

## SOME INTEGRAL INEQUALITIES WITH BOUNDS FOR MOMENTS OF DISTRIBUTION II

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### Abstract

Several generalizations are given of the Gauss-Winckler inequality for the moments of a probability distribution.

### 1. Introduction

Bounds on the moments of probability distributions are important in the area of stochastic analysis generally and also in operations research. In this article we extend several known results.

To fix our notation, let  $Q : [0, \alpha] \rightarrow [0, 1]$  ( $\alpha \in (0, \infty]$ ) be a nondecreasing function with  $Q(0) = 0$  and  $Q(\alpha) = 1$ , that is,  $Q$  is a probability distribution function with support  $[0, \alpha]$ . The  $r$ -th moment  $\nu_r$  of  $Q$  is defined by

$$\nu_r = \int_0^\alpha x^r dQ(x).$$

This is naturally defined for  $r \geq 0$ , though we shall find that a number of results actually hold for  $r > -1$ . In the remainder of this paper we assume that  $Q$  has the above-mentioned properties.

In recent years a rich variety of relations has been uncovered between the moments of distributions. The roots of this go back over a century. Thus, under the additional constraint of a nonincreasing density function, the Gauss-Winckler inequality says that

$$((r + 1)\nu_r)^{1/r} \leq ((s + 1)\nu_s)^{1/s} \quad \text{for } r < s.$$

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When  $r = 1$  and  $s = 2$ , this gives

$$v_2 \geq (4/3)v_1^2,$$

which is a surprising and useful strengthening of the familiar  $v_2 \geq v_1^2$ .

Slightly more involved is the following theorem of the authors [6, 9].

**THEOREM A.** 1° *Let  $\alpha = 1$  and suppose  $Q$  has a derivative of order  $(n + 1)$  such that  $(-1)^{n+1} Q^{(n+1)}$  is nonnegative.*

(a) *For  $n \in \{1, 2\}$  the function  $f_n$  defined by*

$$f_n(r) = \ln \left[ \binom{r+n}{n} v_r \right]$$

*is concave on  $(-1, \infty)$ .*

(b) *If  $n \geq 3$  and  $(-1)^{n-k-1} Q^{(n-k)}(1) \geq 0$  for  $k = 2, \dots, n - 1$ , then  $f_n$  is concave also.*

2° *If  $\alpha \in \mathbf{R}$  and if  $Q'$  is nondecreasing, then  $f_1$  is concave.*

For  $n = 1$ , part (a) says that if a distribution on a bounded interval (which we may take as  $[0, 1]$ ) has a monotone nondecreasing density function, then  $(r + 1)v_r$  is logconcave on  $(-1, \infty)$ . The same conclusion holds (for  $n = 2$ ) if the distribution has a concave density function.

Similarly we derived the following result.

**THEOREM B.** (a) *If  $\alpha = \infty$  and  $(-1)^{n-1} Q^{(n)}$  is positive, continuous and nonincreasing for  $n = 1, 2, \dots, N$ , then  $f_n$  ( $n = 1, 2, \dots, N$ ) is convex.*

(b) *If  $\alpha \in \mathbf{R}$  and  $Q'$  is nonincreasing, then  $f_1$  is convex.*

As a consequence of the concavity of  $f_n$  in Theorem A, we have for  $r \geq 0$  and  $r_1, \dots, r_m \geq 0$  that

$$\left[ \binom{r+n}{n} v_r \right]^{m-1} \binom{r_1 + \dots + r_m + r + n}{n} v_{r_1 + \dots + r_m + r} \leq \prod_{i=1}^m \binom{r_i + r + n}{n} v_{r_i + r}. \tag{1}$$

When  $f_n$  is convex, the reverse of (1) holds. For  $n = 1$ , the reverse of (1) was proved by Pečarić [5] using Chebyshev’s inequality. It is an extension of Stolarsky’s inequality ([7], see also [6, 4] and Section 4).

These results are suggestive of much more. We explore some of the possibilities in this paper.

In Section 2 we give a weighted version of (1) and connect it with the Petrović inequality.

In Section 3 we use a similar idea to derive generalizations of a result related to the Gauss-Winckler inequality. These involve integrals  $\int_0^1 f'(x) dx$ , where  $f(Q(x))/x$  is nondecreasing. In Section 4 we take up the common case of an exponential distribution, standardised to unit mean, to derive new inequalities for the gamma function. A simple choice of distribution on a finite interval similarly gives results for the beta function. These are, of course, intimately associated with the gamma and beta distributions. Finally, in Section 5, we derive an improvement of Stolarsky's inequality.

### 2. Weighted inequality

**THEOREM 1.** *Let  $f_n$  be a concave function (that is, the conclusions of Theorem A hold). If  $r, p_i, r_i$  ( $i = 1, \dots, m$ ) are real numbers such that  $\sum_{i=1}^m p_i r_i + r, r_j + r$  ( $j = 1, \dots, m$ ) and  $r$  are greater than  $-1, p_i \geq 0$  ( $i = 1, \dots, m$ ) and*

$$r_j \cdot \left( \sum_{i=1}^m p_i r_i - r_j \right) \geq 0, \quad j = 1, \dots, m, \tag{2}$$

then

$$\left[ \binom{r+n}{n} v_r \right]^{P_m-1} \binom{\sum_{i=1}^m p_i r_i + r + n}{n} v_{\sum_{i=1}^m p_i r_i + r} \leq \prod_{j=1}^m \left[ \binom{r_j + r + n}{n} v_{r_j + r} \right]^{p_j}, \tag{3}$$

where  $P_m = \sum_{i=1}^m p_i$ .

If  $r_j \cdot (\sum_{i=1}^m p_i r_i - r_j) \leq 0$  ( $j = 1, \dots, m$ ), then the inequality is reversed.

**PROOF.** Suppose (2) holds. Since  $f_n$  is concave, for any  $p, q, s > -1, p \geq q, p \neq s, q \neq s$ , we have

$$\frac{f_n(p) - f_n(s)}{p - s} \leq \frac{f_n(q) - f_n(s)}{q - s}$$

(see [3, p. 2]).

Thus for  $p, q, s > -1, p \geq q, p \neq s, q \neq s$ , we get

$$\left( \frac{\binom{p+n}{n} v_p}{\binom{s+n}{n} v_s} \right)^{1/(p-s)} \leq \left( \frac{\binom{q+n}{n} v_q}{\binom{s+n}{n} v_s} \right)^{1/(q-s)}. \tag{4}$$

If  $r_j > 0$  then  $\sum_{i=1}^m p_i r_i > r_j$  and setting  $p = \sum_{i=1}^m p_i r_i + r, q = r_j + r$  and  $s = r$  in

(4) we obtain

$$\left( \frac{\left( \sum_{i=1}^m p_i r_i + r + n \right) v_{\sum_{i=1}^m p_i r_i + r}}{\binom{r+n}{n} v_r} \right)^{1/\sum_{i=1}^m p_i r_i} \leq \left( \frac{\binom{r_j+r+n}{n} v_{r_j+r}}{\binom{r+n}{n} v_r} \right)^{1/r_j}$$

$$\left( \frac{\left( \sum_{i=1}^m p_i r_i + r + n \right) v_{\sum_{i=1}^m p_i r_i + r}}{\binom{r+n}{n} v_r} \right)^{p_j r_j / \sum_{i=1}^m p_i r_i} \leq \left( \frac{\binom{r_j+r+n}{n} v_{r_j+r}}{\binom{r+n}{n} v_r} \right)^{p_j} \tag{5}$$

If  $r_j < 0$  then  $\sum_{i=1}^m p_i r_i < r_j$  and setting  $p = r_j + r$ ,  $q = \sum_{i=1}^m p_i r_i + r$  and  $s = r$  in (4) we obtain (5).

By multiplication of these inequalities for  $j = 1, \dots, m$  we get (3). The other cases can be proved analogously.

REMARK 1. If  $p_i = 1, r_i \geq 0$  ( $i = 1, \dots, m$ ) then (3) reduces to (1).

REMARK 2. The Petrović inequality (see [3, p. 11]) states the following.

Suppose  $f$  is convex on  $I = [a, b]$  ( $0 \in I$ ) and  $r_i, p_i$  ( $i = 1, \dots, m$ ) are real numbers such that  $r_i \sum_{i=1}^m p_i r_i \in I$  ( $p_i \geq 0$ ) for  $i = 1, \dots, m$ . If (2) holds, then

$$f \left( \sum_{i=1}^m p_i r_i \right) \geq \sum_{i=1}^m p_i f(r_i) + (1 - P_m) f(0).$$

If  $f$  is concave, the inequality is reversed.

In our case  $f_n$  is concave and  $f_n(0) = \ln v_0 = 0$ , so the inequality becomes

$$\sum_{i=1}^m \binom{p_i r_i + n}{n} v_{\sum_{i=1}^m p_i r_i + n} \geq \prod_{i=1}^m \left( \binom{r_i + n}{n} v_{r_i} \right)^{p_i} \tag{6}$$

Also putting  $r = 0$  in (3) gives (6).

REMARK 3. When  $f_n$  is convex, that is, when the conclusion of Theorem B holds, then the inequality is reversed.

### 3. Some further results

The following theorem was proved in [2].

THEOREM 2. Let  $f : [0, 1] \rightarrow \mathbf{R}$  be nondecreasing and positive. If  $f(Q(x))/x$  is nondecreasing, then for  $m \leq r$

$$\frac{v_r^{1/r}}{v_m^{1/m}} \leq \frac{\left( \int_0^1 f^r(x) dx \right)^{1/r}}{\left( \int_0^1 f^m(x) dx \right)^{1/m}}.$$

If  $f(Q(x))/x$  is nonincreasing, then the inequality is reversed.

Using the idea in the proof of Theorem 1, we obtain the following.

**THEOREM 3.** Let  $f : [0, 1] \rightarrow \mathbf{R}$  be nondecreasing and positive. Let  $p_i$  ( $i = 1, \dots, m$ ) be positive real numbers and  $r_i$  ( $i = 1, \dots, m$ ) real numbers such that  $r_i$  ( $i = 1, \dots, m$ ) and  $\sum_{i=1}^m p_i r_i$  are greater than  $-1$ . If  $f(Q(x))/x$  is nondecreasing and if  $r_j(\sum_{i=1}^m p_i r_i - r_j) \geq 0$  ( $j = 1, \dots, m$ ), then

$$\frac{v_{\sum_{i=1}^m p_i r_i}}{\prod_{i=1}^m v_{r_i}^{p_i}} \leq \frac{\int_0^1 f^{\sum_{i=1}^m p_i r_i}(x) dx}{\prod_{i=1}^m \left( \int_0^1 f^{r_j}(x) dx \right)^{p_j}}. \quad (7)$$

If  $r_j(\sum_{i=1}^m p_i r_i - r_j) \leq 0$  ( $j = 1, \dots, m$ ), the inequality (7) is reversed.

If  $f(Q(x))/x$  is nonincreasing, the above results are reversed.

Let us mention some special cases when (2) holds.

1° For  $f \equiv 1$  we have  $v_{p_1 r_1 + \dots + p_m r_m} \geq v_{r_1}^{p_1} \dots v_{r_m}^{p_m}$ .

2° For  $f(x) = x$  and  $Q(x)/x$  nondecreasing, we have

$$\left( \sum_{i=1}^m p_i r_i + 1 \right) v_{\sum_{i=1}^m p_i r_i} \leq \prod_{j=1}^m (r_j + 1)^{p_j} v_{r_j}^{p_j},$$

while if  $Q(x)/x$  is nonincreasing, the opposite inequality holds.

3° For  $f(x) = x^{1/k}$  ( $k > 0$ ) and  $Q(x)/x^k$  nondecreasing, we have

$$\frac{p_1 r_1 + \dots + p_m r_m + k}{k} v_{p_1 r_1 + \dots + p_m r_m} \leq \prod_{j=1}^m \left( \frac{r_j + k}{k} \right)^{p_j} v_{r_j}^{p_j}.$$

When  $Q(x)/x^k$  is nonincreasing the opposite inequality holds. The case  $p_1 = p_2 = \dots = p_m = 1$  can be found in [6].

#### 4. The gamma and beta functions

If  $\alpha = \infty$  and  $Q(x) = 1 - e^{-x}$ , then  $Q$  satisfies the assumptions of Theorem B(a) for any  $N \in \mathbf{N}$ , so  $f_n$  is convex for any  $n \in \mathbf{N}$ . In this case  $v_r = \Gamma(r + 1)$  and we derive the following inequalities for the gamma function.

**COROLLARY 1.** (a) Suppose  $p_i$  ( $i = 1, \dots, m$ ) are positive real numbers and  $r, r_i$  ( $i = 1, \dots, m$ ) are real numbers such that  $r > -1$ ,  $\sum_{i=1}^m p_i r_i + r > -1$  and

$r_j + r > -1$  for  $j = 1, \dots, m$ . If (2) holds, then

$$\begin{aligned} & \left( \binom{r+n}{n} \Gamma(r+1) \right)^{P_m-1} \binom{\sum_{i=1}^m p_i r_i + r + n}{n} \Gamma \left( \sum_{i=1}^m p_i r_i + r + 1 \right) \\ & \geq \prod_{j=1}^m \binom{r_j + r + n}{n}^{P_j} \Gamma^{P_j}(r_j + r + 1). \end{aligned} \quad (8)$$

(b) If  $p_i$  ( $i = 1, \dots, m$ ) are positive real numbers and  $r_i$  ( $i = 1, \dots, m$ ) real numbers such that  $r_i$  ( $i = 1, \dots, m$ ) and  $\sum_{i=1}^m p_i r_i$  are greater than  $-1$ , then

$$\Gamma \left( \sum_{i=1}^m p_i r_i + 1 \right) \geq \prod_{j=1}^m \Gamma^{P_j}(r_j + 1) \quad (9)$$

and

$$\frac{\sum_{i=1}^m p_i r_i + k}{k} \Gamma \left( \sum_{i=1}^m p_i r_i + 1 \right) \geq \prod_{j=1}^m \left( \frac{r_j + k}{k} \right)^{P_j} \Gamma^{P_j}(r_j + 1) \quad (10)$$

for  $k \geq 1$ .

If inequality (2) is reversed, then so are (8)–(10).

If  $\alpha = 1$  and  $Q(x) = 1 - (1-x)^{a+1}$  ( $a > 0$ ), then  $Q$  satisfies the assumptions of Theorem B(b) and we derive inequalities for the beta function.

**COROLLARY 2.** Suppose  $p_i$  ( $i = 1, \dots, m$ ) are positive real numbers and  $r, r_i$  ( $i = 1, \dots, m$ ) are real numbers such that  $r > -1$ ,  $\sum_{i=1}^m p_i r_i + r > -1$  and  $r_j + r > -1$  for  $j = 1, \dots, m$ . If (2) holds, then

$$\begin{aligned} & ((r+1)B(r+1, a+1))^{P_m-1} \left( \sum_{i=1}^m p_i r_i + r + 1 \right) B \left( \sum_{i=1}^m p_i r_i + r + 1, a + 1 \right) \\ & \geq \prod_{j=1}^m (r_j + r + 1)^{P_j} B^{P_j}(r_j + r + 1, a + 1), \end{aligned}$$

where  $B(x+1, y+1) = \int_0^1 t^x (1-t)^y dt$ .

(b) If  $p_i$  ( $i = 1, \dots, m$ ) are positive real numbers and  $r_i$  ( $i = 1, \dots, m$ ) real numbers such that  $r_i$  ( $i = 1, \dots, m$ ) and  $\sum_{i=1}^m p_i r_i$  are greater than  $-1$ , then

$$B \left( \sum_{i=1}^m p_i r_i + 1, a + 1 \right) \geq (a+1)^{P_m-1} \prod_{j=1}^m B^{P_j}(r_j + 1, a + 1)$$

and

$$\left[ \sum_{i=1}^m p_i r_i + 1 \right] B \left( \sum_{i=1}^m p_i r_i + 1, a + 1 \right) \\ \geq (a + 1)^{p_m - 1} \prod_{j=1}^m (r_j + 1)^{p_j} B^{p_j} (r_j + 1, a + 1).$$

Reversal of the direction of inequality (2) reverses those of the above conclusions.

### 5. Integral inequalities involving derivatives

Stolarsky [7] has established the following theorem.

**THEOREM C.** *If  $f : [0, 1] \rightarrow [0, 1]$  is nonnegative and nonincreasing, then for positive  $a$  and  $b$*

$$(a + b) \int_0^1 x^{a+b-1} f(x) dx \geq ab \int_0^1 x^{a-1} f(x) dx \int_0^1 x^{b-1} f(x) dx. \quad (11)$$

In probabilistic terms, this gives the following. Suppose  $\int_0^1 f(x) dx = 1$ , that is,  $f$  is a probability density on  $[0, 1]$ . If this is nondecreasing, then

$$(a + b)v_{a+b-1} \geq abv_{a-1}v_{b-1}.$$

In the first section we mentioned that (1) is an extension of Stolarsky's result (11). In this section we give another improvement of (11).

**THEOREM 4.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be of bounded variation and  $g, h : [a, b] \rightarrow \mathbf{R}$  nonnegative and nondecreasing with continuous first derivative and  $g(a) = h(a) = 0$ . If  $0 \leq f(b) \leq f(x) \leq f(a)$  for all  $x \in [a, b]$ , then*

$$f(a) \int_a^b (g(t)h(t))' f(t) dt \geq \int_a^b g'(t)f(t) dt \int_a^b h'(t)f(t) dt. \quad (12)$$

*If  $0 \leq f(a) \leq f(x) \leq f(b)$ , the inequality is reversed.*

**PROOF.** If  $0 \leq f(b) \leq f(x) \leq f(a)$ , then we may without loss of generality assume that  $f(b) < f(a)$ , because if  $f(b) = f(a)$ , then  $f \equiv C$  and (12) holds. The following inequality of Chebyshev type, established in [1], will be used.

If  $f_1, f_2 : [a, b] \rightarrow \mathbf{R}$  are nondecreasing and continuous and  $f : [a, b] \rightarrow \mathbf{R}$  of bounded variation such that  $f(a) \leq f(x) \leq f(b)$  for all  $x \in [a, b]$ , then

$$\int_a^b df(x) \int_a^b f_1(x)f_2(x)df(x) \geq \int_a^b f_1(x)df(x) \int_a^b f_2(x)df(x). \tag{13}$$

Now integration by parts and (13) give

$$\begin{aligned} & \int_a^b (g(t)h(t))' f(t) dt \tag{14} \\ &= f(b)g(b)h(b) + \int_a^b g(t)h(t)d(-f)(t) \\ &\geq f(b)g(b)h(b) + [f(a) - f(b)]^{-1} \int_a^b g(t)d(-f)(t) \int_a^b h(t)d(-f)(t). \end{aligned}$$

From the identity

$$(a_1 - a_2)(b_1 - b_2)p_1p_2 = (p_1 + p_2)(a_1b_1p_1 + a_2b_2p_2) - (a_1p_1 + a_2p_2)(b_1p_1 + b_2p_2) \tag{15}$$

we have

$$(p_1 + p_2)(p_1a_1b_1 + p_2a_2b_2) \geq (p_1a_1 + p_2a_2)(p_1b_1 + p_2b_2) \tag{16}$$

for  $(a_1 \geq a_2, b_1 \geq b_2)$  or  $(a_1 \leq a_2, b_1 \leq b_2)$  and  $p_1, p_2 \geq 0$ . This is the discrete Chebyshev inequality.

We now set

$$\begin{aligned} p_1 &= f(b), & a_1 &= g(b), & b_1 &= h(b), & p_2 &= f(a) - f(b), \\ a_2 &= \frac{1}{f(a) - f(b)} \int_a^b g(t)d(-f)(t), & b_2 &= \frac{1}{f(a) - f(b)} \int_a^b h(t)d(-f)(t) \end{aligned}$$

in (15). To use (16), we need only check that  $a_1 \geq a_2$  and  $b_1 \geq b_2$ . We have

$$\begin{aligned} (a_1 - a_2)(f(a) - f(b)) &= g(b)(f(a) - f(b)) - g(t)(-f)(t) \Big|_a^b - \int_a^b f(t)g'(t) dt \\ &= f(a)g(b) - \int_a^b f(t)g'(t) dt = \int_a^b (f(a) - f(t))g'(t) dt. \end{aligned}$$

By assumption the last expression is nonnegative, so  $a_1 \geq a_2$ . The other statement is proved similarly.

We combine (14) and (16) to obtain

$$\begin{aligned} & \int_a^b (g(t)h(t))' f(t) dt \\ & \geq \frac{1}{f(a)} \left( f(b)g(b) - \int_a^b g(t)df(t) \right) \left( f(b)h(b) - \int_a^b h(t)df(t) \right) \\ & = \frac{1}{f(a)} \int_a^b g'(t)f(t) dt \int_a^b h'(t)f(t) dt, \end{aligned}$$

which establishes the first part of the proof.

If  $0 \leq f(a) \leq f(x) \leq f(b)$  for all  $x \in [a, b]$ , a similar proof applies, the only difference being that in place of (16) we use

$$(p_1 + p_2)(p_1a_1b_1 + p_2a_2b_2) \leq (p_1a_1 + p_2a_2)(p_1b_1 + p_2b_2)$$

for  $p_1 \geq 0, p_2 \leq 0$  and  $a_1 \geq a_2, b_1 \geq b_2$ . This is a consequence of (15).

REMARK 4. If  $0 \leq f(b) \leq f(x) \leq f(a)$  for all  $x \in [a, b]$  and  $f, g, h$  satisfy the assumptions of Theorem 4, then the condition  $g(a) = h(a) = 0$  can be omitted. For define functions  $\bar{h}$  and  $\bar{g}$  by

$$\bar{h}(x) = h(x) - h(a) \quad \text{and} \quad \bar{g}(x) = g(x) - g(a).$$

Then  $\bar{h}(a) = \bar{g}(a) = 0$  and  $\bar{h}, \bar{g}$  satisfy the assumptions of Theorem 4, so

$$\begin{aligned} & f(a) \int_a^b (g(t)h(t))' f(t) dt \\ & = f(a) \int_a^b (\bar{g}(t)\bar{h}(t))' f(t) dt + \int_a^b (h(a)g'(t) + g(a)h'(t))f(t) dt \\ & \geq f(a) \int_a^b (\bar{g}(t)\bar{h}(t))' f(t) dt \geq f(a) \int_a^b g'(t)f(t) dt \int_a^b h'(t)f(t) dt. \end{aligned}$$

The result below now follows by an easy induction on  $n$ .

COROLLARY 3. Let  $f : [a, b] \rightarrow \mathbf{R}$  be of bounded variation and  $x_i : [a, b] \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) nonnegative, nondecreasing functions with continuous first derivative and  $x_i(a) = 0$  ( $i = 1, \dots, n$ ).

If  $0 \leq f(b) \leq f(x) \leq f(a)$  for all  $x \in [a, b]$ , then

$$(f(a))^{n-1} \int_a^b \left( \prod_{i=1}^n x_i(t) \right)' f(t) dt \geq \prod_{i=1}^n \int_a^b x_i'(t)f(t) dt.$$

If  $0 \leq f(a) \leq f(x) \leq f(b)$  for all  $x \in [a, b]$ , the inequality is reversed.

REMARK 5. Setting  $a = 0$ ,  $b = 1$ ,  $g(x) = x^a$ ,  $h(x) = x^b$ ,  $f(0) = 1$  in (4) gives Stolarsky's inequality, but with a weaker condition on  $f$ . In Theorem C  $f$  is monotonic, but in Theorem 4  $f$  is of bounded variation and  $f(x) \in [f(b), f(a)]$ .

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