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The discrete Orlicz chord Minkowski problem

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Abstract. In this paper, we consider the discrete Orlicz chord Minkowski problem and solve the existence of this problem, which is the nontrivial extension of the discrete L_p chord Minkowski problem for $0 < p < 1$.

1 Introduction

Minkowski problem is one of the cornerstones of the Brunn-Minkowski theory. In the 1890s, Minkowski proposed the Minkowski problem and solved the discrete case. The Minkowski problem was completely solved by Alexsandrov and Fenchel and Jessen.

The L_p Minkowski problem is a part of L_p Brunn-Minkowski theory. Lutwak [19] proposed the L_p Minkowski problem and solved the even L_p Minkowski problem for $p > 1$, but $p \neq n$. After that, the L_p Minkowski problem and related researches can be found in [1, 2, 3, 4, 5, 9, 15, 16, 17].

The Orlicz Brunn-Minkowski theory originated from the work of Lutwak, Yang, and Zhang in 2010 [21]. The development of the Orlicz Brunn-Minkowski theory can be found in [6, 11, 23]. Harbel, Lutwak, Yang, and Zhang [11] first proposed the Orlicz Minkowski problem, which is the extension of the L_p Minkowski problem, and solved the even Orlicz Minkowski problem under some suitable conditions on φ. The existence of the Orlicz Minkowski problem without assuming that μ is the even measure was solved by Huang and He [14], but needing more conditions on φ, the L_p Minkowski problem for $p > 1$ is a special case of this result. For $0 < p < 1$, Wu, Xi, and Leng [22] solved the existence of the discrete Orlicz Minkowski problem. The Orlicz Minkowski problem and related researches can be found in [7, 8, 25, 26].

Recently, a new family of geometric measures was introduced by Lutwak, Xi, Yang, and Zhang [20] through the study of a variational formula with respect to integral geometric invariants of convex bodies called *chord integrals*. Minkowski problems associated with chord measures were posed in [20].

Let $\mathcal{K}^n$ be the collection of convex bodies (compact convex sets with nonempty interior) in $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, the chord integral $I_q(K)$ of K is defined as follows:

$$I_q(K) = \int_{\mathcal{L}^n} |K \cap \ell|^q \, d\ell, \quad q \geq 0,$$

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where $|K \cap \ell|$ denotes the length of the chord $K \cap \ell$, and the integration is with respect to the Haar measure on the Grassmannian $\mathcal{L}^n$ of lines in $\mathbb{R}^n$.

Chord integrals contain volume $V(K)$ and surface area $S(K)$ as two important special cases:

$$I_1(K) = V(K), \quad I_0(K) = \frac{\omega_{n-1}}{n\omega_n} S(K), \quad I_{n+1}(K) = \frac{n+1}{\omega_n} V(K)^2,$$

where ω_n is the volume enclosed by the unit sphere $\mathbb{S}^{n-1}$.

The differential of $I_q(K)$ defines a finite Borel measure $F_q(K,\cdot)$ on $\mathbb{S}^{n-1}$. Precisely, for convex bodies K and L in $\mathbb{R}^n$, Lutwak, Xi, Yang, and Zhang [20] obtained that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} I_q(K+tL) = \int_{\mathbb{S}^{n-1}} h_L(v)\mathrm{d}F_q(K,v), \quad q \geq 0, \tag{1.1}$$

where $F_q(K,\cdot)$ is called the *qth chord measure of* K, and h_L is the support function of L. The cases of $q = 0, 1$ of this formula are classical, which are the variational formulas of surface area and volume,

$$F_0(K,\cdot) = \frac{(n-1)\omega_{n-1}}{n\omega_n} S_{n-2}(K,\cdot), \quad F_1(K,\cdot) = S_{n-1}(K,\cdot).$$

Here, $S_{n-2}(K,\cdot)$ and $S_{n-1}(K,\cdot)$ are the $(n-2)$th order and $(n-1)$th order area measure of K, respectively.

Based on the definition of chord measure, the corresponding chord Minkowski problem was proposed. The solution to the chord Minkowski problem as $q > 0$ was given in [20].

The L_p version of the chord measure was also introduced in [20]; it can be extended from the L_p surface area measure. Correspondingly, the L_p chord Minkowski problem was considered. Xi, Yang, Zhang, and Zhao [24] solved the L_p chord Minkowski problem when $p > 1$, $q > 1$ and the symmetric case of $0 < p < 1$ via the variational method. Guo, Xi, and Zhao [10] solved the L_p chord Minkowski problem for $0 \leq p < 1$ without symmetry assumptions. Li [18] treated the discrete L_p chord Minkowski problem in the condition of $p < 0$ and $q > 0$, as for general Borel measure. Li also gave a proof but need $-n < p < 0$ and $1 < q < n+1$. Hu, Huang, and Lu [12] used flow methods to get regularity of the chord log-Minkowski problem of $p = 0$. On the side, Hu, Huang, Lu, and Wang [13] also found the smooth origin-symmetric solution for the L_p chord Minkowski problem in the case of $\{p > 0, q > 3\} \cup \{-n < p < 0, 3 < q < n+1\}$ by using the same flow as in [12].

The more generalized Orlicz chord Minkowski problem was stated in [27] by the following form:

The Orlicz chord Minkowski problem: *Suppose* $\varphi : (0,\infty) \longrightarrow (0,\infty)$ *is a continuous function. If* μ *is a finite Borel measure on* $\mathbb{S}^{n-1}$ *which is not concentrated on a great subsphere of* $\mathbb{S}^{n-1}$*, what are the necessary and sufficient conditions on* μ *such that there is a convex body* $K \in \mathcal{K}_o^n$ *and a positive constant* c *such that*

$$\mathrm{d}\mu = c\varphi(h_K)\,\mathrm{d}F_q(K,\cdot)?$$

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Due to the lack of homogeneity, the solution to the Orlicz chord Minkowski problem exists as a constant.

In this paper, we consider the existence of the discrete Orlicz chord Minkowski problem, which is an extension of the discrete L_p chord Minkowski problem for $0 < p < 1$ [10]. Our main results can be formulated as follows:

Theorem 1.1 *Let $q > 0$. $\mu = \sum_{i=1}^{N} \alpha_i \delta_{v_i}$ for some $\alpha_i > 0$, and unit vectors $v_1, \ldots, v_N \in \mathbb{S}^{n-1}$ are not contained in any closed hemisphere, where δ_{v_i} is Kronecker delta. Let $\mathcal{P}(v_1, \ldots, v_N) = \{P(z) : z \in \mathbb{R}^N \text{ such that } P(z) \in \mathcal{K}^n\}$. Suppose $\varphi : (0, \infty) \to (0, \infty)$ is differentiable and strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$ exists for every positive t. Then, there exists a polytope $P \in \mathcal{P}(v_1, \ldots, v_N)$ containing the origin in its interior and $c > 0$ such that*

$$c\varphi(h_P) dF_q(P, \cdot) = d\mu.$$

When $\varphi(t) = t^{1-p}$ for $0 < p < 1$, Theorem 1.1 is reduced to Theorem 4.6 of [10]. When $q = 1$, Theorem 1.1 is reduced to Theorem 1.2 of [22].

The paper is organized as follows: In Section 2, we present some notations and basic facts we shall use throughout. The proof of Theorem 1.1 is presented in Section 3.

2 preliminaries

In this section, we present some notations we shall use throughout.

2.1 Basics of convex bodies

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We write $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ for the boundary of the Euclidean unit ball B in $\mathbb{R}^n$.

A *convex body* is a compact convex subset of $\mathbb{R}^n$ with a nonempty interior. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, and the set of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors is denoted by $\mathcal{K}^n_o$.

A compact convex set $K \subset \mathbb{R}^n$ is uniquely determined by its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, where

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

It is trivial that for the support function of the dilate $cK = \{cx : x \in K\}$ of a compact convex set K, we have

$$h_{cK} = ch_K, \quad c > 0.$$

Note that support functions are positively homogeneous of degree 1 and subadditive. It follows immediately from the definition of support functions that for compact convex $K, L \subset \mathbb{R}^n$,

$$K \subseteq L \iff h_K \le h_L.$$

Let $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. The radial function of K with respect to x, denoted by $\rho_{K,x}(u) : \mathbb{S}^{n-1} \to \mathbb{R}$, can be written as

$$\rho_{K,x}(u) = \max\{t : tu + x \in K\}.$$

It is simple to see that when $x \in \mathrm{int}K$, we have that $\rho_{K,x}$ is a positive continuous function on $\mathbb{S}^{n-1}$. For simplicity, we write $\rho_K = \rho_{K,o}$.

The *Hausdorff distance* $d_H(K, L)$ of $K, L \in \mathcal{K}^n$ is defined by

$$d_H(K, L) := \max_{u \in \mathbb{S}^{n-1}} |h_K(u) - h_L(u)|.$$

The set $\mathcal{K}^n$ will be viewed as equipped with the Hausdorff metric. If there exists a sequence K_i of convex bodies in $\mathcal{K}^n$ and a convex body $K \in \mathcal{K}^n$, we say that $\lim_{i\to\infty} K_i = K$ provided

$$\|h_{K_i} - h_K\|_\infty \to 0.$$

Suppose Ω is a compact subset of $\mathbb{S}^{n-1}$ that is not concentrated in any closed hemisphere. The set of continuous functions on Ω will be denoted by $C(\Omega)$. For $h \in C^+(\Omega)$, the *Wulff-shape* $[h]$ is a compact convex set defined by

$$[h] = \{x \in \mathbb{R}^n : x \cdot v \le h(v), \forall v \in \Omega\}.$$

It is simple to see that

$$h_{[h]}(v) \le h(v). \tag{2.1}$$

We shall frequently use the fact that if $h_i \in C(\Omega)$ convergence to $h \in C(\Omega)$ uniformly, then the $[h_i] \to [h]$ in Hausdorff metric.

A useful fact is that, when $[h] \in \mathcal{K}^n$, the support of $S_{n-1}([h], \cdot)$ must be contained in Ω. In particular, let $v_1, \dots v_N$ $(N \ge n+1)$ be unit vectors that are not contained in any closed hemisphere, and let $\Omega = \{v_1, \dots, v_N\}$. For $z = (z_1, \dots, z_N) \in \mathbb{R}^N$, we write

$$[z] = P(z) = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^n : x \cdot v_i \le z_i\}.$$

Define $\mathcal{P}(v_1, \dots, v_N)$ by

$$\mathcal{P}(v_1, \dots, v_N) = \left\{P(z) : z \in \mathbb{R}^N \quad \text{such that } P(z) \in \mathcal{K}^n\right\}.$$

2.2 Chord integral and chord measure

Let $K \in \mathcal{K}^n$. For $z \in \mathrm{int}K$ and $q \in \mathbb{R}$, the qth dual quermassintegral $\widetilde{V}_q(K, z)$ of K with respect to z is

$$\widetilde{V}_q(K, z) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_{K,z}^q(u) \mathrm{d}u,$$

where $\rho_{K,z}(u) = \max\{\lambda > 0 : z + \lambda u \in K\}$ is the radial function of K with respect to z. When z is the origin, it reduces to the radial function $\rho_K(u)$. When $z \in \partial K$, $\widetilde{V}_q(K, z)$

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is defined in the way that the integral is only over those $u \in \mathbb{S}^{n-1}$ such that $\rho_{K,z}(u) > 0$. In other words,

$$\widetilde{V}_q(K,z) = \frac{1}{n} \int_{\rho_{K,z}(u)>0} \rho^q_{K,z}(u)\mathrm{d}u, \quad \text{whenever } z \in \partial K.$$

The integrals of dual quermassintegrals with respect to $z \in K$ naturally give rise to translation invariant quantities. These are known as *chord integrals* in integral geometry. For $K \in \mathcal{K}^n$, the chord integral $I_q(K)$ of K is defined as follows:

$$I_q(K) = \int_{\mathcal{L}^n} |K \cap \ell|^q \, \mathrm{d}\ell, \quad q \geq 0,$$

where $|K \cap \ell|$ denotes the length of the chord $K \cap \ell$, and the integration is with respect to the Haar measure on the Grassmannian $\mathcal{L}^n$ of lines in $\mathbb{R}^n$.

For $q > 0$, the chord integral can be written as the integral of dual quermassintegrals in $z \in K$:

$$I_q(K) = \frac{q}{\omega_n} \int_K \widetilde{V}_{q-1}(K,z)\mathrm{d}z.$$

In analysis, chord integral can be recognized as the Riesz potential: for each $q > 1$, we have

$$I_q(K) = \frac{q(q-1)}{n\omega_n} \int_K \int_K \frac{1}{|x-z|^{n-q+1}} \mathrm{d}x\mathrm{d}z. \tag{2.2}$$

An elementary property of the functional I_q is its homogeneity. If $K \in \mathcal{K}^n$ and $q \geq 0$, then

$$I_q(tK) = t^{n+q-1} I_q(K)$$

for $t > 0$. By compactness of K, it is simple to see that the chord integral $I_q(K)$ is finite whenever $q > 0$.

Let $K \in \mathcal{K}^n$ and $q > 0$. the chord measure $F_q(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$ given by

$$F_q(K,\eta) = \frac{2q}{\omega_n} \int_{\nu_K^{-1}(\eta)} \widetilde{V}_{q-1}(K,z)\mathrm{d}\mathcal{H}^{n-1}(z), \quad \text{for each Borel set } \eta \subset \mathbb{S}^{n-1},$$

where $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is the Gauss map that takes boundary points of K to their corresponding outer unit normals. Note that by convexity of K, its Gauss map ν_K is almost everywhere defined on ∂K with respect to the $(n-1)$-dimensional Hausdorff measure.

The significance of the chord measure $F_q(K,\cdot)$ is that it comes from differentiating, in a certain sense, the chord integral I_q; see [20]. It is simple to see that the chord measure $F_q(K,\cdot)$ is absolutely continuous with respect to the surface area measure $S_{n-1}(K,\cdot)$. In particular, for each $P \in \mathcal{P}(\nu_1,\ldots,\nu_N)$, we have that the chord measure

$F_q(P,\cdot)$ is supported entirely on $\{v_1,\ldots,v_N\}$. It was shown in Theorem 4.3 of [20] that

$$I_q(K) = \frac{1}{n+q-1}\int_{\mathbb{S}^{n-1}} h_K(v)\,\mathrm{d}F_q(K,v).$$

The following lemma shows the variational formula of the chord integral.

Lemma 2.1 [20] *Let $q>0$ and Ω be a compact subset of $\mathbb{S}^{n-1}$ that is not concentrated on any closed hemisphere. Suppose that $g:\Omega\to\mathbb{R}$ is continuous and $h_t:\Omega\to(0,\infty)$ is a family of continuous functions given as follows:*

$$h_t = h_0 + tg + o(t,\cdot),$$

for each $t\in(-\delta,\delta)$ for $\delta>0$. Here, $o(t,\cdot)\in C(\Omega)$ and $o(t,\cdot)/t$ tends to 0 uniformly on Ω as $t\to0$. Let K_t be the Wulff-shape generated by h_t and K be the Wulff-shape generated by h_0. Then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} I_q(K_t) = \int_{\Omega} g(v)\mathrm{d}F_q(K,v).$$

Taking Ω to be a finite set $\{v_1,\ldots v_N\}$, where the $v_i\in\mathbb{S}^{n-1}$ are not contained entirely in any closed hemisphere, we immediately obtain the following corollary for the discrete case.

Corollary 2.2 [10] *Let $q>0$, $z=(z_1,\ldots,z_N)\in\mathbb{R}^N_+$, $\beta=(\beta_1,\ldots,\beta_N)\in\mathbb{R}^N$, and $v_1,\ldots,v_N$ be N unit vectors that are not contained in any closed hemisphere. For sufficiently small $|t|$, consider $z(t)=z+t\beta>0$ and*

$$P_t = [z(t)] = \bigcap_{i=1}^{N}\{x\in\mathbb{R}^n : x\cdot v_i \le z_i(t) = z_i + t\beta_i\}.$$

Then, for $q>0$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} I_q(P_t) = \sum_{i=1}^{N}\beta_i F_q(P_0,v_i). \tag{2.3}$$

Chord measures inherit their translation invariance and homogeneity from chord integrals. The following lemma shows that the chord measure $F_q(K,\cdot)$ is weakly continuous on $\mathcal{K}^n$ with respect to Hausdorff metric.

Lemma 2.3 [24] *Let $q>0$ and $K_i\in\mathcal{K}^n$. If $K_i\to K\in\mathcal{K}^n$, then the chord measure $F_q(K_i,\cdot)$ converges to $F_q(K,\cdot)$ weakly.*

3 The discrete Orlicz chord Minkowski problem

Let μ be a finite discrete Borel measure on $\mathbb{S}^{n-1}$ that is not concentrated in any closed hemisphere; that is,

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$$\mu = \sum_{i=1}^{N} \alpha_i \delta_{v_i}, \tag{3.1}$$

for some $\alpha_i > 0$ and unit vectors $v_1, \dots, v_N \in \mathbb{S}^{n-1}$ not contained in any closed hemisphere, where δ_{v_i} is Kronecker delta.

Suppose $\varphi : (0, \infty) \to (0, \infty)$ is differentiable and strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$ exists for every positive t. For any $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ such that $[z]$ has nonempty interior, we define

$$\Phi_{\phi,\mu}(z, \xi) = \sum_{j=1}^{N} \phi(z_j - \xi \cdot v_j) \cdot \alpha_j$$

for each $\xi \in [z]$. When there is no confusion about what the underlying measure μ is, we shall write $\Phi_\phi = \Phi_{\phi,\mu}$.

In this section, we consider the following extremal problem:

$$\sup_{\xi \in [z]} \Phi_{\phi,\mu}(z, \xi).$$

We will show that the functional $\Phi_{\phi,\mu}(z, \cdot)$ is strictly concave in $\xi \in \mathrm{int}[z]$ and that there exists a unique $\xi_\phi(z) \in \mathrm{int}[z]$ such that

$$\sup_{\xi \in [z]} \Phi_{\phi,\mu}(z, \xi) = \Phi_{\phi,\mu}(z, \xi_\phi(z)).$$

Lemma 3.1 [22] *If $\alpha_1, \dots, \alpha_N \in \mathbb{R}^N_+$, the unit vectors $v_1, \dots, v_N (N \geq n+1)$ are not contained in any closed hemisphere, and ϕ is strictly concave on $[0, \infty)$. Suppose $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ such that $[z]$ has nonempty interior. Then, $\Phi_{\phi,\mu}(z, \cdot)$ is strictly concave in $\xi \in [z]$.*

Then, we give the following lemma to show the existence and uniqueness of $\xi_\phi(z)$.

Lemma 3.2 [22] *Suppose $\alpha_1, \dots, \alpha_N \in \mathbb{R}^N_+$, and the unit vectors $v_1, \dots, v_N (N \geq n+1)$ are not contained in any closed hemisphere. If $\varphi : (0, \infty) \to (0, \infty)$ is differentiable and strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$ exists for every positive t and is unbounded as $t \to \infty$. Suppose $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ such that $[z]$ has nonempty interior. Then, there exists a unique $\xi_\phi(z) \in \mathrm{int}[z]$ such that*

$$\sup_{\xi \in [z]} \Phi_{\phi,\mu}(z, \xi) = \Phi_{\phi,\mu}(z, \xi_\phi(z)).$$

The following lemma shows the continuity of $\xi_\phi(z)$ and $\Phi_\phi(z, \xi_\phi(z))$.

Lemma 3.3 [22] *Suppose $\alpha_1, \dots, \alpha_N \in \mathbb{R}^N_+$, and the unit vectors $v_1, \dots, v_N (N \geq n+1)$ are not contained in any closed hemisphere. If $\varphi : (0, \infty) \to (0, \infty)$ is differentiable and strictly increasing, and $\varphi(s)$ tends to 0 as $s \to 0^+$ such that $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$ exists for every positive t and is unbounded as $t \to \infty$. Let $z^l \in \mathbb{R}^N$ be such that $\lim_{l \to \infty} z^l = z \in \mathbb{R}^N$. If $[z]$ has nonempty interior, then*

$$\lim_{l \to \infty} \xi_\phi(z^l) = \xi_\phi(z)$$

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and

$$\lim_{l\to\infty} \Phi_\phi(z^l, \xi_\phi(z^l)) = \Phi_\phi(z, \xi_\phi(z)).$$

The next lemma shows that $\xi_\phi(z)$ is a differentiable function with respect to vector addition in z.

Lemma 3.4 *Let $z = (z_1, \dots, z_N) \in \mathbb{R}^N_+$, and μ be as given in (3.1). For each $\beta \in \mathbb{R}^N$, consider*

$$z(t) = z + t\beta$$

for sufficiently small $|t|$ so that $z(t) \in \mathbb{R}^N_+$. Denote $\xi_\phi(t) = \xi_\phi(z(t))$. If $\xi_\phi(0) = o$, then $\xi_\phi'(0)$ exists. Moreover,

$$o = \sum_{j=1}^N \frac{1}{\varphi(z_j)} \alpha_j v_j. \tag{3.2}$$

Proof Since $\xi_\phi(t) \in \operatorname{int}[z(t)]$ and maximizes

$$\sup_{\xi\in[z(t)]} \Phi_\phi(z(t), \xi),$$

taking the derivative in ξ shows

$$o = \sum_{j=1}^N \frac{1}{\varphi(z_j(t) - \xi_\phi(t)\cdot v_j)} \alpha_j v_j. \tag{3.3}$$

In particular, at $t = 0$, we have

$$o = \sum_{j=1}^N \frac{1}{\varphi(z_j)} \alpha_j v_j,$$

which establishes (3.2). Set

$$F_\phi(t, \xi) = \sum_{j=1}^N \frac{1}{\varphi(z_j(t) - \xi\cdot v_j)} \alpha_j v_j.$$

Then, (3.3) simply says

$$F_\phi(t, \xi_\phi(t)) = o.$$

By a direct computation, the Jocabian with respect to ξ of F_ϕ at $t = 0$ and $\xi = 0$ is

$$\left.\frac{\partial F_\phi}{\partial \xi}\right|_{(0,0)} = \sum_{j=1}^N \frac{\varphi'(z_j)}{\varphi^2(z_j)} \alpha_j v_j \otimes v_j.$$

Since $v_1, \dots, v_N$ span $\mathbb{R}^n$, we conclude that the Jocabian $\frac{\partial F_\phi}{\partial \xi}$ is positive-definite at $t = 0$ and $\xi = 0$. By the implicit function theorem, we conclude that $\xi_\phi'(0)$ exists. ∎

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For each $q > 0$, we consider the optimization problem:

$$\inf\left\{\Phi_\phi(z,\xi_\phi(z)) : z \in \mathbb{R}^N, I_q([z]) = |\mu|\right\}. \tag{3.4}$$

Lemma 3.5 *Let $q > 0$. If there exists $z \in \mathbb{R}^N_+$ with $\xi_\phi(z) = o$ and $I_q([z]) = |\mu|$ satisfying*

$$\Phi_\phi(z,o) = \inf\left\{\Phi_\phi(z,\xi_\phi(z)) : z \in \mathbb{R}^N, I_q([z]) = |\mu|\right\},$$

then there exists a polytope $P \in \mathcal{P}(v_1,\dots,v_N)$ containing the origin in its interior such that

$$c\varphi(h_P)\mathrm{d}F_q(P,\cdot) = \mathrm{d}\mu,$$

where $P = [z]$.

Moreover, for each $i = 1,\dots,N$, we have

$$h_{[z]}(v_i) = z_i. \tag{3.5}$$

Proof Let $\beta \in \mathbb{R}^N$ be arbitrary and set $z(t) = z + t\beta$. For sufficiently small $|t|$, we have $z(t) \in \mathbb{R}^N_+$. Set

$$\lambda(t) = I_q([z(t)])^{-\frac{1}{n+q-1}}.$$

Note that $\lambda(0) = 1$.

By homogeneity of I_q, it is apparent that $I_q([\lambda(t)z(t)]) = 1$. By (2.3), we have

$$\lambda'(0) = -\frac{1}{n+q-1}\sum_{i=1}^N \beta_i F_q([z],v_i). \tag{3.6}$$

Let $\xi_\phi(t) = \xi_\phi(\lambda(t)z(t)) = \lambda(t)\xi_\phi(z(t))$ and

$$\Psi_\phi(t) = \Phi_\phi(\lambda(t)z(t),\xi_\phi(z(t))).$$

By Lemma 3.4, ξ_ϕ is differentiable at $t = 0$. Moreover, (3.2) holds.

Since z is a minimizer, the fact that $0 = \Psi'_\phi(0)$ shows

$$0 = \lambda'(0)\sum_{j=1}^N \frac{1}{\varphi(z_j)} z_j\alpha_j + \sum_{i=1}^N \frac{1}{\varphi(z_i)}\beta_i\alpha_i - \xi'_\phi(0)\sum_{j=1}^N \frac{1}{\varphi(z_j)} v_j\alpha_j.$$

By (3.2) and (3.6), we have

$$0 = -\frac{1}{n+q-1}\sum_{i=1}^N \beta_i F_q([z],v_i)\sum_{j=1}^N \frac{1}{\varphi(z_j)} z_j\alpha_j + \sum_{i=1}^N \frac{1}{\varphi(z_i)}\beta_i\alpha_i.$$

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Since β is arbitrary, we conclude that

$$\frac{1}{n+q-1}\left(\sum_{j=1}^{N}\frac{1}{\varphi(z_j)}z_j\alpha_j\right)F_q([z],v_i)=\frac{1}{\varphi(z_i)}\alpha_i;$$

that is

$$c\varphi(z_i)F_q([z],v_i)=\alpha_i,$$

where

$$c=\frac{1}{n+q-1}\sum_{j=1}^{N}\frac{1}{\varphi(z_j)}z_j\alpha_j$$

is a constant that only depends on z_j. Let $P=[z]$. Then, the existence of P is proven.

We now show (3.5). Assume that it fails for some i_0. Let $\tilde{z}\in\mathbb{R}^N_+$ be such that $\tilde{z}=h_{[z]}(v_i)$. By $h_{[f]}\le f$, we have $\tilde{z_{i_0}}<z_{i_0}$ and $\tilde{z}_i\le z_i$ for $i\ne i_0$. Note that $[z]=[\tilde{z}]$, and consequently, $I_q([\tilde{z}])=|\mu|$. By definition of Φ_ϕ and ξ_ϕ, we have

$$\Phi_\phi(\tilde{z},\xi_\phi(\tilde{z}))<\Phi_\phi(z,\xi_\phi(\tilde{z}))\le\Phi_\phi(z,\xi_\phi(z))=\Phi_\phi(z,o).$$

This is a contradiction to z being a minimizer. ∎

Theorem 3.6 *Let $q>0$, and μ be as given in (3.1). Suppose $\varphi:(0,\infty)\to(0,\infty)$ is differentiable and strictly increasing, and $\varphi(s)$ tends to 0 as $s\to0^+$ such that $\phi(t)=\int_0^t\frac{1}{\varphi(s)}\mathrm{d}s$ exists for every positive t. Then, there exists a polytope $P\in\mathcal{P}(v_1,\dots,v_N)$ containing the origin in its interior such that*

$$c\varphi(h_P)\mathrm{d}F_q(P,\cdot)=\mathrm{d}\mu.$$

Proof We consider the minimization problem (3.4). Let $z^l\in\mathbb{R}^N$ be a minimizing sequence; that is, $I_q([z^l])=|\mu|$ and

$$\lim_{l\to\infty}\Phi_\phi(z^l,\xi_\phi(z^l))=\inf\left\{\Phi_\phi(z,\xi_\phi(z)):z\in\mathbb{R}^N,I_q([z])=|\mu|\right\}.$$

Note that by translation invariance of I_q and the simple fact that

$$\Phi_\phi(z,\xi)=\Phi_\phi(z',o),$$

where $z'_j=z_j-\xi\cdot v_j$, we can assume without loss of generality that $\xi_\phi(z^l)=o$. Moreover, by the definition of Φ_ϕ, it must be the case that

$$z^l_j=h_{[z^l]}(v_j)$$

by Lemma 3.5. The fact that $o=\xi_\phi(z^l)\in\operatorname{int}[z^l]$ now implies that $z^l_j>0$.

Set $\zeta(r)=(r,\dots,r)\in\mathbb{R}^N$. Then, by the homogeneity of I_q, we may find $r_0>0$ such that

$$I_q([\zeta(r_0)])=|\mu|.$$

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Therefore,

$$\lim_{l\to\infty} \Phi_\phi(z^l, o) \le \Phi_\phi(\zeta(r_0), \xi_\phi(\zeta(r_0)))$$
$$= \sum_{j=1}^N \phi(r_0 - \xi_\phi(\zeta(r_0)) \cdot v_j)\alpha_j$$
$$\le \sum_{j=1}^N \phi(2r_0)\alpha_j < \infty, \tag{3.7}$$

where by Lemma 3.2, we used the fact that $\xi_\phi(\zeta(r_0)) \in \text{int}[\zeta(r_0)]$.

However, if we set $L_l = \max_j z_j^l$, then

$$\Phi_\phi(z^l, o) = \sum_{j=1}^N \phi(z_j^l)\alpha_j \ge \phi(L_l) \min_j \alpha_j. \tag{3.8}$$

By (3.7) and (3.8), z^l is uniformly bounded. Therefore, we may assume that $z^l \to z^0$ for some $z^0 \in \mathbb{R}^N$. By continuity of I_q, we have $I_q([z^0]) = |\mu|$, which implies that $[z^0]$ contains a nonempty interior. Lemma 3.3 now implies that

$$\xi_\phi(z^0) = \lim_{l\to\infty} \xi_\phi(z^l) = o.$$

This and the fact that $\xi_\phi(z^0) \in \text{int}[z^0]$ imply that $z^0 \in \mathbb{R}_+^N$. Moreover, by the definition of Φ_ϕ, we have

$$\Phi_\phi(z^0, o) = \lim_{l\to\infty} \Phi_\phi(z^l, o) = \inf\left\{\Phi_\phi(z, \xi_\phi(z)) : z \in \mathbb{R}^N, I_q([z]) = |\mu|\right\}.$$

Lemma 3.5 now implies the existence of P. ■

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References

[1] G. Bianchi, K. J. Böröczky, A. Colesanti and D. Yang, *The planar L_p-Minkowski problem for $-n < p < 1$*. Adv. Math. **341**(2019), 493–535.

[2] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The logarithmic Minkowski problem*. J. Amer. Math. Soc. **26**(2013), 831–852.

[3] K. J. Böröczky and H. T. Trinh, *The planar L_p-Minkowski problem for* $0 < p < 1$. Adv. in Appl. Math. **81**(2017), 58–81.

[4] S. Chen, Q. Li and G. Zhu, *On the L_p Monge-Ampé re equation*. J. Differential Equations. **263**(2017), 4997–5011.

[5] K.-S. Chou and X.-J. Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*. Adv. Math. **205**(2006), 33–83.

[6] R. J. Gardner, D. Hu and W. Weil, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*. J. Differential Geom. **97**(2014), 427–476.

[7] R. J. Gardner, D. Hug, W. Weil, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I*. Calc. Var. Partial Differential Equations. **58**(2019), 35 pp.

[8] R. J. Gardner, D. Hug, S. Xing and D. Ye, *General volumes in the Orlica-Brunn-Minkowski theory and a related Minkowski problem II*. Calc. Var. Partial Differential Equations **59** (2020), 33 pp.

[9] P. Guan and C. Xia, *The L_p Christoffel-Minkowski problem: the case* $1 < p < k + 1$. Calc. Var. Partial Differential Equations **57**(2018), 23 pp.

[10] L. Guo, D. Xi and Y. Zhao, *The L_p chord Minkowski problem in a critical interval*. Math. Ann. (2023), https://doi.org/10.1007/s00208-023-02721-8.

[11] C. Haberl, E. Lutwak, D. Yang and G. Zhang, *The even Orlicz Minkowski problem*. Adv. Math. **224**(2010), 2485–2510.

[12] J. R. Hu, Y. Huang and J. Lu, *On the regularity of the chord log-Minkowski problem*. Preprint, 2023, arXiv:2304.14220.

[13] J. R. Hu, Y. Huang, J. Lu and S. N. Wang, *The chord gauss curvature flow and its L_p chord Minkowski problem*. Preprint, 2023, arXiv:2305.00453.

[14] Q. Huang and B. He, *On the Orlicz Minkowski problem for polytopes*. Discrete Comput. Geom. **48**(2012), 281–297.

[15] Y. Huang, E. Lutwak, D. Yang and G. Zhang, *The L_p Alesandrov problem for L_p integral curvature*. J. Differential Geom. **110**(2018), 1–29.

[16] Y. Huang and Y. Zhao, *On the L_p dual Minkowski problem*. Adv. Math. **332** (2018), 57–84.

[17] D. Hug, E. Lutwak, D. Yang and G. Zhang, *On the L_p Minkowski problem for polytopes*. Discrete Comput. Geom. **33**(2005), 699–715.

[18] Y. Li, *The L_p chord Minkowski problem for negative p*. J. Geom. Anal. **34**(2024), Paper No. 82, 23.

[19] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*. J. Differential Geom. **38**(1993), 131–150.

[20] E. Lutwak, D. Xi, D. Yang and G. Zhang, *Chord measures in integral geometry and their Minkowski problems*. Comm. Pure Appl. Math. (2023), https://doi.org/10.1002/cpa.22190.

[21] E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies*. Adv. Math. **223**(2010), 220–242.

[22] Y. Wu, D. Xi and G. Leng, *On the discrete Orlicz Minkowski problem*. Trans. Amer. Math. Soc. **371**(2019), 1795–1814.

[23] D. Xi, H. Jin and G. Leng, *The Orlicz Brunn-Minkowski inequality*. Adv. Math. **260**(2014) 350–374.

[24] D. Xi, D. Yang, G. Zhang and Y. Zhao, *The L_p chord Minkowski problem*. Adv. Nonlinear Stud. **23** (2023), 22 pp.

[25] F. Xie, *The Orlicz Minkowski problem for general measures*. Proc. Amer. Math. Soc. 150(2022), 4433–4445.

[26] F. Xie, *The Orlicz Minkowski problem for cone-volume measure*. Adv. in Appl. Math. 149(2023), Paper No. 102523, 18.

[27] X. Zhao and P. Zhao, *Flow by Gauss curvature to the Orlicz chord Minkowski problem*. Preprint, 2023, arXiv.2308.05332v2.

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