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ONE-POINT CONNECTIFICATIONS

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Abstract

A space Y is called an *extension* of a space X if Y contains X as a dense subspace. An extension Y of X is called a *one-point extension* if $Y \backslash X$ is a singleton. Compact extensions are called *compactifications* and connected extensions are called *connectifications*. It is well known that every locally compact noncompact space has a one-point compactification (known as the *Alexandroff compactification*) obtained by adding a point at infinity. A locally connected disconnected space, however, may fail to have a one-point connectification. It is indeed a long-standing question of Alexandroff to characterize spaces which have a one-point connectification. Here we prove that in the class of completely regular spaces, a locally connected space has a one-point connectification if and only if it contains no compact component.

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1. Introduction

Throughout this article by *completely regular* we mean completely regular and Hausdorff (also referred to as Tychonoff).

A space Y is called an *extension* of a space X if Y contains X as a dense subspace. An extension Y of X is called a *one-point extension* if $Y \backslash X$ is a singleton. Compact extensions are called *compactifications* and connected extensions are called *connectifications*.

It is well known that every locally compact noncompact space has a one-point compactification, known as the *Alexandroff compactification* (see [3]). A locally connected disconnected space, however, may fail to have a one-point connectification; trivially, any space with a compact open subspace has no Hausdorff connectification. (The lack of compact open subspaces, however, does not guarantee the existence of a connectification; see [2, 7, 18, 21].) There is indeed an old question of Alexandroff of characterizing spaces which have a one-point connectification.

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This so far has motivated a significant amount of research. The earliest serious work in this direction dates back perhaps to 1945 and is due to Knaster [8]; it presents a characterization of separable metrizable spaces which have a separable metrizable one-point connectification. Knaster's characterization is as follows.

Theorem 1.1 (Knaster [8]). *Let X be a separable metrizable space. Then X has a separable metrizable one-point connectification if and only if it can be embedded in a connected separable metrizable space as a proper open subspace.*

More recently, in [1], Abry *et al.* have given the following alternative characterization of separable metrizable spaces which have a separable metrizable one-point connectification.

Theorem 1.2 (Abry *et al.* [1]). *Let X be a separable metrizable space in which every component is open. Then X has a separable metrizable one-point connectification if and only if X has no compact component.*

Here we characterize locally connected completely regular spaces which have a completely regular one-point connectification. Our characterization resembles that of Abry *et al.* as stated in Theorem 1.2 and, as we will now explain, may be viewed as a dual to Alexandroff's characterization of locally compact Hausdorff spaces having a Hausdorff one-point compactification. Observe that locally compact Hausdorff spaces as well as compact Hausdorff spaces are completely regular. The Alexandroff theorem, now reworded, states that a locally compact completely regular space has a completely regular one-point compactification if and only if it is noncompact. Keeping analogy, we prove that a locally connected completely regular space has a completely regular one-point connectification if and only if it contains no compact component. Our method may also be used to give a description of all completely regular one-point connectifications of a locally connected completely regular space with no compact component. Further, for a locally connected completely regular space with no compact component, we give conditions on a topological property $\mathscr{P}$ which guarantee the space to have a completely regular one-point connectification with $\mathscr{P}$, provided that each component of the space has $\mathscr{P}$. This will conclude Section 2. In Section 3, we will be dealing with T_1-spaces. Results of this section are dual to those we proved in Section 2 rephrased in the context of T_1-spaces. In particular, we will prove that a locally connected T_1-space has a T_1 one-point connectification if it contains no compact component.

We will use some basic facts from the theory of the Stone–Čech compactification. Recall that the *Stone–Čech compactification* of a completely regular space X, denoted by βX, is the Hausdorff compactification of X which is characterized among all Hausdorff compactifications of X by the fact that every continuous mapping $f : X \to [0, 1]$ is continuously extendable over βX. The Stone–Čech compactification of a completely regular space always exists. We will use the following standard properties

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of βX. (By a *clopen* subspace we mean a simultaneously closed and open subspace.)

- A clopen subspace of X has open closure in βX.
- Disjoint zero-sets in X have disjoint closures in βX.
- $\beta T = \beta X$ whenever $X \subseteq T \subseteq \beta X$.

For more information on the subject and other background material, we refer the reader to the texts [5, 6, 17].

2. One-point connectifications of completely regular spaces

The following subspace of βX plays a crucial role throughout our whole discussion.

DEFINITION 2.1. Let X be a completely regular space. Define

$$\delta X = \bigcup\{\mathrm{cl}_{\beta X} C : C \text{ is a component of } X\},$$

considered as a subspace of βX.

Recall that a space X is called *locally connected* if for every x in X, every neighborhood of x in X contains a connected neighborhood of x in X. Every component of a locally connected space is open and thus is clopen, as components are always closed. Observe that any clopen subspace of a completely regular space X has open closure in βX. Therefore, in a locally connected completely regular space X each component of X has open closure in βX; in particular, δX is open in βX.

The following theorem characterizes locally connected completely regular spaces which have a completely regular one-point connectification.

THEOREM 2.2. *A locally connected completely regular space has a completely regular one-point connectification if and only if it contains no compact component.*

PROOF. Let X be a (nonempty) locally connected completely regular space.

Sufficiency. Suppose that X contains no compact component. We show that X has a completely regular one-point connectification. Let C be a component of X. Then $\mathrm{cl}_{\beta X} C \backslash X$ is nonempty, as C is noncompact. Choose an element t_C in $\mathrm{cl}_{\beta X} C \backslash X$. Let

$$P = \{t_C : C \text{ is a component of } X\} \cup (\beta X \backslash \delta X).$$

Note that P misses X, as $\beta X \backslash \delta X$ does so, since X is contained in δX trivially. Also, P is nonempty, as X is so. We show that P is closed in βX. Let t be in $\mathrm{cl}_{\beta X} P$. Obviously, t is contained in P if it is contained in $\beta X \backslash \delta X$. Let t be in δX. Then t is contained in $\mathrm{cl}_{\beta X} D$ for some component D of X. We show that t is identical to t_D. Suppose otherwise. Then

$$U = \mathrm{cl}_{\beta X} D \backslash \{t_D\}$$

is an open neighborhood of t in βX. (Observe that the closure in βX of D is open in βX, as D is a component of X and X is locally connected.) We show that U misses P.

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Let E be a component of X distinct from D. Then E is necessarily disjoint from D. This implies that E and D have disjoint closures in βX, as they are disjoint zero-sets (indeed, disjoint clopen subspaces) in X. Therefore, t_E is not in U, as it is contained in $\mathrm{cl}_{\beta X}E$. It is trivial that U misses $\beta X\backslash\delta X$. Thus, U misses P, which is a contradiction. This shows that P is closed in βX.

Let T be the space which is obtained from βX by contracting the compact subspace P of βX to a point p and let $\phi : \beta X \to T$ denote the corresponding quotient mapping. Consider the subspace $Y = X \cup \{p\}$ of T. Then Y is completely regular, as T is so, and contains X densely, as T does so. That is, Y is a completely regular one-point extension of X. We verify that Y is connected. Note that p is contained in $\mathrm{cl}_Y C$ for every component C of X, as

$$p = \phi(t_C) \in \phi(\mathrm{cl}_{\beta X}C) \subseteq \mathrm{cl}_T\phi(C) = \mathrm{cl}_T C.$$

Since $\mathrm{cl}_Y C$ is the closure of a connected space, it is connected for every component C of X. Therefore,

$$Y = \bigcup\{\mathrm{cl}_Y C : C \text{ is a component of } X\}$$

is connected, as it is the union of a collection of connected subspaces of Y with nonempty intersection.

Necessity. Suppose that X has a completely regular one-point connectification Y. We show that no component of X is compact. Suppose otherwise. Then X contains a compact component C. Trivially, C is closed in Y, as Y is Hausdorff. On the other hand, C is open in Y, as C is open in X, since X is locally connected, and X is open in Y. That is, C is clopen in Y. Since Y is connected, we then have $C = Y$, which is a contradiction. □

REMARK 2.3. Theorem 2.2 is valid if we replace local connectedness of X by the requirement that every component of X is open; this follows trivially by an inspection of the proof.

The method used in the proof of Theorem 2.2 can be modified to give a description of all completely regular one-point connectifications of a locally connected completely regular space X with no compact component; this will be the context of our next theorem.

The following lemma follows from a very standard argument; we therefore omit the proof.

LEMMA 2.4. *Let $Y = X \cup \{p\}$ be a completely regular one-point extension of a space X. Let $\phi : \beta X \to \beta Y$ be the continuous extension of the identity mapping on X. Then βY is the quotient space obtained from βX by contracting $\phi^{-1}(p)$ to p and ϕ is its quotient mapping.*

The following theorem describes, for a locally connected completely regular space which has no compact component, all its completely regular one-point connectifications.

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Theorem 2.5. *Let X be a locally connected completely regular space with no compact component. Let $Y = X \cup \{p\}$ be the quotient space obtained by contracting a nonempty compact subspace of $\beta X \backslash X$ which intersects the closure in βX of each component of X to the point p. Then Y is a completely regular one-point connectification of X. Further, any completely regular one-point connectification of X is obtained in this way.*

Proof. Let P be a nonempty compact subspace of $\beta X \backslash X$ which intersects the closure in βX of every component of X. Let T be the quotient space of βX which is obtained by contracting P to a point p. An argument similar to the one given in the proof of Theorem 2.2 shows that the subspace $Y = X \cup \{p\}$ of T is a completely regular one-point connectification of X.

To show the converse, let $Y = X \cup \{p\}$ be a completely regular one-point connectification of X. Let $\phi : \beta X \to \beta Y$ be the continuous extension of the identity mapping on X. It follows from Lemma 2.4 that βY is the quotient space obtained from βX by contracting $\phi^{-1}(p)$ to p and ϕ is its quotient mapping. We need to show that $\phi^{-1}(p)$ intersects the closure in βX of each component of X. Let C be a component of X. Suppose to the contrary that

$$\phi^{-1}(p) \cap \mathrm{cl}_{\beta X} C = \emptyset.$$

Then p is not contained in $\phi(\mathrm{cl}_{\beta X} C)$ and, since $\mathrm{cl}_{\beta Y} C \subseteq \phi(\mathrm{cl}_{\beta X} C)$, p is not contained in $\mathrm{cl}_Y C$ either. Therefore, C is closed in Y, as it is closed in X. But C is also open in Y, as it is open in X, since X is locally connected (and X is open in Y). This contradicts the connectedness of Y. □

In [10] (see also [9] and [11–15]), we have studied topological properties $\mathscr{P}$ such that any completely regular space which has $\mathscr{P}$ locally has a completely regular one-point extension which has $\mathscr{P}$. Motivated by this, we consider conditions on a topological property $\mathscr{P}$ which guarantee a locally connected completely regular space with no compact component to have a completely regular one-point connectification with $\mathscr{P}$, provided that all its components have $\mathscr{P}$.

We need the following definition.

Definition 2.6. Let $\mathscr{P}$ be a topological property. Then:

(1) $\mathscr{P}$ is *closed hereditary* if any closed subspace of a space having $\mathscr{P}$ has $\mathscr{P}$;
(2) $\mathscr{P}$ is *finitely additive* if any space which is expressible as a finite disjoint union of closed subspaces each having $\mathscr{P}$ has $\mathscr{P}$;
(3) $\mathscr{P}$ is *co-local* if a space X has $\mathscr{P}$ provided that it contains a point p with an open base $\mathscr{B}$ for X at p such that $X \backslash B$ has $\mathscr{P}$ for any B in $\mathscr{B}$.

Remark 2.7. The condition stated in (3) in Definition 2.6 has been introduced by Mrówka [16], where it was called *condition* (W).

Remark 2.8. Some authors call a topological property $\mathscr{P}$ *finitely additive* if any space which is a finite (and not necessarily disjoint) union of closed subspaces each having

$\mathscr{P}$ has $\mathscr{P}$. The reader is warned of the difference between this definition and the definition given in Definition 2.6.

THEOREM 2.9. *Let X be a locally connected completely regular space with no compact component. Let $\mathscr{P}$ be a closed hereditary finitely additive co-local topological property. If every component of X has $\mathscr{P}$ (in particular, if X has $\mathscr{P}$), then X has a completely regular one-point connectification with $\mathscr{P}$.*

PROOF. Note that if X has $\mathscr{P}$, then each of its components has $\mathscr{P}$, as $\mathscr{P}$ is closed hereditary. We may therefore prove the theorem in the case when every component of X has $\mathscr{P}$.

Let P, T, ϕ and Y be as defined in the proof of Theorem 2.2. Since $\mathscr{P}$ is co-local, to show that Y has $\mathscr{P}$ it suffices to show that $Y\backslash U$ has $\mathscr{P}$ for any open neighborhood U of p in Y. Let U be an open neighborhood of p in Y and let U' be an open subspace of T with $U = U' \cap Y$. Then

$$\beta X\backslash\delta X \subseteq \phi^{-1}(p) \subseteq \phi^{-1}(U'),$$

as p is contained in U', and thus

$$\beta X\backslash\phi^{-1}(U') \subseteq \delta X.$$

By compactness (and the definition of δX), it then follows that

$$\beta X\backslash\phi^{-1}(U') \subseteq \mathrm{cl}_{\beta X}C_1 \cup \cdots \cup \mathrm{cl}_{\beta X}C_n, \tag{2.1}$$

where C_i is a component of X for each $i = 1, \ldots, n$. Intersecting both sides of (2.1) with X gives

$$X\backslash U \subseteq C_1 \cup \cdots \cup C_n = D.$$

Note that D has $\mathscr{P}$, as it is a finite disjoint union of closed subspaces each with $\mathscr{P}$ and $\mathscr{P}$ is finitely additive. Thus,

$$Y\backslash U = X\backslash U$$

has $\mathscr{P}$, as it is closed in D and $\mathscr{P}$ is closed hereditary. □

REMARK 2.10. There is a long list of topological properties, mostly covering properties (topological properties described in terms of the existence of certain kinds of open subcovers or refinements of a given open cover of a certain type), satisfying the requirements of Theorem 2.9. Specifically, we mention the Lindelöf property, paracompactness, metacompactness, subparacompactness, the para-Lindelöf property, the σ-para-Lindelöf property, weak θ-refinability, θ-refinability (or submetacompactness), weak $\delta\theta$-refinability and $\delta\theta$-refinability (or the submeta-Lindelöf property). (See [10, Example 2.16] for the proof and [4, 19, 20] for the definitions.)

3. One-point connectifications of T_1-spaces

This section deals with one-point connectifications of T_1-spaces. The results of this section will be dual to those we have obtained in the previous section. We will make critical use of the Wallman compactification; this will replace the Stone–Čech compactification, as used in the previous section.

Recall that the *Wallman compactification* of a T_1-space X, denoted by wX, is the T_1 compactification of X with the property that every continuous mapping $f : X \to K$ of X to a compact Hausdorff space K is continuously extendable over wX. The Wallman compactification is the substitute of the Stone–Čech compactification which is defined for every T_1-space. The Wallman compactification of a T_1-space X is Hausdorff if and only if X is normal and, in this case, it coincides with the Stone–Čech compactification of X. The Wallman compactification has properties which are dual to those of the Stone–Čech compactification. In particular, a clopen subspace of a T_1-space X has open closure in wX, and disjoint zero-sets in X have disjoint closures in wX.

The next theorem is dual to Theorem 2.2.

Theorem 3.1. *A locally connected T_1-space has a T_1 one-point connectification if it contains no compact component.*

Proof. Let X be a (nonempty) locally connected T_1-space with no compact component. Let δX, t_C and P be as defined in (Definition 2.1 and) the proof of Theorem 2.2 with βX substituted by wX in their definitions. As argued in the proof of Theorem 2.2, it follows that P is a nonempty closed subspace of wX which misses X. Let T be the quotient space of wX which is obtained by contracting P to a point p. Then T is a T_1-space, as singletons are all closed in T. As argued in the proof of Theorem 2.2, the subspace $Y = X \cup \{p\}$ of T is a connected T_1 one-point extension of X. □

We do not know whether the converse of Theorem 3.1 holds true; we state this formally as an open question.

Question 3.2. For a locally connected T_1-space, does the existence of a T_1 one-point connectification imply the nonexistence of compact components?

The next two theorems are dual to Theorems 2.5 and 2.9, respectively. We omit the proofs, as they are analogous to the proofs we have already given for Theorems 2.5 and 2.9, respectively (with the use of Theorem 3.1 in place of that of Theorem 2.2).

Theorem 3.3. *Let X be a locally connected T_1-space with no compact component. Let $Y = X \cup \{p\}$ be the quotient space obtained by contracting a nonempty closed subspace of wX, which is contained in $wX \setminus X$ and intersects the closure in wX of each component of X, to the point p. Then Y is a T_1 one-point connectification of X.*

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QUESTION 3.4. For a locally connected T_1-space X with no compact component, does Theorem 3.3 give every T_1 one-point connectification of X?

THEOREM 3.5. *Let X be a locally connected T_1-space with no compact component. Let $\mathscr{P}$ be a closed hereditary finitely additive co-local topological property. If every component of X has $\mathscr{P}$ (in particular, if X has $\mathscr{P}$), then X has a T_1 one-point connectification with $\mathscr{P}$.*

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