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Arising from the 2022 Japan–US Mathematics Institute, this book covers a range of topics in modern algebraic geometry, including birational geometry, classification of varieties in positive and zero characteristic, K-stability, Fano varieties, foliations, the minimal model program, and mathematical physics. The volume includes survey articles providing an accessible introduction to current areas of interest for younger researchers. Research papers, written by leading experts in the field, disseminate recent breakthroughs in areas related to the research of V. V. Shokurov, who has been a source of inspiration for birational geometry over the last 40 years.

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Preface

The Japan–U.S. Mathematics Institute (JAMI) is a series of workshops and conferences that aim to further cooperation in mathematical research through broadly based programs in mathematics. It was founded in 1998 and has been running a series of yearly topical programs at Johns Hopkins University. The 2022 edition of JAMI was devoted to Higher Dimensional Algebraic Geometry and dedicated to Prof. Vyacheslav V. Shokurov on the occasion of his 70th birthday. The conference was delayed due to the global pandemic and finally took place on May 3–8, 2022. Fittingly, the conference was held in Krieger Hall, Johns Hopkins University, where Shokhurov has spent most of his mathematical career.

The first two days of this conference consisted of online lectures by Osamu Fujino (Kyoto University), Antonella Grassi (University of Bologna), Chen Jiang (Fudan University), and Yuri Prokhorov (Steklov Mathematical Institute), whereas the main event was held in Krieger Hall and consisted of 20 talks by Harold Blum (University of Utah), Paolo Cascini (Imperial College London), Ivan Cheltsov (Edinburgh University), Kristin deVleming (University of Massachusetts Amherst), Angela Gibney (University of Pennsylvania), Shihoko Ishii (Tokyo University), Yujiro Kawamata (Tokyo University), János Kollár (Princeton University), Lena Ji (University of Michigan), Jihao Liu (Northwestern University), Yuchen Liu (Northwestern University), James McKernan (UCSD), Joaquín Moraga (Princeton University), Yusuke Nakamura (Tokyo University), Miles Reid (Warwick University), Jihun Park (Pohang University), Giulia Saccá (Columbia University), Vyacheslav Shokurov (Johns Hopkins University), Claire Voisin (Centre National de la Recherche Scientifique), and Ziquan Zhuang (MIT).

The conference focused on topics closely related to Shokurov's scientific contribution, including Fano varieties, the theory of complements, boundedness of varieties, birational classification of algebraic varieties in characteristic zero and $p > 0$, and K-stability. Despite difficulties caused by the coronavirus pandemic, many top national and international experts were able to participate.

The conference was preceded by a lecture series "Classification Theory of Algebraic Varieties" by Caucher Birkar (Cambridge University), which was delivered online on February 16, 18, and 19, 2021.

This volume is dedicated to V. V. Shokurov on the occasion of his 70th birthday.

1

Birational Geometry of Algebraic Varieties and Shokurov's Work

Chistopher Hacon

Vyacheslav Vladimirovich Shokurov attended the Faculty of Mechanics and Mathematics of Moscow State University between 1967 and 1972 and obtained his PhD from Moscow State University in 1975. His research advisor was Yuri I. Manin. After receiving his PhD, Shokurov started work at the Yaroslavl' Pedagogical Institute (1982–90); he visited the Institute for Advanced Studies in Princeton (1990–91) and has been at Johns Hopkins University since 1991.

Throughout his career, Shokurov has advised nine PhD students: Terutake Abe, Florin Ambro, Caucher Birkar, Ivan Cheltsov, Yifei Chen, Sung Rak Choi, Joseph Cutrone, Nicholas Marshburn, and Jihun Park.

1.1 Early Works

Among Shokurov's first results in birational geometry [29, 30] were positive answers to two important conjectures of Iskovskikh about Fano varieties. Recall that X is Fano if $-K_X$ is ample and l is a straight line if it is a rational curve such that $-K_X \cdot l = 1$.

Theorem 1.1.1 (General elephant conjecture) *If X is a non-singular Fano threefold, then a general divisor $S \in |-K_X|$ is a smooth K3 surface.*

Theorem 1.1.2 (Lines on Fano solids) *Every non-singular Fano three-fold X of index 1 contains a straight line except for $X \cong \mathbb{P}^1 \times \mathbb{P}^2$.*

While these results have a classical flavor, they are also closely related to important questions and techniques in the minimal model program to which Shokurov would later make many fundamental contributions.

C. H. was partially supported by the NSF research Grants Numbers DMS-1952522, and DMS-2301374.

In 1983, Shokurov published the paper "Prym varieties: theory and applications" [31]. In this paper he performs a detailed study of Beauville's generalized Prym varieties and classifies generalized Prym varieties that are isomorphic (as principally polarized varieties) to a product of Jacobians of smooth curves. One of the main applications of this result is a proof of Iskovskikh's celebrated rationality criterion for a standard conic bundle under the additional assumption that the base of the bundle is a minimal rational surface.

1.2 Early Contributions to the Minimal Model Program

The minimal model program for threefolds was established in the 1980s. This program aims to generalize to arbitrary dimension the classification of surfaces obtained by the Italian school of algebraic geometry in the early 1900s. The first breakthroughs were obtained by S. Mori, who introduced the cone theorem and proved the existence of flips for terminal threefolds [23, 24]. Y. Kawamata then proved the contraction of negative extremal rays for terminal threefolds [16], and the cone theorem for klt pairs in any dimension [17]. This latest result was in part inspired by Shokurov's preprint on nonvanishing theorems. Shokurov's non-vanishing theorem appears in [32].

Theorem 1.2.1 (Non-vanishing theorem) *Let X be a smooth variety, A a $\mathbb{Q}$-divisor with simple normal crossings such that $\lceil A \rceil \geq 0$, D a nef Cartier divisor such that $aD+A-K_X$ is big for some $a \in \mathbb{Q}^{>0}$, then $H^0(X, \mathcal{O}_X(bD+\lceil A \rceil)) > 0$ for all $b \gg 0$.*

This is one of Shokurov's best-known results and it is absolutely essential in the higher dimensional minimal model program as it is a key ingredient in the proof of the cone theorem, and the existence of divisorial and flipping contractions, and has a slew of other important applications. For example, it implies that the index of the canonical divisor of a Fano variety is at most $\dim X + 1$ [32].

In 1992, Shokurov published his most-cited paper [33], where he proves the existence of three-dimensional log flips for klt pairs (X, B) (see also [20], a Summer seminar at the University of Utah (1991) where Shokurov's ideas are discussed in detail. See Chapters 16–22). This paper is a tour de force that not only completes one of the last steps of the threefold minimal model program but also introduces many new techniques as well as important results such as Shokurov's connectedness theorem, special termination, n-complements, limiting flips, (sub-)adjunction to divisors, and results on the inversion of adjunction

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that hold in arbitrary dimension. The termination of three-dimensional log flips was proven by Kawamata [18].

This completed the main objectives of the three-dimensional minimal model program and established a clear program in higher dimensions (cf. [28]). The main obstructions to completing this program were the existence and termination of flips and the abundance conjecture.

Conjecture 1.2.2 (Existence of flips) *Let (X,B) be a klt (or log canonical) pair and $f\colon X \to Z$ be a flipping contraction, then the flip $f^+\colon X^+ \to Z$ exists.*

Recall that a flipping contraction $f\colon X \to Z$ is a small birational morphism (i.e., an isomorphism in codimension 1) such that X is $\mathbb{Q}$-factorial, the relative Picard number is $\rho(X/Z) = 1$, and $-(K_X + B)$ is ample over Z. The corresponding flip $f^+\colon X^+ \to Z$ is a small birational morphism such that $\rho(X^+/Z) = 1$, and $K_{X^+} + B^+$ is ample over Z. The flip exists if and only if the relative canonical ring

$$R(K_X + B/Z) := \oplus_{m\geq 0} f_*\mathcal{O}_X(m(K_X + B))$$

is a finitely generated sheaf of $\mathcal{O}_Z$-algebras. In this case, $X^+ = \mathrm{Proj}R(K_X + B/Z)$.

Conjecture 1.2.3 (Termination of flips) *There does not exist an infinite sequence of flips $(X_1,B_1) \dashrightarrow (X_2,B_2) \dashrightarrow (X_3,B_3) \dashrightarrow \ldots$, where each $\phi_i\colon (X_i,B_i) \dashrightarrow (X_{i+1},B_{i+1} = \phi_{i*}B_i)$ is a flip of log canonical pairs.*

If Conjectures 1.2.2 and 1.2.3 hold, then for any lc pair (X,B), there is a sequence of flips and divisorial contractions $(X,B) = (X_1,B_1) \dashrightarrow \ldots \dashrightarrow (X_n,B_n) =: (X',B')$ such that

(1) if K_X+B is not pseudo-effective, then there is a Mori fiber space $(X',B') \to Z$ such that $\dim(X') > \dim(Z)$, $\rho(X'/Z) = 1$, and $-(K_{X'} + B')$ is ample over Z, and
(2) if $K_X + B$ is pseudo-effective, then (X',B') is a minimal model, that is, $K_{X'} + B'$ is nef.

In case (2), if $K_{X'} + B'$ is also big, then Theorem 1.2.1 implies that $K_{X'} + B'$ is semiample. The following conjecture (which holds in dimension 3 by [19]) predicts that this always holds (even if $K_{X'} + B'$ is not big).

Conjecture 1.2.4 (Abundance conjecture) *If* (X,B) *is a log canonical pair such that* K_X+B *is nef, then* K_X+B *is semiample so that* $|m(K_X+B)|$ *is base point free for* $m>0$ *sufficiently divisible.*

Shokurov introduced many new ideas toward an inductive proof in all dimensions for the existence and termination of flips. In this approach, log canonical (or klt) pairs with $\mathbb{R}$-coefficients play a fundamental role (as opposed to varieties with terminal singularities and no boundary divisors). Many of these ideas went on to be the backbone of recent celebrated progress in the minimal model program. The emphasis on log pairs is essential for proofs by induction; so much so that proving results for log pairs in dimension d is sometimes thought of being equivalent in difficulty and importance to proving the same result for varieties in dimension $d+\frac{1}{2}$.

1.3 Existence of Higher Dimensional Flips

The first substantial progress toward the higher dimensional minimal model program ($\dim X \geq 4$) was achieved in 2001, when Shokurov introduced a program for proving the existence of flips in any dimension, and he announced the proof of the existence of flips in dimension 4. This spurred an intense amount of activity, including two special programs: one at the Steklov Mathematical Institute of the Russian Academy of Sciences (in December 2001) and one at the Isaac Newton Institute in Cambridge (in 2002). Both of these programs produced volumes, including a proof of the existence of four-fold flips; the first one (published in Proc. Steklov Inst. Math. in 2003) followed Shokurov's ideas closely and included [36], while the second one, edited by A. Corti [7], includes a more substantial reworking of Shokurov's ideas and features the paper [10], which shows that the minimal model program for d-dimensional klt pairs implies the existence of flips for klt pairs in dimension $d+1$. This result would be the key new ingredient in the groundbreaking papers [5, 11], where the induction is completed and it is shown that klt flips exist in all dimensions and moreover minimal models exist for any klt pair (X,B) such that K_X+B is pseudo-effective and either B is big or K_X+B is big, and that Mori fiber spaces exist for any klt pair (X,B) such that K_X+B is not pseudo-effective. In particular, this implies the finite generation of the canonical ring

$$R(K_X+B)=\oplus_{m\geq 0}H^0(X,\mathcal{O}_X(m(K_X+B)))$$

for any klt pair (X,B), a result which was also proven independently by Y.-T. Siu [41].

It should be emphasized that many of the ideas in [5, 11] heavily rely on previous work of Shokurov (especially [33]), and Siu's work on deformation invariance of plurigenera ([40]).

(1) Following ideas of Shokurov (see also [7, section 4]), the existence of flips is reduced to the existence of pl-flips. In this case, we have a plt pair $(X, S+B)$ and a flipping contraction $f\colon X \to Z$. We may assume that Z is affine. By adjunction, $K_S + B_S := (K_X + S + B)|_S$ is klt and

$$R_S(X, S+B) = \operatorname{Im}(R(K_X + S + B) \to R(K_S + B_S))$$

is the restricted algebra. Shokurov shows that $R(K_X+S+B)$ is finitely generated if and only if the restricted algebra $R_S(X, S+B)$ is finitely generated (see also [7, §2]).

(2) Using results of Siu on the extension of pluricanonical forms, it is shown in [11] that the restricted algebra $R_S(X, S+B)$ is isomorphic to the log canonical ring $R(K_S+\Theta)$ of a klt pair (S, Θ), where Θ is an $\mathbb{R}$-divisor such that $0 \leq \Theta \leq B_S$ (and in particular (S, Θ) is klt).

(3) It is then necessary to deduce that Θ is a $\mathbb{Q}$-divisor and hence $R(K_S + \Theta)$ is finitely generated by induction on the dimension. This is achieved using ideas inspired by Shokurov's use of diophantine approximation arguments (see also [7, §2]) and by the theory of Shokurov polytopes [15, §3].

(4) Finally, there is the issue of termination of relevant flips. This step makes use of Shokurov's special termination arguments as well as Shokurov polytopes.

Finally, it should be noted that some key arguments of [11] have their origin in [9], which in turn is motivated by a conjecture of Shokurov [35].

1.4 Termination of Flips

Termination of three-fold terminal flips was proved by Shokurov in [32], where the notion of difficulty is introduced. After the proof of termination of three-fold flips for log canonical pairs (which follows from [18] in the klt case, and in the general lc case by applying Shokurov's special termination arguments), it is natural to wonder if termination of flips holds for four-folds and indeed for higher dimensional varieties. The traditional approach is to seek an invariant (often called the difficulty) that improves (strictly increases) after each flip and then show that the possible values of this invariant satisfy the ascending chain condition or ACC so that there can be no infinite increasing sequence of these invariants, and hence the sequence of flips must terminate. Versions of this approach were successful for canonical four-folds [8] and pseudo-effective (or anti-effective) log canonical four-folds [1, 22, 38]. In particular, [38] shows the termination of ordered flips (flips with scaling) for pseudo-effective log canonical four-folds (note that ordered termination is known in any dimension for klt pairs such that $K_X + B$ or B is big [5]; however, the non-big case is quite subtle even in dimension 4 and currently out of reach in dimension ≥ 5).

Another inspiring idea of Shokurov is the idea that two fundamental conjectures on minimal log discrepancies (namely, Shokurov's conjecture on the acc for mld's and Ambro's conjecture on the semicontinuity of mld's; see Sections 2.4.1 and 2.4.2) imply the termination of flips [37]. This is a beautiful argument that reduces a global phenomenon to questions of a local (or even formal) nature. Even though both conjectures have proven to be extremely difficult (they are only understood in very low dimension or in special cases such as for toric varieties), they have inspired a remarkable body of work and are currently active areas of research.

In more precise terms, recall that a set $I \subset \mathbb{R}$ satisfies the ascending chain condition or acc (resp. the descending chain condition or dcc) if every non-decreasing (resp. non-increasing) sequence of elements of I is eventually constant. Let (X,B) be a log pair, so that X is normal, $B = \sum b_i B_i$ is an effective $\mathbb{R}$-divisor, and $K_X + B$ is $\mathbb{R}$-Cartier. For any resolution $f\colon X' \to X$ we define $K_{X'} + B' = f^*(K_X + B)$, and for any divisor E on X' we let $a(X,B;E) = 1 - \mathrm{mult}_E(B')$ be the log discrepancy of (X,B) along E. The minimal log dicrepancy $\mathrm{mld}(X)$ is the infimum of the log discrepancies $a(X,B;E)$, where E runs over all divisors over X, and for any closed point $x \in X$ we let $\mathrm{mld}(X,B;x)$ be the minimal log discrepancy at x, which is the infimum of the log discrepancies $a(X,B;E)$, where E runs over all divisors over X with center x.

Conjecture 1.4.1 (Acc for minimal log discrepancies [38]) *Fix $d \in \mathbb{N}$ and a dcc set $I \subset [0,1]$, then the set of all minimal log discrepancies*

$$\{\mathrm{mld}(X,B) \mid \dim X = d,\ (X,B) \text{ a pair s.t. } \mathrm{coeff}(B) \in I\}$$

satisfies the acc.

Conjecture 1.4.2 (Semicontinuity for mld's [2, Conjecture 2.4]) *Let (X,B) be a log pair, then the function $x \to \mathrm{mld}(X,B;x)$ is lower semicontinuous on closed points $x \in X$.*

This conjecture reflects the fact that points with bigger log discrepancies are expected to be less singular and we expect the singularities to be worse

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at special points. Therefore, the main result of [38] can be rephrased as follows.

Theorem 1.4.3 *Conjectures 1.4.1 and 1.4.2 imply Conjecture 1.2.3.*

In [6], a closely related result is proven, namely that termination of flips (Conjecture 1.2.3) for $d+1$-dimensional varieties follows from the log minimal model program, the ascending chain condition for minimal log discrepancies, and the BAB (or Borisov–Alekseev–Borisov) conjecture for varieties in dimension d. Notice that much of the log minimal model program is known to hold by [5] and the BAB conjecture is known by [4].

Another closely related conjecture of Shokurov, which has played an important role in higher dimensional birational geometry, is the acc for log canonical thresholds. If (X, B) is a log canonical pair and $M \geq 0$ is an $\mathbb{R}$-Cartier divisor, then the log canonical threshold (or lct) of M with respect to (X, B) is

$$\mathrm{lct}(X, B; M) := \sup\{t \in \mathbb{R} | (X, B + tM) \text{ is log canonical}\}.$$

Let $I \subset [0, 1]$ be a dcc set and J a dcc set or positive real numbers, then

$$\mathrm{LCT}_d(I, J) = \{\mathrm{lct}(X, B; M) | (X, B) \text{ is lc},\ \dim X = d,\ B \in I,\ M \in J\},$$

where $B \in I$, and $M \in J$ means that the coefficients of B are in I and the coefficients of M are in J. The following result conjectured in [33] is known as Skokhurov's acc for lcts conjecture.

Theorem 1.4.4 ([12]) *Fix $d = \dim X$ and dcc sets $I \subset [0, 1]$ and $J \subset \mathbb{R}^{>0}$. Then the set $\mathrm{LCT}_d(I, J)$ satisfies the acc.*

Birkar and Shokurov show that a stronger variant of this conjecture (for a-lc thresholds) is implied by the acc for mld's [6] and building on this result J. Liu has shown that for $a \in [0, 1)$ the acc for mld's is in fact equivalent to the acc for a-lc thresholds [21]. For other closely related questions we refer the reader to [20, 18.14].

1.5 Complements

Another important theory developed by Shokurov is the theory of complements which addresses the problem of finding a well-behaved divisor in the anticanonical (or in a multiple of the anticanonical) linear system. This theory made one of its first appearances in Iskovskikh's work on Fano three-folds [14] (as already mentioned (Theorem 1.1.1), the main question of [14], related to the smoothness of a general anticanonical divisor, is answered by Shokurov in

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[29]). These ideas were further refined and successfully applied by Kawamata, Mori, Shokurov, and others. Shokurov used it in his proof of the existence of three-fold flips [33]. The case of surfaces is worked out in detail in [34] (see also [25]). If (X, B) is a log Fano pair so that (X, B) has klt (lc or ϵ-lc) singularities and $-(K_X + B)$ is ample, then one knows that there is a divisor $\Delta \sim_{\mathbb{R}} -(K_X + B)$ such that $K_X + B + \Delta$ is also klt (lc or ϵ-lc); however, it is often important to "control the coefficients" of the divisor in the new pair $(X, B + \Delta)$. To this end, Shokurov introduces the following precise definition (see [33] and [25, 4.1.3]).

Definition 1.5.1 Let $(X, S+B)$ be a subpair such that B and S have no common components, S is a reduced divisor, and $\lfloor B \rfloor \leq 0$. Then an n-complement is given by a $\mathbb{Q}$-divisor D^+ such that

(1) $n(K_X + D^+) \sim 0$ (and in particular, nD^+ is integral divisor);
(2) $K_X + D^+$ is lc;
(3) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$.

The difficulty here is of course to control the integer n and the singularities of (X, D^+) (e.g., if $(X, S+B)$ is ϵ-klt, then also (X, D^+) should be ϵ-klt). Shokurov conjectures that bounded complements exist (meaning that one can control the index n as well as the singularities of (X, D^+) in terms of the coefficients of B and the singularities of $(X, S+B)$). Important results in this direction were then proved by Shokurov and Prokhorov in [26, 27]. Later on, in his breakthrough work on the boundedness of Fano varieties ([3, 4]), Birkar proved an important case of Shokurov's conjecture on bounded complements.

Theorem 1.5.2 (Boundedness of complements [3, Theorem 1.7]) *Fix $d \in \mathbb{N}$ and $I \subset [0, 1] \cap \mathbb{Q}$, a finite set, then there exists $n \in \mathbb{N}$ such that if (X, B) is a projective d-dimensional lc pair of Fano type,* coeff$(B) \in I$ *and $-(K_X + B)$ is nef, then there is an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.*

There remain many interesting open questions about complements and there is active ongoing research in this area, including recent results by Han-Liu-Shokurov [13] and Shokurov [39].

In conclusion, it is a pleasure to congratulate Prof. Vyacheslav Vladimirovich Shokurov on a long and inspiring career. His many results and conjectures have been, and continue to be, a true inspiration to many of us working in birational algebraic geometry.

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