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EXTREME POSITIVE CONTRACTIONS ON FINITE DIMENSIONAL l^p-SPACES

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In this paper we give a characterization of the extreme positive contractions on finite dimensional l^p-spaces for $1 < p < \infty$. This is related to the characterization of the extreme doubly stochastic operators. In Section 2 we present the basic properties of the facial structure of the set of doubly stochastic $n \times m$ matrices. In Section 3 we use these facts for description of the facial structure of the set of positive contractions on finite dimensional l^p-space. Next is considered stability of the positive part of the unit ball of operators (Section 5). In Section 7 we prove that extreme positive contractions on l^p_n are strongly exposed.

1. Terminology and notation. Let $(X, \mathscr{A}, m)$ be a σ-finite measure space. As usual, we denote by $L^p(m)$, $1 < p < \infty$, the Banach lattice of all p-summable real-valued functions on X with standard norm and order. If $X = \{1, 2, \dots, n\}$ $n < \infty$, and m is a counting measure we write l^p_n instead of $L^p(m)$. If $X = [0, 1]$ and m is Lebesgue measure we write briefly L^p. $L^p_+(m)$ denotes the cone of positive functions ($f \geqq 0$) in $L^p(m)$. The adjoint space $[L^p(m)]'$ is identified with $L^{p'}(m)$, where $1/p + 1/p' = 1$. For $f \in L^p(m)$ we denote

$$f^{(p-1)}(x) = |f(x)|^{p-1} \operatorname{sign} f(x).$$

Let $1 < r < \infty$ and let $(Y, \mathscr{B}, n)$ be a σ-finite measure space. We denote by $\mathscr{L}(L^p(m), L^r(n))$ the Banach space of all linear bounded operators from $L^p(m)$ into $L^r(n)$. An operator T is said to be *positive* ($T \geqq 0$) if $Tf \geqq 0$ whenever $f \geqq 0$. The set of all positive operators (contractions) is denoted by $\mathscr{L}_+(L^p(m), L^r(n))$ $(\mathscr{P})$.

To every operator $T \in \mathscr{L}(l^p_m, l^r_n)$ $(m, n \leqq \infty)$ there corresponds a unique matrix (t_{ij}) , $i = 1, \dots, n, j = 1, \dots, m$ with real entries, such that

$$(Tx)_i = \sum_{j=1}^{m} t_{ij} x_j.$$

Clearly the adjoint operator

$$T^* \in \mathscr{L}(l^{r'}_n, l^{p'}_m)$$

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with $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$, is determined in the same manner by the transposed matrix (t_{ji}).

If an operator $T \in \mathscr{L}(L^p(m), L^r(n))$ attains its norm on f and $\|T\| = \|f\| = 1$, then from the strict convexity of L^p-spaces it follows that

$$T^*(Tf)^{(r-1)} = f^{(p-1)}.$$

We define the support of a positive operator T, denoted by supp T, as a maximal set $A \subset X$ such that $T1_{A^c} = 0$. For a matrix (t_{ij}) let

$$\text{supp}(t_{ij}) = \{j: \text{there exists } i_0 \text{ such that } t_{i_0 j} \neq 0\}.$$

The support of an operator T will be identified with support of a matrix, which corresponds to T.

For $T \in \mathscr{L}_+(L^p(m), L^r(n))$ we have

$$\text{supp } T \supset \text{supp } T^*g \quad \text{where } g \in L^{r'}(n).$$

In particular, if $g > 0$, then

$$\text{supp } T = \text{supp } T^*g$$

(note that supp $f = \{x \in X: f(x) \neq 0\}, f \in L^p(m)$, should be read modulo m-null sets). If T attains its norm at f then supp $f \subset$ supp T.

Let $T \in \mathscr{L}(L^p(m), L^r(n))$. Let $T = \sum T_k$ be a decomposition of T into the operators T_k (i.e., supp T_k disjoint and supp T_k^* disjoint). Then

$$\|T\| = \sup \|T_k\|.$$

Moreover, $T \geqq 0$ if and only if $T_k \geqq 0$ for all k. Furthermore T is an extreme positive contraction if and only if the T_n's are extreme positive contractions.

Let $(t_{ij}) = T$ be a matrix. By T^t we denote transposed matrix. We say that T is an elementary matrix if there are no nonzero matrices T_1, T_2 such that $T = T_1 + T_2$ and

$$\text{supp } T_1 \cap \text{supp } T_2 = \text{supp } T_1^t \cap \text{supp } T_2^t = \emptyset$$

(see **[6]**). If T is a finite matrix, then we can represent T as a finite sum of some elementary matrices T_k,

$$T = \sum_{k=1}^{k_0} T_k,$$

with supp T_k disjoint and supp T_k^t disjoint. In such case we will say that the matrix T can be decomposed into k_0 elementary matrices.

Let K be a convex set. We say that a subset F of K is a *face* if

$$x + (1 - \alpha)y \in F \quad \text{with } x, y \in K, 0 < \alpha < 1,$$

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implies $x, y \in K$. Note that ext $F \subset$ ext K. For $x \in K$ we define a face generated by x as follows

$$F_x = \{ y \in K: \text{there exist } z \in K \text{ and } 0 < \alpha \leqq 1 \text{ such that } x = \alpha y + (1 - \alpha)z \}.$$

We also define the dimension $\dim_K x$ of the point x in K, as the affine dimension $\dim F_x$ of F_x. The point $x \in K$ is extreme if and only if $\dim_K x = 0$.

Throughout this paper we assume $1 < p < \infty$ and $1 < r < \infty$.

2. Doubly stochastic matrices. Doubly stochastic matrices have been extensively studied by a number of authors (see for example [**12**] and reference there). In this section we consider the facial structure of the convex set of doubly stochastic matrices. All these properties can be obtained by modifying the arguments presented in [**2**] (see also [**4**]). For the convenience of the reader we present below proofs of the basic properties which we use in the next sections.

Let μ, ν be finite measures on $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$, respectively, such that

$$\mu(\{1, 2, \ldots, n\}) = \nu(\{1, 2, \ldots, m\}).$$

Let (p_{ij}) be a positive measure on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ with marginal distributions μ, ν. The matrix $P = (p_{ij})$ is called *doubly stochastic* with respect to μ and ν. We write $P \in \mathscr{D}(\mu, \nu)$.

PROPERTY 1. *Let $P \in \mathscr{D}(\mu, \nu)$ be an $n \times m$ elementary matrix. And let*

$$\operatorname{supp} \mu = \{1, \ldots, n\}, \operatorname{supp} \nu = \{1, \ldots, m\}.$$

Then

$$\dim_{\mathscr{D}(\mu,\nu)} P = mn - m - n + 1 - z$$

where z denotes the number of null entries of the matrix P.

Proof. The set

$$D_P = \{P + R: P \pm R \in \mathscr{D}(\mu, \nu)\}$$

is included in the face F_P generated by P in $\mathscr{D}(\mu, \nu)$ and

$$\dim_{\mathscr{D}(\mu,\nu)} P = \dim D_P.$$

For $R = (r_{ij})$ we have $P \pm R \in \mathscr{D}(\mu, \nu)$ if and only if

$$\sum_{\nu=1}^{m} r_{\nu j} = 0, \quad \sum_{\nu=1}^{m} r_{i\nu} = 0 \quad \text{and } |r_{ij}| \leqq p_{ij}.$$

Thus

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$$\dim_{\mathscr{D}(\mu,\nu)} P = \dim\{ (r_{ij}) : \sum_{\nu=1}^{m} r_{i\nu} = \sum_{\nu=1}^{m} r_{\nu j} = (1 - \operatorname{sign} p_{ij}) r_{ij} = 0, \quad i = 1, \ldots, m, j = 1, 2, \ldots, n\}.$$

Let

$$\varphi_i(R) = \sum_{j=1}^{n} r_{ij},$$

$$\psi_j(R) = \sum_{i=1}^{m} r_{ij},$$

$$\mathscr{H}_{ij}(R) = (1 - \operatorname{sign} p_{ij}) r_{ij}$$

be defined on the space of $n \times m$ matrices $R = (r_{ij})$. We have

$$\dim_{\mathscr{D}(\mu,\nu)} P = \dim\{R : \varphi_i(R) = \psi_j(R) = \mathscr{H}_{ij}(R) = 0, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}.$$

Therefore $\dim_{\mathscr{D}(\mu,\nu)} P$ is equal to nm minus the number of linearly independent functionals in the set

$$\{\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n, \mathscr{H}_{11}, \ldots, \mathscr{H}_{mn}\}.$$

Since

$$\sum_{i=1}^{m} \varphi_i = \sum_{j=1}^{n} \psi_j,$$

the functional φ_1 depends linearly on $\varphi_2, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$. We denote

$$Z = \{ (i, j) : p_{ij} = 0\}.$$

Now it is sufficient to show that the functionals

$$(*) \quad \begin{cases} \varphi_i & (i = 2, \ldots, m), \\ \psi_j & (j = 1, \ldots, n), \\ \mathscr{H}_{ij} & (i, j) \in Z \end{cases}$$

are linearly independent. Suppose

$$\xi(R) = \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij}(\alpha_i + \beta_j + \gamma_{ij}(1 - \operatorname{sign} p_{ij})) = 0,$$

(where $\alpha_1 = 0$), for some α_i, β_j, γ_{ij} in $\mathbf{R}$. Now we choose inductively a

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sequence $\{ (i_k, j_k) \}_{k=1}^{n+m-1}$ where $(i_k, j_k) \notin Z$, satisfying:

(a) $i_1 = 1$ and $(i_1, j_1) \notin Z$ (i.e., $p_{i_1 j_1} > 0$ and $\gamma_{i_1 j_1} = 0$)

(b) $$\begin{cases} i_{k+1} \in \{i_1, \dots, i_k\} \text{ and } j_{k+1} \notin \{j_1, \dots, j_k\} \\ \text{or} \\ i_{k+1} \notin \{i_1, \dots, i_k\} \text{ and } j_{k+1} \in \{j_1, \dots, j_k\}. \end{cases}$$

We can construct the sequence $\{ (i_k, j_k) \}$, since the matrix P is an elementary matrix. Note that

$$\text{card}\{i_1, \dots, i_k\} + \text{card}\{j_1, \dots, j_k\} = k + 1$$

$(k = 1, \dots, m + n - 1)$, so

$$\{i_1, \dots, i_{n+m-1}\} = \{1, \dots, m\},$$

$$\{j_1, \dots, j_{n+m-1}\} = \{1, \dots, n\}.$$

Let $S^{ij} = (s_{kl})$ be the $n \times m$ matrix such that

$$s_{kl} = \delta_{ik}\delta_{jl}.$$

The conditions $\xi(S^{ij}) = 0$ imply

$$\alpha_{i_k} + \beta_{j_k} = 0 \quad (k = 1, \dots, n + m - 1).$$

Clearly $\beta_{j_1} = 0$, since $i_1 = 1$ and $\alpha_1 = 0$. We obtain

$$\alpha_{i_2} = \beta_{j_2} = 0$$

since $\alpha_{i_1} = \beta_{j_1} = 0$ and $i_1 = i_2$ or $j_1 = j_2$. Continuing in this way, we obtain

$$\alpha_{i_1} = \dots = \alpha_{i_{n+m-1}} = \beta_{j_1} = \dots = \beta_{j_{n+m-1}} = 0.$$

Thus $\gamma_{ij} = 0$ for all $(i, j) \in Z$, since

$$\mathcal{H}_{ij}(S^{kl}) = \delta_{ik}\delta_{jl}.$$

Therefore the functionals (*) are linearly independent.

PROPERTY 2. *Let $P = (p_{ij}) \in \mathcal{D}(\mu, \nu)$ be an $n \times m$ matrix which can be decomposed into k_0 elementary matrices. Then*

$$\dim_{\mathcal{D}(\mu,\nu)} P = nm + k_0 - n_0 - m_0 - z$$

where

$$z = \text{card}\{p_{ij} : p_{ij} = 0\},$$

$$n_0 = \text{card supp } \mu = \text{card supp } P,$$

$$m_0 = \text{card supp } \nu = \text{card supp } P^t.$$

Proof. Let

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$$P = \sum_{k=1}^{k_0} P_k$$

be a decomposition of P into elementary matrices P_k. We have

$$\dot{\cup} \text{ supp } P_k = \text{supp } P \quad \text{and } \dot{\cup} \text{ supp } P_k^t = \text{supp } P^t.$$

The matrices P_k are doubly stochastic with respect to

$$(\mu|_{\text{supp}P_k}, \nu|_{\text{supp}P_k^t}).$$

By Property 1 we have

$$\dim_{\mathscr{D}(\mu,\nu)}[P_k - n_k m_k - n_k - m_{k'}] = z_k'$$

where

$$n_k = \text{card supp } P_k, \ m_k = \text{card supp } P_k^t \quad \text{and}$$

$$z_k' = \text{card}\{p_{ij}: j \in \text{supp } P_k, \ i \in \text{supp } P_k^t, \ p_{ij} = 0\}.$$

If $R = (r_{ij}) \in \{S: P \pm S \in \mathscr{D}(\mu, \nu)\}$, then $|r_{ij}| \leqq p_{ij}$. We have

$$R = \sum_{k=1}^{k_0} R_k,$$

where

$$R_k = I_{\text{supp}P_k^t} R I_{\text{supp}P_k}.$$

Hence

$$\begin{aligned}
\dim_{\mathscr{D}(\mu,\nu)} P &= \dim\{R: \sum P_k \pm R_k \in \mathscr{D}(\mu, \nu)\} \\
&= \sum_{k=1}^{k_0} \dim\{R_k: P_k \pm R_k \in \mathscr{D}(\mu|_{\text{supp}P_k}, \nu|_{\text{supp}P_k^t})\} \\
&= \sum_{k=1}^{k_0} \dim_{\mathscr{D}(\mu|_{\text{supp}P_k}, \nu|_{\text{supp}P_k^t})} P_k \\
&= \sum_{k=1}^{k_0} (n_k m_k - n_k - m_k + 1 - z_k') \\
&= nm - z - n_0 - m_0 + k_0
\end{aligned}$$

because

$$z = nm - \sum_{k=1}^{k_0} (n_k m_k - z_k'), \ \sum_{k=1}^{k_0} n_k = n_0, \ \sum_{k=1}^{k_0} m_k = m_0.$$

Birkhoff [**1**] has shown that in the case

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$$\mu(\{i\}) = \nu(\{i\}) = 1 \quad \text{for } i = 1, 2, \dots, n$$

the set of extreme doubly stochastic matrices coincides with the set of all permutation matrices. This result was generalized to the infinite case by Kendal [**10**] (see also [**9**], [**14**]). For arbitrary measure μ, ν such that

$$n = \text{card supp } \mu, \quad m = \text{card supp } \nu$$

a matrix $(p_{ij}) \in \mathscr{D}(\mu, \nu)$ is *extreme* if and only if for every $k \times k$ submatrix T of the matrix p_{ij} the number of positive entries of T is less than $2k$, $k = 2, 3, \dots, \min(n, m)$ (see [**11**], Proposition 2).

With each matrix $P = (p_{ij})$ we associate a graph $G(P)$ as follows. Corresponding to row i we have a node x_i in $G(P)$ and corresponding to column j we have a node y_j in $G(P)$. There is edge joining x_i and y_j if and only if $p_{ij} > 0$. Then a matrix $P \in \mathscr{D}(\mu, \nu)$ is extreme if and only if the connected components of $G(P)$ are trees (see for example [**3**], Theorem 2.1). For generalization to the infinite dimensional case see [**7**].

Note that P is an elementary matrix if and only if the graph $G(P)$ is connected. A matrix P can be decomposed into k_0 elementary matrices if and only if the graph $G(P)$ has k_0 connected components.

Suppose that $P \in \mathscr{D}(\mu, \nu)$ can be decomposed into k_0 elementary matrices. Then, by Property 2, $P \in \text{ext } \mathscr{D}(\mu, \nu)$ if and only if

$$k_0 + nm = n_0 + m_0 + z$$

where

$$z = \text{card}\{p_{ij}:p_{ij} = 0\},\ n_0 = \text{card supp } P,\ m_0 = \text{card supp } P^t.$$

3. Extreme positive contractions in the finite dimensional case.

THEOREM 1. *Let* $1 < r \leqq p < \infty$ *and let* $f \in L^p_+(m)$, $g \in L^r_+(n)$ *be functions with* $\|f\|_p = \|g\|_r = 1$. *Then the set*

$$\mathscr{A}_{f,g} = \{T \in \mathscr{L}_+(L^p(m), L^r(n)):Tf = g,\ T^*g^{r-1} = f^{p-1},$$
$$\text{supp } T = \text{supp} f\}$$

is a weak operator closed face in the positive part of the unit ball of $\mathscr{L}(L^p(m), L^r(n))$.

Proof. By Proposition 1 in [**6**] we have $\|T\| = 1$ if $T \in \mathscr{A}_{f,g}$. We claim that, if

$$T = \alpha T_1 + (1 - \alpha)T_2 \in \mathscr{A}_{f,g}$$

for some $0 < \alpha \leqq 1$ and $T_1, T_2 \in \mathscr{P}$, then $T_1 \in \mathscr{A}_{f,g}$. Indeed, since $\|Tf\| = \|f\|$, by the strict convexity of L^p-space we have $T_1 f = Tf = g$. Similarly $T_1^* g^{r-1} = f^{p-1}$. Since T_1 attains its norm on f, we have supp $T_1 \supset$ supp f. Since $T_1 \leqq T/\alpha$, we obtain

$$\text{supp } T_1 \subset \text{supp } T = \text{supp} f.$$

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Therefore $T_1 \in \mathscr{A}_{f,g}$ and $\mathscr{A}_{f,g}$ is a face.

Now, let T be weak operator limit of a net $T_\alpha \in \mathscr{A}_{f,g}$. Obviously $T \in \mathscr{P}$. Since

$$\langle Tf, g^{r-1}\rangle = \lim\langle Tf, g^{r-1}\rangle = \langle g, g^{r-1}\rangle = 1$$

we have $Tf = g$. Similarly $T^*g^{r-1} = f^{p-1}$, supp $T \subset$ supp f. Thus $T \in \mathscr{A}_{f,g}$, so $\mathscr{A}_{f,g}$ is closed in the weak operator topology.

Note that if $T \in \mathscr{A}_{f,g}$ then supp $T^* =$ supp g. Indeed,

$$\text{supp } T^* = \bigcup_{h \in L^p(m)} \text{supp } Th.$$

Since $T1_{(\text{supp} f)^c} = 0$, we have

$$\text{supp } Th \subset \text{supp } Tf = \text{supp } g.$$

Hence supp $T^* =$ supp g. We can write

$$\mathscr{A}_{f,g} = \{T \in \mathscr{L}_+(L^p(m), L^r(n)) : Tf = g, T^*g^{r-1} = f^{p-1},$$
$$\text{supp } T^* = \text{supp } g\}.$$

THEOREM 2. *Let* $1 < r \leqq p < \infty$, *and let* $f \in L^p_+(m)$, $g \in L^r_+(n)$ *be such that* $\|f\| = \|g\| = 1$. *Then* $\mathscr{A}_{f,g}$ *is affinely isomorphic to* $\mathscr{D}(\mu, \nu)$ *where* $d\mu = f^p dm$, $d\nu = g^r dn$.

Proof. For every $T \in \mathscr{A}_{f,g}$ we define an operator

$$P_T : L^\infty(\mu) \to L(\nu)$$

by

$$(1) \quad P_T h = \frac{T(hf)}{g}.$$

It is easy to see that P_T extends to an operator $P_T \in \mathscr{D}(\mu, \nu)$. Conversely, if $Q \in \mathscr{D}(\mu, \nu)$ then

$$Th = gQ\left(\frac{h}{f}\right)$$

defines an operator $T \in \mathscr{L}_+(L^p(m), L^r(n))$. Moreover $T \in \mathscr{A}_{f,g}$. Indeed, since P acts on classes of functions modulo μ-null sets, supp $T \subset$ supp f. We have

$$\int (T^*v)udm = \int vTudn$$
$$= \int vg^{1-r}Q(u/f)d\nu$$
$$= \int f^{p-1}Q^*(vg^{1-r})udm.$$

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Thus

$$T^*v = f^{p-1}Q(vg^{1-r}).$$

Now it is easy to see that $T \in \mathscr{A}_{f,g}$.

Therefore $T \to P_T$ is an affine bijection from $\mathscr{A}_{f,g}$ onto $\mathscr{D}(\mu, \nu)$.

LEMMA 1. *Let* $T \in \mathscr{L}_+(l_n^p, l_m^p)$ *be an elementary operator. Then*

$$\dim_{\mathscr{P}} T = \begin{cases} nm - n_0 - m_0 - z + 1 & \text{if } \|T\| = 1 \\ nm - z & \text{if } \|T\| < 1 \end{cases}$$

where

$$n_0 = \text{card supp } T,\ m_0 = \text{card supp } T^*,\ z = \text{card}\{t_{ij}:t_{ij} = 0\}$$

with $T = (t_{ij})$.

Proof. Let $\|T\| = 1$. There exists a unique positive vector $f = (f_j)$ of norm 1 such that T attains its norm on f and supp T = supp f (see Theorem 4 in [**6**]). The face generated by T in $\mathscr{P}$ is included in $\mathscr{A}_{f,Tf}$. Hence the face generated by T in $\mathscr{P}$ coincides with the face generated by T in $\mathscr{A}_{f,Tf}$. Since the face $\mathscr{A}_{f,Tf}$ is affine isomorphic to $\mathscr{D}(\mu, \nu)$, where

$$\mu(\{j\}) = f_j^p, \quad \nu(\{i\}) = (Tf)_i^p$$

we have

$$\dim_{\mathscr{P}} T = \dim_{\mathscr{D}(\mu,\nu)} P,$$

where P is defined by formula (1). Operators in $\mathscr{D}(\mu, \nu) \subset \mathscr{L}(l_n^1, l_m^1)$ are identified with doubly stochastic matrices. Then to the operator T there corresponds a matrix (p_{ij}) such that

$$p_{ij} = (Tf)_i^{p-1} t_{ij} f_j.$$

We have

$$\dim_{\mathscr{P}} T = \dim_{\mathscr{D}(\mu,\nu)} (p_{ij})$$

and (p_{ij}) is an elementary matrix, too (i.e., k_0 from Property 2 is equal to 1), so

$$\dim_{\mathscr{P}} T = nm - n_0 - m_0 - z + 1.$$

Let $\|T\| < 1$. Now

$$\dim_{\mathscr{P}} T = \dim \text{lin}\{S:\|T \pm S\| < 1,\ T \pm S \geqq 0\}$$
$$= \dim \text{lin}\{S:T \pm S \geqq 0\}.$$

The condition $T \pm S \geqq 0$ is equivalent to $|s_{ij}| \leqq t_{ij}$ for all (i, j). Thus $s_{ij} = 0$ if $t_{ij} = 0$. Therefore

$$\dim_{\mathscr{P}} T = mn - z.$$

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As a consequence of Lemma 1 we obtain:

THEOREM 3. *Let* $0 \neq T \in \mathscr{L}_+(l^p_n, l^p_m)$, $n, m \in \mathbf{N}$, *be an elementary operator. Then* T *is an extreme positive contraction if and only if*

$$\|T\| = 1 \quad \textit{and} \quad nm + 1 = n_0 + m_0 + z,$$

where

$$n_0 = \text{card supp } T,\ m_0 = \text{card supp } T^*,\ z = \text{card}\{t_{ij}:t_{ij} = 0\}.$$

Let $T = \sum T_k$, where T_k are elementary operators. Then the operators $T \in \text{ext}\,\mathscr{P}$ if and only if every $T_k \in \text{ext}\,\mathscr{P}$. Therefore the above theorem is a characterization of the extreme positive contractions in the finite dimensional case.

For an operator $T \in \mathscr{L}(L^p(m), L^r(n))$ we define

$$J(T) = \{\text{supp} f : \|Tf\| = \|T\|\,\|f\|\}$$

(see [6]). Note that if $p = r$, one of L^p spaces is finite dimensional and $T \in \text{ext}\,\mathscr{P}$, then

$$\text{supp } T \in J(T) \quad \text{and} \quad \text{supp } T^* \in J(T^*).$$

Remark. Now we give the value of $\dim_{\mathscr{P}} T$. Let $T \in \mathscr{L}_+(l^p_n, l^p_m)$ be a contraction, which can be decomposed into k_0 elementary operators,

$$T = \sum_{k=1}^{k_0} T_k.$$

We may assume that $\|T_k\| = 1$ for $k = 1, \ldots, k_s$ and $\|T_k\| < 1$ for $k = k_s + 1, \ldots, k_0$. Let

$$n_k = \text{card supp } T_k,\ m_k = \text{card supp } T^*_k$$

and let

$$N = \sum_{k=1}^{k_s} n_k,\ M = \sum_{k=1}^{k_s} m_k.$$

The numbers N and M are the cardinalities of the maximal elements of $J(T)$ and $J(T^*)$, respectively. Then

$$\dim_{\mathscr{P}} T = nm + k_s - N - M - z,$$

where

$$z = \text{card}\{t_{ij}:t_{ij} = 0\}.$$

The above equality follows from the following equalities:

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$$\dim {}_{\mathscr{P}}T = \sum_{k=1}^{k_0} \dim {}_{\mathscr{P}}T_k$$

$$\begin{aligned} \dim {}_{\mathscr{P}}T_k &= nm - n_k - m_k + 1 - z_k \quad \text{for } k = 1, \dots, k_s \\ \dim {}_{\mathscr{P}}T_k &= nm - z_k \quad \text{for } k = k_s + 1, \dots, k_0 \end{aligned}$$

$$z = nm - \sum_{k=1}^{k_0} mn - z_k$$

where z_k denote the number of null entries in the matrix corresponding to T_k.

Example. There exists an extreme positive contraction T such that T does not attain its norm on some vector. Let $T \in L_+(l^2, l^2)$ be defined by

$$T(x_1, x_2, x_3, \dots) = \left(\frac{1}{\sqrt{2}} x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_2 + x_3), \dots\right).$$

We define

$$\varphi(x) = \|x\|^2 - \|Tx\|^2 = \frac{1}{4}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + \dots\right].$$

Obviously $\varphi(x) \geqq 0$, hence $\|T\| \leqq 1$. For every nonzero vector $x \in l^2$ there exists an i such that $x_i \neq x_{i+1}$. Then $\varphi(x) \neq 0$ i.e., T does not attain its norm on any vector in l^2. We claim that $T \in \operatorname{ex} \mathscr{P}$. Let $R \in \mathscr{L}(l^2, l^2)$ be such that

$$T \pm R \geqq 0 \quad \text{and} \quad \|T \pm R\| \leqq 1.$$

It follows that

$$Rx = (r_{11}x_1, r_{21}x_1 + r_{22}x_2, r_{32}x_2 + r_{33}x_3, \dots).$$

Since

$$2\|Rx\|^2 + 2\|Tx\|^2 = \|(T + R)x\|^2 + \|(T - R)x\|^2 \leqq 2\|x\|^2,$$

we obtain

(a) $\|Rx\|^2 \leqq \varphi(x)$

and analogously

(b) $\|R^*x\|^2 \leqq \varphi^*(x)$

where

$$\begin{aligned} \varphi^*(x) &= \|x\|^2 - \|T^*x\|^2 \\ &= \frac{1}{4}\left[(\sqrt{2}x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + \dots\right]. \end{aligned}$$

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We put

$$y^{(n)} = \left(\underbrace{1, 1, \ldots, 1}_{n}, 1 - \frac{1}{n}, 1 - \frac{2}{n}, \ldots, 0, 0, \ldots\right),$$

$$z^{(n)} = \left(\frac{\sqrt{2}}{2}, \underbrace{1, 1, \ldots, 1}_{n}, 1 - \frac{1}{n}, 1 - \frac{2}{n}, \ldots, 0, 0, \ldots\right)$$

for $n \in \mathbf{N}$. If $x = y^{(n)}$ in (a) then we obtain

$$r_{11}^2 + (r_{21} + r_{22})^2 + \ldots \leqq \frac{1}{4}\left[(1 - 1)^2 + \ldots\right] = \frac{1}{4n}.$$

Thus

$$r_{11} = 0 \quad \text{and} \quad r_{k,k-1} = -r_{kk} \text{ for } k = 2, 3, \ldots.$$

Now we use the inequality (b). By a similar calculation for $z^{(n)}$ we obtain

$$\frac{\sqrt{2}}{2} r_{11} = -r_{21} \quad \text{and} \quad r_{kk} = -r_{k+1,k} \quad \text{for } k = 2, 3, \ldots.$$

It follows that $R = 0$.

An analogous example in $L^2[0, 1]$ is obtained by letting

$$(Tf)(t) = \begin{cases} \frac{\sqrt{2}}{2} f(t) & t \in [0, 1/2) \\ \frac{\sqrt{2}}{2} f(2t - 1) + \frac{1}{2} f(t) & t \in [1/2, 1). \end{cases}$$

The operator T is an extreme positive contraction on $L^2[0, 1]$ which does not attain its norm on any unit vector. Some additional information about extreme positive contraction on l^p can be found in [**8**].

4. Extreme positive contractions in $\mathscr{L}(l_n^p, l_m^r)$.

LEMMA 2. *Let $T \in \mathscr{L}_+(l_n^p, l_m^r)$ where $1 < r < p < \infty$ and $n, m \in \mathbf{N}$. Then*

$$\dim_{\mathscr{P}} T = \begin{cases} nm + k_0 - n_0 - m_0 - z & \text{if } \|T\| = 1 \\ nm - z & \text{if } \|T\| < 1, \end{cases}$$

where

$$n_0 = \text{card supp } T, \; m_0 = \text{card supp } T^*, \; z = \text{card}\{t_{ij} : t_{ij} = 0\},$$

k_0 is the number of elementary operators into which the operator T can be decomposed.

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Proof. Let $||T|| = 1$. Then there exists a unique vector $f = (f_j) \geqq 0$ such that $||f|| = 1$, T attains its norm at f and supp T = supp f. Let us put

$$p_{ij} = (Tf)_i^{r-1} t_{ij} f_j.$$

By an argument analogous to the one in the proof of Lemma 1 we have

$$\dim_{\mathscr{P}} T = \dim_{\mathscr{D}(\mu,\nu)}(p_{ij})$$

where μ, ν are measures such that

$$\mu(\{j\}) = f_j^p, \nu(\{i\}) = (Tf)_i^r$$

and we use Property 2.

If $||T|| < 1$ it is easy to see that $\dim T = nm - z$.

COROLLARY. *Let* $1 < r < p < \infty$, $n, m \in \mathbf{N}$ *and* $0 \neq T \in \mathscr{L}_+(l_n^p, l_m^r)$. *Then* $T \in \text{ex}\,\mathscr{P}$ *if and only if*

$$||T|| = 1 \quad \textit{and} \quad mn + k_0 = n_0 + m_0 + z,$$

where

$$n_0 = \text{card supp } T, m_0 = \text{card supp } T^*, z = \text{card}\{t_{ij}:t_{ij} = 0\}$$

and k_0 *denote the number of elementary operators into which* T *can be decomposed.*

5. Skeletons in the set of positive contractions. The k-skeleton of a convex set Q is the set of all points $x \in Q$ such that $\dim_Q x \leqq k$. A convex compact set Q in Euclidean space is called *stable* if all the k-skeletons of Q are closed (see **[13]**). The set of extreme points is the 0-skeleton.

Example. Let $T \in \mathscr{L}(l_2^p, l_2^p)$, $1 < p < \infty$, be defined by

$$T(x_1, x_2) = \left(ax_1 + bx_2, \frac{1}{2} x_2\right).$$

For every $0 \leqq a \leqq 1$ there exists $b \in [0, 1]$ such that $||T|| = 1$. Let $0 < a_1 < a_2 < \ldots < 1$ with $\lim a_n = 1$ and $0 \leqq b_n \leqq 1$ be such that $||T_n||_p = 1$, where

$$T_n = \begin{bmatrix} a_n & b_n \\ 0 & 1/2 \end{bmatrix}.$$

Fix $k \in \mathbf{N}$. Let $\alpha, \beta \geqq 0$ be such that T_n attains its norm on $(\alpha, \beta) \in l_2^p$ and $||(\alpha, \beta)||_p = 1$. We have

$$1 = ||T_{k+1}||_p^p \geqq (a_{k+1}\alpha + b_{k+1}\beta)^p + \left(\frac{1}{2}\beta\right)^p$$

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$$= [\,(a_k\alpha + b_k\beta) + (a_{k+1} - a_k)\alpha + (b_{k+1} - b_k)\beta]^p + \left(\frac{1}{2}\,\beta\right)^p.$$

Hence $b_{k+1} \leqq b_k$. Since $b_n \geqq 0$, there exists $\lim b_n = b_0$. Moreover $b_0 = 0$ since otherwise we would have

$$1 = \lim\|T_n\| = \left\|\begin{bmatrix} 1 & b_0 \\ 0 & 1/2 \end{bmatrix}\right\| > 1.$$

The operators T_n are extreme positive contractions (Theorem 3) and

$$\lim T_n = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

is not extreme.

Using the above example it is not hard to see that the set of extreme positive contractions in $\mathscr{L}(l_n^p, l_m^p)(n, m \geqq 2, 1 < p < \infty)$ is not a closed set. Therefore the set of positive contractions in $\mathscr{L}(l_n^p, l_m^p)(n, m \geqq 2, 1 < p < \infty)$ is not stable.

PROPOSITION. *Let $1 < r < p < \infty$ $n, m \in \mathbf{N}$. The set of all positive contractions in $\mathscr{L}(l_n^p, l_m^r)$ is stable.*

Proof. We need to prove that for every positive contraction T there exists $\epsilon > 0$ such that for every $S \in \mathscr{P}$ the condition $\|S - T\| < \epsilon$ implies $\dim_{\mathscr{P}} S \geqq \dim_{\mathscr{P}} T$ (see [**13**], Theorem 2.3).

Let $\|T\| < 1$ and

$$\epsilon = \min(1 - \|T\|, \{t_{ij}:t_{ij} \neq 0\}\,).$$

If $\|S - T\| < \epsilon$ then

$$\|S\| < 1 \quad \text{and} \quad |s_{ij} - t_{ij}| < \epsilon.$$

The numbers of the null entries of $S = (s_{ij})$ are less than or equal to the number of the null entries of $T = (t_{ij})$.

By Lemma 2 we obtain $\dim_{\mathscr{P}} S \geqq \dim_{\mathscr{P}} T$.

Let $\|T\| = 1$. We put

$$\epsilon = \min\{t_{ij}:t_{ij} \neq 0\}.$$

If $\|S - T\| < \epsilon$ then null entries of matrix (s_{ij}) can be only on that place on which are null entries of matrix (t_{ij}). If $\|S\| < 1$ then $\dim_{\mathscr{P}} S \geqq \dim_{\mathscr{P}} T$ (Lemma 2).

Now suppose that $\|S\| = 1$. The number $d = nm + k_0 - n_0 - m_0 - z$ from Property 2 denote $\dim_{\mathscr{P}}$ of operators T and S. Consider now the value of d, when we change one of the null entries in an $n \times m$ matrix to a non zero entry. Let $A = (a_{ij})$ be fixed $n \times m$ matrix. We put

$$n_0 = \text{card supp } A,\ m_0 = \text{card supp } A^t,\ z = \text{card}\{a_{ij}:a_{ij} = 0\}.$$

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k_0 denotes the number of elementary matrices A_k into which matrix A can be decomposed,

$$A = \sum_{k=1}^{k_0} A_k.$$

Let (i, j) be the index of a null entry which we change to a non zero entry. We consider the following five cases:

1^0. $i \in \operatorname{supp} A_k$, $j \in \operatorname{supp} A_k^t$. Then n_0, m_0, k_0 do not change. z decreases by one. Hence d increases.

2^0. $i \in \operatorname{supp} A_{k_1}, j \in \operatorname{supp} A_{k_2}^t$, $k_1 \neq k_2$. Then n_0, m_0 do not change. k_0 decreases by one, since in place of two elementary matrices A_{k_1}, A_{k_2} there appears one elementary matrix. Hence d does not change.

3^0. $i \in \operatorname{supp} A_k, j \in (\operatorname{supp} A^t)^c$. Then n_0, k_0 do not change. m_0 increases by one. Hence d does not change.

4^0. $i \in (\operatorname{supp} A)^c$, $j \in \operatorname{supp} A_k^t$. Then m_0, k_0 do not change. n_0 increases by one. Hence d does not change.

5^0. $i \in (\operatorname{supp} A)^c, j \in (\operatorname{supp} A^t)^c$. Then m_0, n_0, k_0 increase by one, since there appears a new elementary matrix which possesses only this non zero entry. Hence d does not change.

Therefore we obtain $\dim_{\mathscr{P}} S \geqq \dim_{\mathscr{P}} T$.

Note that the unit ball of $\mathscr{L}(l_n^p, l_n^p)$, $n \geqq 2$, is stable if $p = 2$ and is not stable if $p \neq 2$, $1 < p < \infty$ (see **[5]**).

6. The case of l^1- and l^∞-spaces. Assume that $1 < p < \infty$. For a matrix T we put

$$\sigma(T) = \sum_{i,j} \operatorname{sign}|t_{ij}| = mn - z.$$

It is not hard to see that the following equalities hold:

If $T \in \mathscr{L}(l_n^1, l_m^1)$, then

$$\dim_{\mathscr{P}} T = \sigma(T) - N,$$

where

$$N = \operatorname{card}\{j: \sum_{i=1}^{m} t_{ij} = 1\}.$$

If $T \in \mathscr{L}(l_n^1, l_m^p)$, then

$$\dim_{\mathscr{P}} T = \sum_{j \in J} \sum_{i=1}^{m} \operatorname{sign} t_{ij}$$

where

$$J = \left\{ j: \sum_{i=1}^{m} t_{ij}^p = 1 \right\}.$$

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If $T \in \mathscr{L}(l^1_n, l^\infty_m)$, then

$$\dim_{\mathscr{P}} T = \operatorname{card}\{ (i, j): t_{ij} \in (0, 1) \}.$$

If $T \in \mathscr{L}(l^p_n, l^1_m)$, then

$$\dim_{\mathscr{P}} T = \begin{cases} \sigma(T) - n_0 & \text{if } \|T\| = 1 \\ \sigma(T) & \text{if } \|T\| < 1 \end{cases}$$

where $n_0 = \operatorname{card}\operatorname{supp} T$. Note that $\|T\| = \|T^*1\|_{p'}$. If $T \in \mathscr{L}(l^p_n, l^\infty_m)$, then

$$\dim_{\mathscr{P}} T = \sum_{j=1}^{n} \sum_{i \in I_2} \operatorname{sign} t_{ij}$$

where

$$I_2 = \left\{ i: \sum_{j=1}^{n} t_{ij}^{p'} < 1 \right\}.$$

If $T \in \mathscr{L}(l^\infty_n, l^1_m)$, then

$$\dim_{\mathscr{P}} T = \begin{matrix} \sigma(T) & \text{if } \|T\| = \sum_{i,j} t_{ij} < 1 \\ \sigma(T) - 1 & \text{if } \|T\| = \sum_{i,j} t_{ij} = 1. \end{matrix}$$

If $T \in \mathscr{L}(l^\infty_n, l^p_m)$, then

$$\dim_{\mathscr{P}} T = \begin{cases} \sigma(T) & \text{if } \|T\| < 1 \\ \sigma(T) - m_0 & \text{if } \|T\| = 1 \end{cases}$$

where $m_0 = \operatorname{card}\operatorname{supp} T^t$. Note that

$$\|T\| = \|T1\|_p = \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} t_{ij} \right)^p \right)^{1/p}.$$

If $T \in \mathscr{L}(l^\infty_n, l^\infty_m)$, then

$$\dim_{\mathscr{P}} T = \sigma(T) - M,$$

where

$$M = \operatorname{card}\left\{ i: \sum_{j=1}^{n} t_{ij} = 1 \right\}.$$

Remark. Using the above equalities and arguments similar to that which we use in Section 5 it is not difficult to check that the set of positive contractions is stable for all cases which we consider above.

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7. Strongly exposed points. A point x_0 in a convex set K is called *exposed* if there exists a linear functional ξ such that $\xi(x_0) > \xi(x)$ for all $x \in K\setminus\{x_0\}$. An exposed point $x_0 \in K$ is called *strongly exposed* if for any sequence $x_n \in K$ the condition $\xi(x_n) \to \xi(x_0)$ implies $x_n \to x_0$.

THEOREM 4. *Let $1 < r \leqq p < \infty$. Each extreme positive contraction in $\mathscr{L}(l_n^p, l_m^r)$ is strongly exposed.*

Proof. In a compact convex set each exposed point is strongly exposed. Therefore we need to show that each extreme operator is exposed.

Let $0 \neq T = (t_{ij}) \in \mathscr{L}(l_n^p, l_m^r)$ be an extreme positive contraction. Then there exists a vector $f = (f_j) \geqq 0$ such that $\|f\| = 1$, T attains its norm at f and $\operatorname{supp} T = \operatorname{supp} f$. We define a functional ξ by

$$\xi(S) = \sum_{i=1}^{m} (Tf)_i^{(r-1)}(Sf)_i - \sum_{i,j} s_{ij}\,(1 - \operatorname{sign} t_{ij}),$$

$S = (s_{ij}) \in \mathscr{L}(l_n^p, l_m^r)$. Suppose that S is a positive contraction. Then, by Hölder's inequality,

$$\xi(S) \leqq \|Tf^{(r-1)}\|_{r'}\|Sf\|_r - \sum_{i,j} s_{ij}(1 - \operatorname{sign} t_{ij}) \leqq 1,$$

and

$$\xi(T) = \|Tf^{(r-1)}\|_{r'}\|Tf\|_r = 1.$$

Now suppose that $\xi(S) = 1$ for some positive contraction S. Then $s_{ij} = 0$ if $t_{ij} = 0$, and

$$\sum_i (Tf)_i^{(r-1)}\,(Sf)_i = \|Tf^{(r-1)}\|_{r'}\|Sf\|_r = 1.$$

Therefore the zero entries of (s_{ij}) and (t_{ij}) coincide and $Tf = Sf$. Because the graph of an extreme doubly stochastic matrix determines this matrix (see [**3**], Theorem 2.1), and positive contractions are related to a doubly stochastic matrix (cf. proof of Theorem 2). We obtain $S = T$, i.e, T is exposed by ξ.

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