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INTEGRAL EXTENSIONS OF COMMUTATIVE BANACH ALGEBRAS

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Dedication. This paper is dedicated to my father on the occasion of his 80th birthday.

Introduction. In this paper, we continue the study of integral extensions begun in [**7**]. Whereas in the previous paper, we dealt exclusively with the extension $A[x]/(\alpha(x))$, $\alpha(x)$ a monic polynomial over A, we now deal with arbitrary integral extensions. Applications of the results presented herein will be made in subsequent papers.

To simplify our presentation, we make the following conventions. By an algebra, we will always mean a commutative complex algebra with an identity element, usually denoted by e. If A and B are algebras, then B will be called an extension of A if there is an isomorphism of A into B that carries the identity of A onto the identity of B. When convenient, we simply view A as a subalgebra of B that contains the identity of B. B is said to be integral over A if every element of B satisfies a monic polynomial over A. If $(A, ||\cdot||_A)$ and $(B, ||\cdot||_B)$ are normed algebras with B an extension of A, then $(B, ||\cdot||_B)$ is called a normed extension of $(A, ||\cdot||_A)$ if the given isomorphism of A into B is also norm preserving.

The paper has been divided into four sections. In section 1, we study the relationship between the carrier space Φ_B of B and the carrier space Φ_A of A. If $\pi_A{}^B$ denotes the natural mapping of Φ_B into Φ_A ($\pi_A{}^B(\tilde{\varphi}) = \tilde{\varphi}|A$, $\tilde{\varphi} \in \Phi_B$), then $\pi_A{}^B$ is onto (Theorem 1.1). We further show that Φ_B is compact if and only if Φ_A is compact. The Šilov boundary $\partial\hat{B}$ of B always contains $(\pi_A{}^B)^{-1}(\partial\hat{A})$, $\partial\hat{A}$ the Šilov boundary of A, and examples can easily be given which show that the inclusion can be proper. A necessary and sufficient condition (Proposition 1.4) that $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial A)$ is given in terms of the Šilov boundary of the simple extensions $A[b]$ for $b \in B$. The remainder of section 1 is given to the study of $\partial A[b]\hat{}$, where b is an element integral over A.

The second section is concerned with the application of analytic functions to integral extensions of Banach algebras. Specifically, we show that if B is either a normed or a semi-simple integral extension of a Banach algebra, then B is closed under the application of analytic functions of several variables in the usual sense, even though the algebras might not be complete under any norm. This allows us to conclude that many of the standard theorems for Banach algebras which rely on analytic functions remain valid for such integral

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extensions of Banach algebras. (Perhaps it is worthwhile mentioning here that not every integral extension of a Banach algebra is normable as a normed extension of that algebra—an example will be given in [**10**].)

In section 3, we study a class of integral extensions we have called standard extensions. This class includes the simple extensions $A_\alpha = A[x]/(\alpha(x))$, where $\alpha(x)$ is a monic polynomial over A. The extension A_α is known to be normable as a normed extension of A and is complete in this norm precisely when A is complete in its given norm (see [**1**] for details). Using this technique for norming A_α, a special class of norms $||\cdot||_B$ on standard extensions that render $(B, ||\cdot||_B)$ a normed extension of $(A, ||\cdot||_A)$ can be constructed. We have called such norms standard norms. In Theorem 3.2 we give a necessary and sufficient condition for a standard extension to be complete in a standard norm. However, under standard norms, standard extensions are always Q-algebras; that is, the group of units is open in the norm topology on B.

In the final section of the paper, we show that every Banach algebra possesses a normed extension $(C, ||\cdot||_C)$ that is complete and integrally closed. This extends a theorem of B. Cole [**3**] who proved that every uniform algebra has a normed extension that is also a uniform algebra and closed under square roots. We also show that if A is indecomposable, then we can take C to be indecomposable.

The main technique of the paper is to reflect the problem at hand back into the subalgebras of B that are singly or finitely generated over A, that is, the subalgebras $A[b_1, \ldots, b_k]$ of polynomials in the elements $b_1, \ldots, b_k \in B$, coefficients in A.

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1. Carrier space of integral extensions. If B is an integral extension of A, then it is well-known that an ideal M in A is a maximal ideal in A if and only if there is a maximal ideal N in B such that $M = N \cap A$ (see [**14**, Chapter V, Section 2]). From this it easily follows that $R(A) = R(B) \cap A$, where $R(A)$ denotes the radical of A. In this section, we are interested only in those maximal ideals of A that are kernels of complex homomorphisms. For an algebra A, we denote the space of non-trivial complex homomorphisms on A by Φ_A. Rickart in [**13**] calls Φ_A the carrier space of A. As usual, for $a \in A$, $\hat{a}$ denotes the Gelfand transform on Φ_A of a, and $\hat{A}$ the algebra of such functions. The weakest topology induced on Φ_A by $\hat{A}$ will be called the Gelfand topology. The neighborhood

$$\{\theta \in \Phi_A : |\theta(b_i) - \varphi(b_i)| < \epsilon, i = 1, 2, \ldots, k\}$$

of $\varphi \in \Phi_A$ will be denoted by $V_A(\varphi; b_1, \ldots, b_k; \epsilon)$, $\epsilon > 0$, $b_1, \ldots, b_k \in A$.

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For an extension B of A, we set $\pi_A{}^B(\tilde{\varphi}) = \tilde{\varphi}|A$, $\tilde{\varphi} \in \Phi_B$. Clearly, $\pi_A{}^B$ is a continuous mapping with respect to the Gelfand topologies on Φ_B and Φ_A.

PROPOSITION 1.1. *Let B be an integral extension of A. Then*
(i) *$\pi_A{}^B$ is onto; and*
(ii) $\hat{a} \to \hat{a} \circ \pi_A{}^B$ is an isomorphism of $\hat{A}$ into $\hat{B}$; thus, $\hat{B}$ is an integral extension of $\hat{A}$.

Proof. (i) Suppose $M = \varphi^{-1}(0)$, $\varphi \in \Phi_A$. Then there exists a maximal ideal N in B such that $N \cap A = M$ (see [**14**, p. 259]). Let ψ denote the natural mapping of B onto B/N. Then, for $a \in A$, $a = \varphi(a)e + m$, $m \in M$, we have $\psi(a) = \varphi(a)\psi(e)$. Hence, $\psi(A) = \mathbf{C}\psi(e)$. Since B is integral over A, B/N is integral over $\psi(A)$ so that $B/N = \psi(A) = \mathbf{C}\psi(e)$. If we set $\tilde{\varphi}(b) = \lambda_b$, where $\psi(b) = \lambda_b\psi(e)$, we have that $\tilde{\varphi} \in \Phi_B$, $\tilde{\varphi}|A = \varphi$ and $\tilde{\varphi}^{-1}(0) = N$. Hence, $\pi_A{}^B$ is onto.
(ii) follows immediately from (i).

It follows from the proof of part (i) of the above proposition that if every maximal ideal of A is the kernel of a complex homomorphism, then the same is true for every maximal ideal in B.
For a polynomial $\beta(x) = \sum \beta_i x^i \in A[x]$, set $\beta_\varphi(x) = \sum \varphi(\beta_i)x^i$.

THEOREM 1.2. *Let B be an integral extension of A. Then Φ_B is compact if and only if Φ_A is compact.*

Proof. Since $\pi_A{}^B$ is continuous and onto, Φ_B compact forces Φ_A to be compact.

Conversely, suppose Φ_A is compact. Let $||\cdot||_\infty$ denote the uniform norm over Φ_A. Then $||\hat{a}||_\infty < +\infty$ for each $a \in A$. We now show that every element of B has a bounded transform. Let $b \in B$, and $\beta(x) = x^n + \sum_{\gamma=0}^{n-1} \beta_\gamma x^\gamma$ be any monic polynomial over A such that $\beta(b) = 0$. If $t > 0$ is any positive number satisfying $t^n \geqq \sum_{\gamma=0}^{n-1} ||\hat{\beta}_\gamma||_\infty t^\gamma$, then for $\tilde{\varphi} \in \Phi_B$, $|\tilde{\varphi}(b)| \leqq t$ since $\beta_\varphi(\tilde{\varphi}(b)) = 0$. Thus, $|\hat{b}(\tilde{\varphi})| \leqq t$ for all $\tilde{\varphi} \in \Phi_B$, and $\hat{B}$ is a normed algebra with respect to the uniform norm (over Φ_B) so that Φ_B is compact in the Gelfand topology.

By a natural algebra, we shall mean an algebra in which every maximal ideal is the kernel of a complex homomorphism and for which Φ_A is compact. Combining the comments following Proposition 1.1 and Theorem 1.2 we have

COROLLARY 1.3. *If B is an integral extension of A, then B is a natural algebra if and only if A is a natural algebra.*

Before we continue, we introduce the following notation: If A is an algebra and $\alpha(x) \in A[x]$ is monic, then A_α will denote $A[x]/(\alpha(x))$. $A_{\alpha_1 \ldots \alpha_k}$ will denote the repeated extension $(\ldots((A_{\alpha_1})_{\alpha_2})\ldots)_{\alpha_k}$, where $\alpha_i(x)$ is a monic polynomial over $(\ldots(A_{\alpha_1})_{\alpha_2}\ldots)_{\alpha_{i-1}}$. For an ideal I in A, $H(I) = \{\theta \in \Phi_A : \theta(i) = 0, i \in I\}$ will be called the hull of I. For a subset $F \subset \Phi_A$, $K(F) = \{a \in A : \varphi(a) = 0, \varphi \in F\}$ will be called the kernel of F. $K(\Phi_A)$ will be simply denoted by $K(A)$. For A, a Banach algebra, $K(A)$ is the usual radical.

We conclude this section with a discussion of the Šilov boundary of integral extensions. Let $\partial\hat{A}$ denote the Šilov boundary of $\hat{A}$ when Φ_A is compact. For the extension $B = A[x]/(\alpha(x))$, $\alpha(x)$ monic, and Φ_A compact, it is known that $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$ (see [**7**, Theorem 3.1]). For general integral extensions, we have

PROPOSITION 1.4. *Let B be an integral extension of A, with Φ_A compact. Then*
(i) $\partial\hat{B} \supseteq (\pi_A{}^B)^{-1}(\partial\hat{A})$;
(ii) $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$ *if and only if* $\partial A[b]\hat{} = (\pi_A{}^{A[b]})^{-1}(\partial\hat{A})$ for every $b \in B$.

Proof. To prove the first assertion of the proposition, suppose $\tilde{\varphi}_0 \in (\pi_A{}^B)^{-1}(\varphi_0)$, where $\varphi_0 \in \partial\hat{A}$. Let $V = V_B(\tilde{\varphi}_0; b_1, \ldots, b_k; \epsilon)$ denote any basic neighborhood of $\tilde{\varphi}_0$, and let $B' = A[b_1, \ldots, b_k]$. We will show that there exists $b \in B$ such that $|\hat{b}|$ maximizes on V and is less than that maximum on $\Phi_B \backslash V$. Let B_k denote a repeated extension of the form $A_{\alpha_1 \ldots \alpha_k}$ for which there is a homomorphism μ_k of B_k onto B'. (To construct B_k, simply take $\alpha_i(x) \in A[x]$ monic such that $\alpha_i(b_i) = 0$.) We will view $\Phi_{B'}$ and $H(\mu_k{}^{-1}(0))$ as the same set. Then $\hat{B}' = \hat{B}_k|H(\mu_k{}^{-1}(0))$, and $\partial\hat{B}' \supseteq \partial\hat{B}_k \cap H(\mu_k{}^{-1}(0))$. By Theorem 3.1 (loc. cit.), we have that $\partial\hat{B}_k = (\pi_A{}^{B_k})^{-1}(\partial\hat{A})$, and it follows that if $\varphi_0 \in \partial\hat{A}$, then

$$(\pi_A{}^{B'})^{-1}(\varphi_0) = (\pi_A{}^{B_k})^{-1}(\varphi_0) \cap H(\mu_k{}^{-1}(0)) \subseteq \partial\hat{B}'.$$

Since $V' = \pi_{B'}{}^B(V) = V_{B'}(\varphi_0'; b_1, \ldots, b_k; \epsilon)$, where $\varphi_0' = \pi_{B'}{}^B(\tilde{\varphi}_0)$, is an open neighborhood in $\Phi_{B'}$ of φ_0', there exists $b \in B'$ such that $|\hat{b}|$ assumes its maximum $||\hat{b}||_\infty$ on V' and $|\hat{b}| < ||\hat{b}||_\infty$ on $\Phi_B \backslash V'$ (see [**13**, p. 138]). When $\hat{b}$ is viewed as a transform on Φ_B (actually, we are considering $\hat{b} \circ \pi_{B'}{}^B$), we have that $|\hat{b}|$ maximizes only on $V = (\pi_{B'}{}^B)^{-1}(V')$. Hence, $V \cap \partial\hat{B} \neq \emptyset$. It follows that $\tilde{\varphi}_0 \in \partial\hat{B}$ since $\partial\hat{B}$ is closed in Φ_B. Thus, $\partial\hat{B} \supseteq (\pi_A{}^B)^{-1}(\partial\hat{A})$. (For use in the remainder of the proof of the theorem, note that $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$ if and only if $\pi_A{}^B(\partial\hat{B}) = \partial\hat{A}$. This follows immediately from the above.)

To prove the second assertion, suppose first that $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$, and that $b \in B$. Then, by part (i), we have that $\partial A[b]\hat{} \supseteq (\pi_A{}^{A[b]})^{-1}(\partial\hat{A})$, and $\partial\hat{B} \supseteq (\pi_{A[b]}{}^B)^{-1}(\partial A[b]\hat{})$. Therefore,

$$\partial\hat{A} = \pi_A{}^B(\partial\hat{B}) \supseteq (\pi_A{}^{A[b]})(\partial A[b]\hat{}) \supseteq \partial\hat{A}$$

so that equality holds, and $\partial A[b]\hat{} = (\pi_A{}^{A[b]})^{-1}(\partial\hat{A})$ for every $b \in B$.

Next, suppose that $\partial A[b]\hat{} = (\pi_A{}^{A[b]})^{-1}(\partial\hat{A})$ holds for every $b \in B$. Let $\tilde{\varphi} \in \partial\hat{B}$ and let V be any neighborhood in Φ_B of $\tilde{\varphi}$. Then there exists $b \in B$ such that $|\hat{b}|$ assumes its maximum modulus on V, say $||\hat{b}||_\infty = 1$, and $|\hat{b}| < 1/2$ on $\Phi_B \backslash V$. The set

$$W = \{\theta' \in \Phi_{A[b]} : |\theta'(b)| > 1/2\}$$

is a neighborhood in $\Phi_{A[b]}$ of $\varphi' = \pi_{A[b]}{}^B(\tilde{\varphi})$ and $\pi_{A[b]}{}^B(V) \supset W$. When $|\hat{b}|$ is viewed as a function on $\Phi_{A[b]}$, $|\hat{b}|$ maximizes only on the set W, and hence there is a $\theta' \in W \cap \partial A[b]\hat{}$. By hypothesis, $\theta = \pi_A{}^{A[b]}(\theta') \in \partial\hat{A}$ so that $(\pi_A{}^B)^{-1}(\theta) \cap V \neq \emptyset$ forces $V \cap (\pi_A{}^B)^{-1}(\partial\hat{A}) \neq \emptyset$, and since V is an arbitrary neighborhood of $\tilde{\varphi} \in \partial\hat{B}$, the fact that $(\pi_A{}^B)^{-1}(\partial\hat{A})$ is closed forces $\tilde{\varphi} \in (\pi_A{}^B)^{-1}(\partial\hat{A})$.

The preceding proposition suggests that we consider conditions under which $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$, when B is the simple integral extension $A[b]$. For a Banach algebra A and $f \in C(\Phi_A)$, f integral over $\hat{A}$, Björk in [**2**] has shown that $\partial\hat{A}[f] = (\pi_{\hat{A}}{}^{\hat{A}[f]})^{-1}(\partial\hat{A})$ whenever the mapping $\pi_{\hat{A}}{}^{\hat{A}[f]}$ is open. His proof can be used to show

1.5. *If A is a Banach algebra and $B = A[b]$ is a simple integral extension such that $\pi_A{}^B$ is open, then $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$.*

We now use 1.5 to prove

THEOREM 1.6. *Let B be an integral extension of A, with Φ_A compact. If $\pi_A{}^B$ is open, then $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$.*

Proof. Since $\pi_A{}^{A[b]}(V) = \pi_A{}^B((\pi_{A[b]}{}^B)^{-1}(V))$ and since $\pi_{A[b]}{}^B$ is continuous, the assumption that $\pi_A{}^B$ is open forces $\pi_A{}^{A[b]}$ to be open for each $b \in B$. Without loss of generality, we can assume $\hat{A}$ is a Banach algebra under some norm. (For otherwise, we can replace $\hat{A}$ by its uniform closure $\hat{A}^-$ in $C(\Phi_A)$ and $A[b]\hat{}$ by $\hat{A}^-[\hat{b}]$.) Then 1.5 implies $\partial A[b]\hat{} = (\pi_A{}^{A[b]})^{-1}(\partial\hat{A})$. Since $b \in B$ is arbitrary, the theorem follows from Proposition 1.4.

We should note that $\pi_A{}^B$ open is not a necessary condition for $\partial\hat{B}$ to coincide with $(\pi_A{}^B)^{-1}(\partial\hat{A})$. Observe that if $\partial\hat{A} = \Phi_A$, then $\partial\hat{B} = (\pi_A{}^B)^{-1}(\partial\hat{A})$, without any conditions on $\pi_A{}^B$.

2. Analytic functions. For an algebra B, let $\sigma_B(b_1, \ldots, b_k)$ denote the set $\{(\varphi(b_1), \ldots, \varphi(b_k)) \in \mathbf{C}^k : \varphi \in \Phi_B\}$, where $b_1, \ldots, b_k \in B$. Of course, if B is a Banach algebra, then $\sigma_B(b_1, \ldots, b_k)$ is the usual joint spectrum of the elements $b_1, \ldots, b_k$. If D is an open set in $\mathbf{C}^k$, let $\mathscr{O}_D$ denote the algebra of all functions analytic on D, let $\mathbf{1}(x) = 1$, $x \in D$, and let $z_1, \ldots, z_k$ denote the coordinate functions. If B is a (normed) algebra, we say that B is closed under the (continuous) application of analytic functions if for each $b_1, \ldots, b_k \in B$ and for each open set $D \supset \sigma_B(b_1, \ldots, b_k)$, there is a (continuous) homomorphism $\Psi : \mathscr{O}_D \to B$ such that $\Psi(\mathbf{1}) = e$, $\Psi(z_i) = b_i$, $i = 1, 2, \ldots, k$ and $\varphi(\Psi(f)) = f(\varphi(b_1), \ldots, \varphi(b_k))$ for $\varphi \in \Phi_B$ and $f \in \mathscr{O}_D$. (By Ψ continuous, we mean that if $f_n \to f$ uniformly on the compact subsets of D, $\{f_n\} \subset \mathscr{O}_D$, then $\Psi(f_n) \to \Psi(f)$ in the norm on B.) If B is a Banach algebra, then it is well-known that B is closed under the continuous application of analytic functions (see, for example, [**4**, pp. 76-84]). If B is assumed to be an integral extension of A such that for each finite set $b_1, \ldots, b_k \in B$, $A[b_1, \ldots, b_k]$ is a Banach algebra under some norm, then B is closed under the application of analytic functions. This follows immediately from the Banach algebra case and the fact that $\sigma_B(b_1, \ldots, b_k) = \sigma_{B'}(b_1, \ldots, b_k)$, where $B' = A[b_1, \ldots, b_k]$. We now prove

THEOREM 2.1. *Let B be an integral extension of A, a Banach algebra. If B is semi-simple, then B is closed under the application of analytic functions, and if B is a normed extension of A, then B is closed under the continuous application of analytic functions.*

Proof. Let $\{b_1, \ldots, b_k\}$ be an arbitrary finite subset of B. Let B_k denote a repeated extension of the form $A_{\alpha_1 \ldots \alpha_k}$ for which there is a homomorphism μ_k of B_k onto $B' = A[b_1, \ldots, b_k]$ such that $\mu_k(a) = a$, $a \in A$. B_k can be made into a Banach algebra, and a normed extension of A, by the repeated application of the technique of Arens and Hoffman (see [**1**]). Now, assume B is semi-simple. Then $B' = A[b_1, \ldots, b_k]$ is semi-simple since $R(B') = R(B) \cap B' = (0) \cap B' = (0)$. The semi-simplicity of B' in turn implies that $\mu_k^{-1}(0) = K(H(\mu_k^{-1}(0)))$, which is closed with respect to any complete norm on B_k. Hence, $B_k/\mu_k^{-1}(0)$ is a Banach algebra under the induced quotient norm. Since $B' \cong B_k/\mu_k^{-1}(0)$, so is B' a Banach algebra. By the observation made earlier, B is closed under the application of analytic functions.

Suppose, next, that B is a normed extension of A, say with respect to the norm $|| \cdot ||_B$. Let B', μ_k and B_k be as in the opening of the above paragraph. A complete norm $|| \cdot ||_k$ can be selected for B_k in such a way that $||a||_k = ||a||$ for all $a \in A$ and μ_k is a norm decreasing homomorphism of B_k onto B', the latter being given the norm $|| \cdot ||_B$. For $b \in B'$, set

$$||b||_Q = \inf_{j \in \mu_k^{-1}(0)} ||b' + j||_k,$$

where $\mu_k(b') = b$. Then $|| \cdot ||_Q$ is a complete norm on B' satisfying $||b||_B \leqq ||b||_Q$, $b \in B'$, since μ_k is norm decreasing. This shows that B' is a Banach algebra under $|| \cdot ||_Q$ so that $(B', || \cdot ||_Q)$ is closed under the continuous application of analytic functions. Since $|| \cdot ||_Q \geqq || \cdot ||_B$ on B', the same is true for the normed algebra $(B', || \cdot ||_B)$. Since $b_1, \ldots, b_k \in B$ was arbitrary, $(B, || \cdot ||_B)$ is closed under the continuous application of analytic functions.

As we pointed out in the introduction, the significance of the above theorem is that many of the standard theorems for Banach algebras hold for certain integral extensions of Banach algebras. For example, if B is either a semi-simple, or a normed, integral extension of a Banach algebra, then the Šilov Idempotent Theorem holds for such B's. Furthermore, the factorization theorems in [**9**] and [**11**] also hold. For later use, we record two specific results.

2.2. *Suppose B is either a semi-simple, or a normed, integral extension of a Banach algebra. If $\alpha(x)$ is a monic polynomial over B and if B is indecomposable, then there exists a monic polynomial $\alpha_1(x) \in B[x]$ such that $\alpha_1(x)$ is a factor of $\alpha(x)$ and $B[x]/(\alpha_1(x))$ is indecomposable.*

The proof of 2.2 follows from Theorem 2.1 and the proof of Corollary 5.2 in [**11**].

2.3. *Let* $(B, || \cdot ||_B)$ *be a normed integral extension of a Banach algebra, and let* $f \in C(\Phi_B)$ *such that* $\exp f = \hat{a}$ *(convergence in the uniform norm),* $a \in B$. *Then there exists* $b \in B$ *such that* $\exp b = a$ *(convergence in* $|| \cdot ||_B$*).*

See Corollary 6.2 in [**4**] for the Banach algebra version of 2.3.

3. Standard extensions. The type of extension embodied in the following definition (3.1) can be viewed as a natural generalization of the extension $A_\alpha = A[x]/(\alpha(x))$. If B_1 and B_2 are both extensions of A and if there is an isomorphism ψ of B_1 onto B_2 such that $\psi(a) = a$, for all $a \in A$, then we write $B_1 \cong_A B_2$.

Definition 3.1. An extension B of A is called a standard extension of A if there exists a well-ordered set $\mathfrak{A}$ and a family $\{B_i\}_{i \in \mathfrak{A}}$ of intermediate subalgebras such that $B = \bigcup_i B_i$ and

(i) $i, j \in \mathfrak{A}$ and $i \leqq j$ implies $B_i \subseteq B_j$, and

(ii) $i \in \mathfrak{A}$ implies $B_i \cong_{\tilde{B}_i} \tilde{B}_i[x]/(\alpha_i(x))$, where $\tilde{B}_i = \bigcup_{j<i} B_j$, $\alpha_i(x)$ is a monic polynomial over $\tilde{B}_i$, and for the first element $i_0 \in \mathfrak{A}$, $\tilde{B}_{i_0} = A$.

Standard extensions are easily seen to exist. For example, repeated extensions $A_{\alpha_1 \ldots \alpha_k}$ are standard extensions. We next describe a "construction" for infinitely generated standard extensions. This is modelled after the proof given in [**5**] for the existence of an algebraic closure of a field. Let $\Lambda \subset A[x]$ be a set of monic polynomials and suppose that Λ is well-ordered by $\leqq$. When $\alpha(x) \in \Lambda$ is used as an index, we will simply write α instead of $\alpha(x)$. Now, let X be any set with cardinality greater than that of A and containing A as a subset. Let $\mathscr{B}$ denote the set of all mappings f of initial intervals of Λ (with respect to $\leqq$) into the set of subsets of X such that

(i) $f(\alpha) \supset A$ and is an integral extension of A, $\alpha \in \operatorname{dom}(f)$;

(ii) $f(\beta)$ contains $f(\alpha)$ and is an integral extension of the latter for $\alpha \leqq \beta$, $\alpha, \beta \in \operatorname{dom}(f)$; and

(iii) for $\beta \in \operatorname{dom}(f)$,

$$f(\beta) \cong_{\tilde{f}(\beta)} \tilde{f}(\beta)[x]/(p_\beta(x)),$$

where $\tilde{f}(\beta) = \bigcup_{\alpha<\beta} f(\alpha)$ (if $\alpha_0 \in \Lambda$ is the first element with respect to $\leqq$, then $\tilde{f}(\alpha_0) = A$) and $p_\beta(x) \in \tilde{f}(\beta)[x]$ is a monic factor of least degree of $\beta(x)$.

If we write $f \prec g$, $f, g \in \mathscr{B}$, if and only if $\operatorname{dom}(f) \subset \operatorname{dom}(g)$ and $f = g|\operatorname{dom}(g)$, then there exists $f_0 \in \mathscr{B}$ which is maximal with respect to $\prec$ and satisfying (i), (ii) and (iii); this follows from Zorn's Lemma. Thus, each $\alpha(x) \in \Lambda$ has a zero in $B_\Lambda = \bigcup_{\alpha \in \operatorname{dom}(f_0)} f_0(\alpha)$, since assuming otherwise contradicts the maximality of f_0. (Note that since we are indexing by a subset of $A[x]$, the cardinality $\tilde{f}(\beta)$ is always the same as the cardinality of A.)

The condition of $p_\beta(x)$ in (iii) above was introduced in the interest of economy. It has the further implication that if A is an indecomposable Banach algebra, then B_Λ is also indecomposable, as we will show in the next section.

Suppose that B is a standard extension of A relative to $\{B_i\}_{i \in \mathfrak{A}}$ and that $(A, \|\cdot\|_A)$ is a normed algebra. Then $\|\cdot\|_A$ can be extended to a norm $\|\cdot\|_B$ on B in such a way that the isomorphism of B_i onto $\tilde{B}_i[x]/(\alpha_i(x))$ is an isometry, the latter being given a norm of the Arens-Hoffman type extending the norm on $\tilde{B}_i$. A norm $\|\cdot\|_B$ with this property will be called a standard norm and $(B, \|\cdot\|_B)$ is called a standard normed extension of $(A, \|\cdot\|_A)$.

THEOREM 3.2. *Let $(B, \|\cdot\|_B)$ be a standard normed extension of $(A, \|\cdot\|_A)$ with respect to the family $\{B_i\}_{i \in \mathfrak{A}}$. If $(A, \|\cdot\|_A)$ is a Banach algebra, then $(B, \|\cdot\|_B)$ is complete if and only if there exists $l \in \mathfrak{A}$ such that $B_l = B$ and $\{B_j : j \leqq l\}$ is a finite set of intermediate subalgebras.*

Proof. Suppose $l \in \mathfrak{A}$ has the property in the statement of the theorem. Then $B = B_l \cong_A A_{\alpha_1 \ldots \alpha_k}$ for some collection of polynomials $\alpha_1(x), \ldots, \alpha_k(x)$. Since $\|\cdot\|_B$ is a standard norm on B, $(B, \|\cdot\|_B)$ will be complete if and only if $\tilde{B}_{l_1}$ is complete with respect to $\|\cdot\|_B$, where l_1 is the largest index in $\mathfrak{A}$ less than l and $\tilde{B}_{l_1} \subsetneqq B_l$. This follows from the form of the Arens-Hoffman type norm (see [**1**]). Hence, by repeating the argument a finite number of times, we obtain $l, l_1, \ldots, l_k$ in $\mathfrak{A}$ such that $\tilde{B}_{l_k} = A \subsetneqq \tilde{B}_{l_{k-1}} \subsetneqq \ldots \subsetneqq B_l = B$ and $(\tilde{B}_{l_j}, \|\cdot\|_B)$ is complete if and only if $(\tilde{B}_{l_{j+1}}, \|\cdot\|_B)$ is complete. Since $(A, \|\cdot\|_A)$ is complete and $\|\cdot\|_A = \|\cdot\|_B$ restricted to A, it follows that $(B, \|\cdot\|_B)$ is complete.

Suppose next that the condition of the theorem fails. Then there exists a least $\omega \in \mathfrak{A}$ such that $\tilde{B}_\omega$ is a standard extension with respect to $\{B_\alpha\}_{\alpha<\omega}$ and $\{B_\alpha : \alpha < \omega\}$ is a countable set. Now, restrict $\|\cdot\|_B$ to $\tilde{B}_\omega$, using the same notation for the restriction. Then $(\tilde{B}_\omega, \|\cdot\|_B)$ is a standard normed extension of $(A, \|\cdot\|_A)$ with respect to $\{B_\alpha : \alpha < \omega\}$. Now, enumerate the distinct elements in $\{B_\alpha : \alpha < \omega\}$ by the positive integers: $A = B_0 \subsetneqq B_1 \subsetneqq B_2 \subsetneqq \ldots$, and $\tilde{B}_\omega = \bigcup_{n=1}^{\infty} B_n$. For each positive integer n, $B_{n+1} \cong_{B_n} B_n[x]/(\alpha_n(x))$, $\alpha_n(x) \in B_n[x]$, the isomorphism being an isometry when the latter is given an Arens-Hoffman type norm. Thus, if

$$B_n' = \{b_1\mathfrak{x} + \ldots + b_{k-1}\mathfrak{x}^{k-1}, b_1, \ldots, b_{k-1} \in B_n\},$$

k = degree of $\alpha_n(x)$, then $B_{n+1} = B_n \oplus B_n'$, the direct sum being topological; indeed, for $b \in B_{n+1}$, $\|b\|_B = \|b_0\|_B + \|b - b_0\|_B$ where $b_0 \in B_n$, $b - b_0 \in B_n'$. Now, for each positive integer n, let $b_n \in B_n'$, $\|b_n\|_B = 1/2^n$. Thus, the sequence

$$\left\{\sum_{n=1}^{N} b_n\right\}_{N=1}^{+\infty}$$

is a $\|\cdot\|_B$-Cauchy sequence. Suppose $\sum_{n=1}^{\infty} b_n$ converges to b in $\tilde{B}_\omega$. Let n_0 be a positive integer such that $b \in B_{n_0}$. Now, by the requirement that $b_n \in B_n'$ and by the norm condition on the direct sum $B_{n+1} = B_n \oplus B_n'$, we have that

$$\left\|b - \sum_{n=1}^{N} b_n\right\|_B = \left\|b - \sum_{n=1}^{n_0-1} b_n\right\|_B + \sum_{n=n_0}^{N} \|b_n\|_B$$

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for $N > n_0$. Hence, $\sum_{n=n_0}^{N} ||b_n||_B \to 0$ and consequently, $b_n = 0$, for all $n > n_0$. This is a contradiction, so that $\sum_{n=1}^{\infty} b_n$ does not converge in $\tilde{B}_\omega$. We will show that this series does not converge in B by showing that $\tilde{B}_i$ is $||\cdot||_B$-closed in B for every $i \in \mathfrak{A}$. First note that $\tilde{B}_i$ is $||\cdot||_B$-closed in B_i. Hence, suppose B_i is $||\cdot||_B$-closed in B_j for all $j < j_0$. Then B_i remains $||\cdot||_B$-closed in $\tilde{B}_{j_0}$, and by the above, $||\cdot||_B$-closed in B_{j_0} as well. By transfinite induction, B_i is closed in B_j for all $j \geqq i$. Hence, B_i is $||\cdot||_B$-closed in $B = \bigcup_{j\in\mathfrak{A}} B_j$. Thus, the same is true for $\tilde{B}_i$. By combining this with the fact that $(\tilde{B}_\omega, ||\cdot||_B)$ is incomplete, we have that $(B, ||\cdot||_B)$ is necessarily incomplete.

If B is a standard extension of A and incomplete with respect to a standard norm $||\cdot||_B$, then it is plausible that B is complete with respect to some other norm (not a standard norm!). For separable extensions, this can not happen. (B is called a separable extension of A if B is a standard extension with respect to $\{B_j\}_{j\in\mathfrak{A}}$, where the polynomial $\alpha_j(x)$ generating B_j over $\tilde{B}_j$ has an invertible discriminant in $\tilde{B}_j$.) If infinitely many of the B_j's are distinct, then it is easily seen that $(\pi_A{}^B)^{-1}(\varphi)$ is an infinite subset of Φ_B, and it follows from the next result that such a B can never be a Banach algebra under any norm.

THEOREM 3.3. *Let A be an algebra and B an integral extension of A. If $(\pi_A{}^B)^{-1}(\varphi)$ is an infinite subset of Φ_B for some $\varphi \in \Phi_A$, then B is incomplete under any norm.*

Proof. Set $X = (\pi_A{}^B)^{-1}(\varphi)$. Since X is hull-kernel closed in Φ_B, $\Phi_{B/K(X)}$ can be identified with X. It is easily seen that $B/K(X)$ is an integral extension of $A/(K(X) \cap A) \cong \mathbf{C}$, the complex numbers. Hence, for $b \in B$, $\hat{b}|X$ has finite range. Suppose, now, that B is a Banach algebra under some norm. Then $K(X)$ is closed in this norm and $B/K(X)$ is a Banach algebra under the induced quotient norm. Since $\Phi_{B/K(X)} = X$ is infinite by assumption, $B/K(X)\hat{} \cong \hat{B}|X$ is infinite dimensional and hence $B/K(X)$ contains an element $b + K(X)$ with infinite spectrum $\sigma_{B/K(X)}(b + K(X))$ (see [**6**, Lemma 7, p. 376]). Since the latter set coincides with the range of $\hat{b}|X$, we have a contradiction. Thus, B can never be a Banach algebra if the condition of the theorem holds.

When A is a Banach algebra, the standard normed extensions $(B, ||\cdot||_B)$ are Q-algebras, a property enjoyed by all Banach algebras. A normed algebra is called a Q-algebra if the group of units is open; this is equivalent to every maximal ideal being closed. Thus, a natural normed algebra is a Q-algebra if and only if every complex homomorphism is continuous.

THEOREM 3.4. *Let A be a Banach algebra and let $(B, ||\cdot||_B)$ be a standard normed extension of A. Then $(B, ||\cdot||_B)$ is a Q-algebra.*

Proof. Since B is a natural algebra by Corollary 1.3, we need only show that $\tilde{\varphi} \in \Phi_B$ is $||\cdot||_B$-continuous. If $\tilde{\varphi}$ is $||\cdot||_B$-discontinuous, then there is a least $k \in \mathfrak{A}$ such that $\tilde{\varphi}$ is $||\cdot||_B$-discontinuous on B_k. Since A is a Banach algebra and $||\cdot||_B$ extends $||\cdot||_A$, $\tilde{\varphi}$ is $||\cdot||_B$-continuous on A. Thus, it follows that

$A \subsetneq B_k$. Since $B_k \cong {}_{\tilde{B}_k}B_k[x]/(\alpha_k(x))$, $\tilde{\varphi}$ must also be $\|\cdot\|_B$-discontinuous on $\tilde{B}_k$ because of the form of the norm $\|\cdot\|_B$ on $\tilde{B}_k[x]/(\alpha_k(x))$ (see comments on bottom of [**7**, p. 582]). This implies there exists $b \in \tilde{B}_k$ such that $|\tilde{\varphi}(b)| > \|b\|_B$ and consequently $\tilde{\varphi}$ is $\|\cdot\|_B$-discontinuous on B_j for some $j < k$, where $b \in B_j$. This is a contradiction to the assumption concerning k so that $\tilde{\varphi}$ is $\|\cdot\|_B$-continuous on B_i for all $i \in \mathfrak{A}$; hence, $\tilde{\varphi}$ is $\|\cdot\|_B$-continuous on B itself.

Without the assumption that $\|\cdot\|_B$ is a standard norm, the above theorem does not necessarily hold.

The next theorem is motivated by the work of B. Cole in [**3**].

THEOREM 3.5. *Let B be a standard extension of A, A a complex algebra with Φ_A compact. Then the following hold*:

(i) *$\pi_A{}^B$ is an open mapping*;

(ii) *$\partial\hat{B} = (\pi_A{}^B)^{-1}(\hat{A})$; and*

(iii) *$\hat{B}$ is $\|\cdot\|_\infty$-dense in $C(\Phi_B)$ if and only if $\hat{A}$ is $\|\cdot\|_\infty$-dense in $C(\Phi_A)$.*

Proof. (i) Let B be a standard extension with respect to $\{B_i\}_{i\in\mathfrak{A}}$. Let $V_1 = V_B(\tilde{\varphi}; b_1, \ldots, b_k; \epsilon)$ be a basic neighborhood of $\tilde{\varphi}$ in Φ_B, and let $b_1, \ldots, b_k \in B_i$, $i \in \mathfrak{A}$. Then $\pi_{B_i}{}^B(V_1) = V_{B_i}(\pi_{B_i}{}^B(\tilde{\varphi}); b_1, \ldots, b_k; \epsilon)$ so that $\pi_{B_i}{}^B(V_1)$ is a neighborhood of $\pi_{B_i}{}^B(\tilde{\varphi})$ in Φ_{B_i}. Since $\pi_{\tilde{B}_i}{}^{B_i}$ is an open mapping (see [**9**, 1.2, p. 358]), $\pi_{\tilde{B}_i}{}^B(V_1)$ is a neighborhood of $\pi_{\tilde{B}_i}{}^B(\tilde{\varphi})$ in $\Phi_{\tilde{B}_i}$. By repeating the argument, we can find a $j \in \mathfrak{A}$, $j < i$ such that $\pi_{B_j}{}^B(V_1)$ is a neighborhood of $\pi_{B_j}{}^B(\tilde{\varphi})$. Clearly, there is a least l such that $\pi_{B_l}{}^B(V_1)$ is a neighborhood of $\pi_{B_l}{}^B(\tilde{\varphi})$. By the above arguments, l is the least element of $\mathfrak{A}$, that is, $B_l = A$, and $\pi_A{}^B(V_1)$ is a neighborhood of $\pi_A{}^B(\tilde{\varphi})$. Thus, $\pi_A{}^B$ is an open mapping.

(ii) This part follows immediately from Theorem 1.6 and (i) above.

(iii) Assume that $\hat{A}$ is $\|\cdot\|_\infty$-dense in $C(\Phi_A)$. Then $\hat{B}_k$ is $\|\cdot\|_\infty$-dense in $C(\Phi_{B_k})$ where $k \in \mathfrak{A}$ is the least index such that $B_k \neq A$. This follows from Corollary 4.2 in [**7**]. Now, assume that $\hat{B}_i$ is $\|\cdot\|_\infty$-dense in $C(\Phi_{B_i})$ for all $i < j$. When $\hat{B}_i$ is viewed as a subalgebra of $\tilde{B}_j{}^\wedge$, the uniform closure $\hat{B}_i{}^-$ of $\hat{B}_i$ in $C(\Phi_{\tilde{B}_j})$ is conjugate closed as well as contained in the uniform closure $\overline{\tilde{B}_j{}^\wedge}$ of $\tilde{B}_j{}^\wedge$. Thus,

$$\tilde{B}_j{}^\wedge \subseteq \bigcup_{i<j} \hat{B}_i{}^- \subseteq \overline{\tilde{B}_j{}^\wedge}.$$

Since $\tilde{B}_j{}^\wedge$ is separating on $\Phi_{\tilde{B}_j}$ and $\bigcup_{i<j}\hat{B}_i{}^-$ is conjugate closed, it follows from the Stone-Weierstrass Theorem that $\overline{\tilde{B}_j{}^\wedge} = C(\Phi_{\tilde{B}_j})$. But $\tilde{B}_j{}^\wedge$ dense in $C(\Phi_{\tilde{B}_j})$ implies the same is true for $\hat{B}_j$. Thus, $\hat{B}_j$ is $\|\cdot\|_\infty$-dense in $C(\Phi_{B_j})$ for all $j \in \mathfrak{A}$. By another application of the preceding argument, $\hat{B}$ is $\|\cdot\|_\infty$-dense in $C(\Phi_B)$.

Now, suppose that $\hat{B}$ is $\|\cdot\|_\infty$-dense in $C(\Phi_B)$. It suffices to show that if $f \in C(\Phi_A)$ and $\epsilon > 0$ are given, then there exists $g \in A$ such that $\|f - g\|_\infty < \epsilon$. Since $C(\Phi_A)$ can be viewed as a subalgebra of $C(\Phi_B)$, there exists $g \in \hat{B}$ such that $\|f - g\|_\infty < \epsilon$. Now, there exists a least $l \in \mathfrak{A}$ such that there exists $g \in \hat{B}_l$ satisfying $\|f - g\|_\infty < \epsilon$. We will show that $B_l = A$. Let $\alpha(x) = \alpha_l(x)$

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be the monic polynomial generating B_l over $\tilde{B}_l$, and assume $B_l \neq A$. Then

$$g = \left(\sum_{j=0}^{k-1} b_j \mathfrak{x}^j\right)^{\wedge}, \quad b_j \in \tilde{B}_l,$$

where $k =$ degree $\alpha(x)$ over $\tilde{B}_l$. Now, consider the system of equations:

$$g(\varphi, \lambda_i) = \sum_{j=0}^{k-1} \varphi(b_j)\lambda_i^{\,j},$$

where $\varphi \in \Phi_{\tilde{B}_l}$ and λ_i runs through all the roots of $\alpha_\varphi(x) = 0$, each repeated according to its multiplicity. Set

$$\tilde{g}(\varphi) = \sum_{i=1}^{k} g(\varphi, \lambda_i), \quad \varphi \in \Phi_{\tilde{B}_l}.$$

Thus $\tilde{g}$ as defined is a function on $\Phi_{\tilde{B}_l}$ and moreover, $\tilde{g} \in \tilde{B}_l{}^\wedge$. Now, for $\varphi \in \Phi_{\tilde{B}_l}$,

$$\begin{aligned} |\tilde{g}(\varphi) - kf(\varphi)| &= \left|\sum_{i=1}^{k} g(\varphi, \lambda_i) - kf(\varphi)\right| \\ &\leqq \sum_{i=1}^{k} |g(\varphi, \lambda_i) - f(\varphi)| \\ &< k\epsilon. \end{aligned}$$

Since Φ_{B_l} is compact, $||g - kf||_\infty < k\epsilon$, or $||(1/k)\tilde{g} - f||_\infty < \epsilon$. But $(1/k)\tilde{g} \in \tilde{B}_l{}^\wedge$ implies there exists $j < l$ such that $(1/k)\tilde{g} \in B_j{}^\wedge$. This contradicts the assumption that l is the least index such that there exists $g \in \hat{B}_i$ and $||g - f||_\infty < \epsilon$. Thus, $B_l = A$. This establishes the denseness of $\hat{A}$ in $C(\Phi_A)$.

4. Integral closure of a Banach algebra. Part of the motivation for the preceding section is that we wish to extend a result of B. Cole. In [**3**], he showed that if A is a uniform algebra, then there exists a uniform algebra C that is a normed extension of A and closed under square roots. The same techniques show that there is such a C that is also integrally closed. For Banach algebras, we have

THEOREM 4.1. *Let $(A, ||\cdot||_A)$ be a Banach algebra. Then there exists a complete normed extension $(C, ||\cdot||_C)$ of A such that*
- (i) *C is integrally closed;*
- (ii) *$\pi_A{}^C$ is onto and an open mapping;*
- (iii) $\partial\hat{C} = (\pi_A{}^C)^{-1}(\partial\hat{A})$;
- (iv) *$\hat{C}$ is dense in $C(\Phi_C)$ if and only if $\hat{A}$ is dense in $C(\Phi_A)$; and*
- (v) *if A is indecomposable, then C can also be taken indecomposable.*

Proof. Let C_1 denote the standard extension B_Λ of A constructed in the last section, where Λ is the set of all monic polynomials in $A[x]$ and let $||\cdot||_1$ be a standard norm on C_1. Properties (ii)–(iv) hold for C_1 by Theorem 3.5. We next show that C_1 is indecomposable when A is indecomposable. To this end, let $\{B_\alpha\}$ be the family of intermediate extensions satisfying conditions (i) and (ii) of the construction of C_1. Suppose that for β, B_α is indecomposable for all $\alpha < \beta$. Then clearly $\tilde{B}_\beta$ is indecomposable. Let $p_\beta(x)$ be the monic factor of least degree of $\beta(x)$ such that $B_\beta \cong {}_{\tilde{B}_\beta}\tilde{B}_\beta[x]/(p_\beta(x))$. Then, by 2.2, there exists a monic factor $q(x)$ of $p_\beta(x)$ such that $\tilde{B}_\beta[x]/(q(x))$ is indecomposable. If deg $q(x) <$ deg $p_\beta(x)$, then $p_\beta(x)$ is not a monic factor of $\beta(x)$ of least degree, which contradicts our assumption concerning $p_\beta(x)$. Therefore, deg $q(x) =$ deg $p_\beta(x)$, so that $q(x) = p_\beta(x)$. Hence, B_β is also indecomposable. Thus, by transfinite induction, B_β is indecomposable for all β so that C_1 itself is indecomposable. Now, let $\bar{C}_1$ denote the $\|\cdot\|_1$-completion of C_1. Since $(C_1, \|\cdot\|_1)$ is a Q-algebra by Theorem 3.4, $\Phi_{\bar{C}_1} = \Phi_{C_1}$, and it follows that $\bar{C}_1$ is also indecomposable whenever C_1 is indecomposable. Thus, properties (ii)–(v) hold for the extension $\bar{C}_1$. We may and do consider A as a subset of $\bar{C}_1$.

If ω_1 denotes the first uncountable ordinal, then by the Principle of Transfinite Recussion we can find a family

$$\{(C_\alpha, \|\cdot\|_\alpha)\}_{\alpha<\omega_1}$$

of Banach algebras such that $(C_0, \|\cdot\|_0) = (A, \|\cdot\|_A)$ and

(i)′ for $\alpha < \beta$, $C_\alpha \subseteq C_\beta$ and $(C_\beta, \|\cdot\|_\beta)$ a normed extension of $(C_\alpha, \|\cdot\|_\alpha)$ with respect to the identity mapping of C_α into C_β;

(ii)′ every monic polynomial over the closure $\bar{D}_\beta$ (in C_β) of $D_\beta = \bigcup_{\alpha<\beta} C_\alpha$ has a zero in C_β;

(iii)′ C_β satisfies properties (ii)–(iv) with respect to D_β; and

(iv)′ C_α are taken indecomposable when A is indecomposable. Now, let $C = \bigcup_{\alpha<\omega_1} C_\alpha$ and introduce the obvious definitions of the algebraic operations into C. For a norm on C, we take $\|b\|_C = \|b\|_\alpha$, if $b \in C_\alpha$. Clearly $(C, \|\cdot\|_C)$ is a normed algebra, and is complete by a theorem of Pym (see [**12**]). It is easily seen that C is integrally closed. To prove property (ii), we first show that $\pi_A{}^C$ is an open mapping. Suppose $\beta < \omega_1$ and $\pi_A{}^{C_\alpha}$ is open for all $\alpha < \beta$. Then $\pi_A{}^{D_\beta}$ is open (see the proof of Theorem 3.5). Now, let $\bar{D}_\beta$ denote the $\|\cdot\|_C$-closure of D_β in C. Clearly, $\bar{D}_\beta \subset C_\beta$, and $\pi_{\bar{D}_\beta}{}^{C_\beta}$ is open by (iii)′ above. But D_β is a Q-algebra under $\|\cdot\|_C$ (restricted to it) since $\varphi \in \Phi_{D_\beta}$ implies that $|\varphi(b)| \leqq \|b\|_\alpha = \|b\|_C$ whenever $b \in C_\alpha \subset D_\beta, \alpha < \beta$. Hence $\Phi_{\bar{D}_\beta} = \Phi_{D_\beta}$ so that $\pi_A{}^{\bar{D}_\beta} = \pi_A{}^{D_\beta}$. Since

$$\pi_A{}^{C_\beta} = \pi_A{}^{D_\beta} \circ \pi_{D_\beta}{}^{C_\beta},$$

$\pi_A{}^{C_\beta}$ is also an open mapping. Thus, by the Principle of Transfinite Induction, $\pi_A{}^{C_\alpha}$ is open for all $\alpha < \omega_1$. Thus, it follows that $\pi_A{}^C$ is an open mapping. Now, to show $\pi_A{}^C$ is onto, consider $X = \pi_A{}^C(\Phi_C)$. Since $\pi_A{}^C$ is continuous, X is both open and closed. By Theorem 3.3.25 in [**13**], $\partial A \subset X$ so that $X = \Phi_A$. This shows that (ii) holds for the C we constructed. It follows also that $\pi_{C_\alpha}{}^C$ and $\pi_{D_\alpha}{}^C$ are also onto, open mappings.

We next establish property (iii). First note that $\partial\hat{C}_1 = (\pi_A{}^{C_1})^{-1}(\partial\hat{A})$. Next assume that $\beta < \omega_1$ has the property that $\partial\hat{C}_\alpha = (\pi_A{}^{C_\alpha})^{-1}(\partial\hat{A})$ for all $\alpha < \beta$.

To show that the same property holds for β, begin by assuming $\tilde{\varphi} \notin \partial\hat{D}_\beta$. Then there exists $\alpha < \beta$ and an open set $W \subset \Phi_{D_\beta}$ containing $\tilde{\varphi}$ such that

$$(\pi_{C_\alpha}{}^{D_\beta})^{-1}(\pi_{C_\alpha}{}^{D_\beta}(W)) = W \quad \text{and} \quad W \cap \partial\hat{D}_\beta = \emptyset.$$

Since $\pi_{C_\alpha}{}^{D_\beta}$ is onto, $\pi_{C_\alpha}{}^{D_\beta}(\partial\hat{D}_\beta) \supseteq \partial\hat{C}_\alpha$ so that $\pi_{C_\alpha}{}^{D_\beta}(W) \cap \partial\hat{C}_\alpha = \emptyset$. Thus, $\pi_{C_\alpha}{}^{D_\beta}(\tilde{\varphi}) \notin \partial\hat{C}_\alpha$ and since $\alpha < \beta$, we have that $\pi_A{}^{D_\beta}(\tilde{\varphi}) \notin \partial\hat{A}$. Hence, we have that $\partial\hat{D}_\beta \supseteq (\pi_A{}^{D_\beta})^{-1}(\partial\hat{A})$. On the other hand,

$$(\pi_A{}^{D_\beta})^{-1}(\partial\hat{A}) = \bigcup_{\alpha<\beta} (\pi_{C_\alpha}{}^{D_\beta})^{-1}(\partial\hat{C}_\alpha)$$

and the latter set is a closed maximizing set for $\hat{D}_\beta$. Thus, $(\pi_A{}^{D_\beta})^{-1}(\partial\hat{A}) \supseteq \partial\hat{D}_\beta$ so that we can conclude that $\partial\hat{D}_\beta = (\pi_A{}^{D_\beta})^{-1}(\partial\hat{A})$. Since $\partial\hat{C}_\beta = (\pi_{D_\beta}{}^{C_\beta})^{-1}(\partial\hat{D}_\beta)$, we have that $\partial\hat{C}_\beta = (\pi_A{}^{C_\beta})^{-1}(\partial\hat{A})$. By the Principle of Transfinite Induction, this equality must hold for all $\beta < \omega_1$, and a repetition of the above proof with β replaced by ω_1 and D_β replaced by C yields the conclusion in part (iii) of the theorem.

Property (iv) is established as in the proof of the corresponding statement of Theorem 3.5 and property (v) follows immediately from (iv)′.

By a proper extension C of A, we mean an extension such that every idempotent in C must be in A.

COROLLARY 4.2. *If A is a Banach algebra, and if $A = \sum_{i=1}^n \oplus A_i$, A_i indecomposable for each i, then there exists a complete normed extension $(C, ||\cdot||_C)$ of A satisfying* (i)–(iv) *of the theorem and C is a proper extension of A.*

Proof. Let $e_1, \ldots, e_k$ be mutually orthogonal idempotents such that $e_iA = A_i$, $i = 1, 2, \ldots, n$. For each i, there is an extension $(C_i, ||\cdot||_i')$ of $(A_i, ||\cdot||_i)$, $||e_ia||_i = ||e_ia||_A$, satisfying the properties (i)–(v) of the theorem. Let $C = \sum_{i=1}^n \oplus C_i$ with $||\cdot||_C'$ defined by

$$||b_1 \oplus \ldots \oplus b_n||_C' = \sum_{i=1}^n ||b_i||_i'.$$

Then the algebra C is an extension of A satisfying properties (i)–(iv). Furthermore, C is a proper extension since $u \in C$ an idempotent implies that u is a sum of idempotents in A (note: $ue_i = e_i$ or 0). Since

$$||a||_A \leqq \sum_{i=1}^n ||e_ia||_A = ||a||_C' \leqq \left(\max_{1\leqq i\leqq n} ||e_i||_A\right)||a||_A,$$

there is a norm $||\cdot||_C$ equivalent to $||\cdot||_C'$ such that $(C, ||\cdot||_C)$ is a complete normed extension of A (see [**8**, Lemma 1]).

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