$ is a descending sequence of sets of finite measure (μ^+) converging to $E \in \mathbf{R}^+$, then

$$\lim_n \mu^+(E_n) = \mu^+(E).$$

We first prove the theorem for the case $E = 0$. Let $\epsilon > 0$ be arbitrary. Each E_n covers a set $F_n \in \mathbf{R}$ such that

$$\mu^+(E_n - F_n) = \mu^+(E_n) - \mu^+(F_n) < 2^{-n}\epsilon.$$

(Since $F_n \in \mathbf{R}$, $E_n - F_n \in \mathbf{R}^+$, and μ^+ is subtractive on $\mathbf{R}^+$.) Set

$$G_n = \bigcap_1^n F_i;$$

then $G_n \in \mathbf{R}$ and $G_n \leqslant E_n$, so that $\lim G_n = 0$. Consequently, $\lim \mu(G_n) = 0$. Since

$$\mu^+(E_n - G_n) = \mu^+\left(\bigcup_{i=1}^{n} (E_n - F_i)\right) \leqslant \mu^+\left(\bigcup_{i=1}^{n} (E_i - F_i)\right) \leqslant \sum_{i=1}^{n} \mu^+(E_i - F_i) < \epsilon,$$

we have

$$\mu^+(E_n) - \mu(G_n) < \epsilon, \ \lim_n \mu^+(E_n) \leqslant \epsilon.$$

Since ϵ is arbitrary, $\lim \mu^+(E_n) = 0$. In the general case, $E = \lim E_n$ is any set of finite measure (μ^+). There exists an ascending sequence $\{F_n\}$ $(F_n \in \mathbf{R})$ converging to E. $\{E_n - F_n\}$ is a descending sequence $(E_n - F_n \in \mathbf{R}^+)$ converging to 0; therefore

$$\lim_n \mu^+(E_n) - \lim_n \mu^+(F_n) = \lim_n \mu^+(E_n - F_n) = 0,$$
$$\lim_n \mu^+(E_n) = \lim_n \mu(F_n) = \mu^+(E).$$

THEOREM 2.3. *If μ is a measure on the ring $\mathbf{R}$ there exists a measure μ^0 on $\mathbf{R}^0$ which is an extension of μ.*

Proof. Let E be any set of $\mathbf{R}^0$: if every $E^+ \in \mathbf{R}^+$ which covers E is of infinite measure (μ^+), we set $\mu^0(E) = \infty$. Suppose now that there exists a set of $\mathbf{R}^+$ of finite measure (μ^+) covering E; then there exists a descending sequence $\{E_n\}$ of sets of $\mathbf{R}^+$, all of finite measure (μ^+), converging to E. If $\{F_n\}$ is a second sequence, with the properties of $\{E_n\}$, converging to E, then it is easily verified, as in the proof of 2.1, using the fact that μ^+ is continuous from above on $\mathbf{R}^+$, that $\lim \mu^+(E_n) = \lim \mu^+(F_n)$. Hence we may define, without ambiguity, $\mu^0(E) = \lim \mu^+(E_n)$. If $E, F \in \mathbf{R}^0$ and $E \leqslant F$, then $\mu^0(E) \leqslant \mu^0(F)$. For it suffices to consider the case $\mu^0(E) < \infty$, $\mu^0(F) < \infty$, and then the proof is analogous to the case treated in the proof of 2.1. To prove that $\mu^0(E + F) = \mu^0(E) + \mu^0(F)$, it suffices to consider the case $\mu^0(E) < \infty$, $\mu^0(F) < \infty$. Let $\{E_n\}$, $\{F_n\}$ be descending sequences of sets of $\mathbf{R}^+$, all of finite measure (μ^+), converging to E, F respectively. We have

$$\mu^+(E_n \cup F_n) = \mu^+(E_n) + \mu^+(F_n) - \mu^+(E_n \cap F_n),$$

and since $\lim (E_n \cap F_n) = 0$ by (2.2 (4)), $\lim \mu^+(E_n \cap F_n) = 0$. In the limit we have the required equation.

Suppose that

$$E = \sum_1^\infty E_n \in \mathbf{R}^0 (E_n \in \mathbf{R}^0).$$

It follows from the monotone property that

$$\mu^0(E) \geqslant \sum_1^\infty \mu^0(E_n).$$

It remains to prove the inverse inequality for the case $\mu^0(E_n) < \infty$ $(n = 1, 2, \ldots)$. Let $\epsilon > 0$ be arbitrary. For each index n there exists $F_n \in \mathbf{R}^+$, covering E_n, such that $\mu^+(F_n) - \mu^0(E_n) < 2^{-n}\epsilon$. We have, by 2.2 (3),

$$\mu^0\left(\sum_1^\infty E_n\right) \leqslant \mu^+\left(\bigcup_1^\infty F_n\right) \leqslant \sum_1^\infty \mu^+(F_n) < \sum_1^\infty \mu^0(E_n) + \epsilon,$$

so that

$$\mu^0\left(\sum_1^\infty E_n\right) \leqslant \sum_1^\infty \mu^0(E_n),$$

and the proof is complete.

The measure μ^0 on $\mathbf{R}^0$ is an extension of μ^+ on $\mathbf{R}^+$, which in turn is an extension of μ on $\mathbf{R}$. In what follows, μ^0 denotes the extension to $\mathbf{R}^0$ of μ on $\mathbf{R}$ according to Theorem 2.3. Consider the transfinite sequence of rings (1.3):

$$\mathbf{R} = \mathbf{R}_0 \leqslant \mathbf{R}_1 \leqslant \mathbf{R}_2 \leqslant \ldots \leqslant \mathbf{R}_\alpha \leqslant \ldots \leqslant \mathbf{R}_{\omega_1};$$

an extension of μ on $\mathbf{R}$ to a measure μ_α on $\mathbf{R}_\alpha$ is called a "normal extension" if for every ordinal β of the first kind $(0 \leqslant \beta \leqslant \alpha)$,

$$\mu_\beta = \overset{0}{\mu}_{\beta-1}.$$

We see directly that the normal extension μ_α, if it exists, is unique. The following theorem will serve as a lemma for the transfinite induction.

THEOREM 2.4. *Suppose that the normal extension μ_α exists for a given ordinal α. Let $\epsilon > 0$ be arbitrary; for every $E \in \mathbf{R}_\alpha$ of finite measure (μ_α), there exists $E^+ \in \mathbf{R}_0{}^+ = \mathbf{R}^+$, covering E, such that $\mu_\alpha(E^+) - \mu_\alpha(E) < \epsilon$.*

Proof. It suffices to prove the theorem for α of the first kind, assuming the theorem for $\alpha - 1$. Let $E \in \mathbf{R}_\alpha$ be of finite measure (μ_α), and suppose in the first place that $E \in \mathbf{R}_{\alpha-1}{}^+$. We may express E as a union

$$E = \bigcup_1^\infty E_n (E_n \in \mathbf{R}_{\alpha-1}),$$

and by the induction hypothesis there exists $E_n{}^+ \in \mathbf{R}_0{}^+$, covering E_n, such that

$$\mu_\alpha(E_n^+) - \mu_\alpha(E_n) < 2^{-n}\epsilon.$$

The set

$$E_0^+ = \bigcup_1^\infty E_n^+ \in \mathbf{R}_0^+$$

covers E, and

$$\mu_\alpha(E_0^+) - \mu_\alpha(E) = \mu_\alpha(E_0^+ - E) = \mu_\alpha\Big(\bigcup_1^\infty (E_n^+ - E)\Big)$$
$$\leqslant \mu_\alpha\Big(\bigcup_1^\infty (E_n^+ - E_n)\Big) \leqslant \sum_1^\infty \mu_\alpha(E_n^+ - E_n) < \epsilon.$$

The theorem is proved for $\mathbf{R}_{\alpha-1}{}^+$, then it is evidently also true for $\mathbf{R}_\alpha$, and the proof is complete.

We now prove the fundamental theorem of the paper:

THEOREM 2.5. *If μ is a measure on a ring $\mathbf{R}$ there exists a measure $\bar{\mu}$ on the generated σ-ring $\mathbf{S}(\mathbf{R})$, which is the normal extension of μ.*

Proof. Let α be any ordinal such that $0 < \alpha \leqslant \omega_1$. It suffices to prove the existence of the normal extension μ_α, assuming the existence of all the normal extensions μ_β for $\beta < \alpha$. If α is of the first kind we set $\mu_\alpha = \mu^0{}_{\alpha-1}$. Suppose that α is of the second kind. Let E be any set of $\mathbf{R}_\alpha$, then there exists $\beta < \alpha$ such that $E \in \mathbf{R}_\beta$, and we set $\mu_\alpha(E) = \mu_\beta(E)$. Then μ_α is defined, without ambiguity, as an additive, monotone function on $\mathbf{R}_\alpha$, which is an extension of μ_β for every $\beta < \alpha$. It remains to prove the countable additivity of μ_α. Suppose that

$$E = \sum_1^\infty E_n \in \mathbf{R}_\alpha (E_n \in \mathbf{R}_\alpha).$$

It will be sufficient to show that

$$\mu_\alpha(E) \leqslant \sum_1^\infty \mu_\alpha(E_n)$$

for the case that $\mu_\alpha(E_n) < \infty$, $(n = 1, 2, \ldots)$ and α is of the second kind. For each index n there exists an ordinal $\beta(n) < \alpha$ such that $E_n \in R_{\beta(n)}$. Since $\mu_{\beta(n)}$ is normal there exists $E_n{}^+ \in \mathbf{R}_0{}^+$ covering E_n such that

$$\mu_\alpha(E_n^+) - \mu_\alpha(E_n) < 2^{-n}\epsilon,$$

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ϵ being an arbitrary positive number, (2.4). Since $\bigcup E_n^+ \in \mathbf{R}_0^+$,

$$\mu_\alpha(E) \leqslant \mu_\alpha\left(\bigcup_1^\infty E_n^+\right) \leqslant \sum_1^\infty \mu_\alpha(E_n^+) < \sum_1^\infty \mu_\alpha(E_n) + \epsilon,$$

so that

$$\mu_\alpha(E) \leqslant \sum_1^\infty \mu_\alpha(E_n),$$

and this completes the proof.

Henceforth $\bar{\mu}$ will denote the (unique) normal extension to $\mathbf{S}(\mathbf{R})$ of the measure μ on the ring $\mathbf{R}$. It follows from Theorems 2.4 and 2.5 that, $\epsilon > 0$ being arbitrary, for every $E \in \mathbf{S}(\mathbf{R})$ of finite measure $(\bar{\mu})$, there exists a set $E^+ \in \mathbf{R}^+$, covering E, such that $\bar{\mu}(E^+) - \bar{\mu}(E) < \epsilon$. We prove next the dual of Theorem 2.4.

THEOREM 2.6. *Let $\epsilon > 0$ be arbitrary. For every $E \in \mathbf{S}(\mathbf{R})$ of finite measure $(\bar{\mu})$ there exists $E^- \in \mathbf{R}^-$, covered by E, such that $\bar{\mu}(E) - \bar{\mu}(E^-) < \epsilon$.*

Proof. It suffices to prove the theorem for $\mathbf{R}_\alpha$, α of the first kind, under the supposition that the theorem is true for $\mathbf{R}_{\alpha-1}$. Then the theorem is evident for $\mathbf{R}_{\alpha-1}^+$. Let E be any set of $\mathbf{R}_\alpha$ of finite measure $(\bar{\mu})$: there exists a descending sequence $\{E_n\}$ $(E_n \in \mathbf{R}_{\alpha-1}^+)$ of sets of finite measure $(\bar{\mu})$ converging to E. For each n there exists $F_n^- \in \mathbf{R}^-$ covered by E_n such that

$$\bar{\mu}(E_n) - \bar{\mu}(F_n^-) < 2^{-n-1}\epsilon.$$

Set

$$H_n = \bigcap_1^n F_i^-$$

so that $\{H_n\}$ is a descending sequence $(H_n \in \mathbf{R}^-)$, whose limit

$$H = \bigcap_1^\infty H_n$$

belongs to $\mathbf{R}^-$, such that $H_n \leqslant E_n$, $H \leqslant E$.

$$\bar{\mu}(E_n - H_n) = \bar{\mu}\left(\bigcup_{i=1}^n (E_n - F_i^-)\right) \leqslant \bar{\mu}\left(\bigcup_{i=1}^n (E_i - F_i^-)\right)$$
$$\leqslant \sum_1^n \bar{\mu}(E_i - F_i^-) < 2^{-1}\epsilon.$$

Therefore $\bar{\mu}(E_n) - \bar{\mu}(H_n) < 2^{-1}\epsilon$, and in the limit, $\bar{\mu}(E) - \bar{\mu}(H) < \epsilon$.

The dual of the above method—passing by $\mathbf{R}^-$ instead of by $\mathbf{R}^+$—does not work in the general case. To see why, it suffices to consider the case of a set $E \in \mathbf{R}^- - \mathbf{R}$ such that for every ascending sequence $\{E_n\}$ $(E_n \in \mathbf{R})$ converging to E from below, $\lim \mu(E_n) < \infty$, and for every descending sequence $\{F_n\}$ $(F_n \in \mathbf{R})$ converging to E from above, $\mu(F_n) = \infty$ $(n = 1, 2, \ldots)$. For the same reason, the direct induction from $\mathbf{R}$ to $\mathbf{R}^0$ by the formula

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$$\lim_n \mu(E_n) = \mu^0(E) \qquad (E_n \in \mathbf{R}, \lim_n E_n = E \in \mathbf{R}^0)$$

is not applicable to the general case. However, if μ is finite and bounded on $\mathbf{R}$, it can be shown that each of these alternative methods is applicable. Then, each must give the normal extension, since, in this case, the extension from $\mathbf{R}$ to $\mathbf{S}(\mathbf{R})$ is unique.

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