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LIMITATION THEOREMS FOR SOME METHODS OF SUMMABILITY

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The object of this paper is to establish limitation theorems for the ordinary and also absolute generalized Nörlund methods which include some known results as special cases. We shall give a different proof of the recent result of S. Narang (*Proc. Indian Acad. Sci. Sect. A* 88 (1979), 115-123), and we get a generalization of the result of G. Das (*J. London Math. Soc.* 41 (1966), 685-692) which states the summability factors of the absolute Nörlund methods.

1. Introduction

The object of this paper is to establish limitation theorems for the (N, p, α) and $|\mathrm{N}, p, \alpha|$ methods which include some known results as special cases. In Theorem 2 we shall give a different proof of a recent result of Narang ([*6*], Theorem 1). It is worth noting that in this theorem we cannot omit the condition $(i) : \Delta(p * \alpha)_n \leq 0$, which was not mentioned in [*6*]. A counterexample is the case $(\mathrm{N}, p, \alpha) = (E, \lambda)$; in fact we may not apply the theorem to (E, λ) . Theorem 3 is a generalization of the result of Das ([*1*], Theorem 1) which states the summability factors of the absolute Nörlund methods.

Let $\{p_n\}$ and $\{\alpha_n\}$ be given sequences of real numbers such that

$$(p * \alpha)_n = \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu \neq 0 \quad \text{for all} \quad n \geq 0 ,$$

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and let $\sum a_n$ be a given infinite series with its partial sum s_n. If $t_n \to s$ as $n \to \infty$, where

$$(1.1) \qquad t_n = t_n^{p,\alpha} = \left(1/(p * \alpha)_n\right) \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu s_\nu ,$$

then the series $\sum a_n$ is said to be summable (N, p, α) to s and we write $\sum a_n = s(\mathrm{N}, p, \alpha)$ (see Das [2]). Also if the sequence $\left\{t_n^{p,\alpha}\right\}$ is of bounded variation

$$\sum \left|t_n^{p,\alpha} - t_{n+1}^{p,\alpha}\right| < \infty ,$$

the series $\sum a_n$ is said to be summable $|\mathrm{N}, p, \alpha|$ and we write $\sum a_n \in |\mathrm{N}, p, \alpha|$. The method (N, p, α) reduces to the Nörlund method (N, p) when $\alpha_n = 1$, to the method $(\overline{\mathrm{N}}, \alpha)$ when $p_n = 1$, and to the method (E, λ) when $p_n = (\delta\lambda)^n/n!$ and $\alpha_n = \delta^n/n!$ $(\lambda > 0, \delta > 0)$ (see Hardy [3], p. 179).

Throughout this paper we use the following notations. If $p_0 \neq 0$, we define for $\{p_n\}$ a sequence $\{c_n\}$ such that

$$(1.2) \qquad (c * p)_n = \delta_{n,0} \quad \text{(Kronecker delta)}.$$

We shall write $\{p_n\} \in M$ if $p_n > 0$, $p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$ for all $n \geq 0$. We denote $\Delta a_n = a_n - a_{n+1}$, $\nabla a_n = a_n - a_{n-1}$, $\Delta_n a_{n,\nu} = a_{n,\nu} - a_{n+1,\nu}$ and $a_{-1} = 0$. A capital letter K is an absolute constant, not necessarily the same at each occurrence.

2. The main theorems

Concerning the (N, p, α) method, we have

THEOREM 1. *Let $\{p_n\}$ and $\{\alpha_n\}$ be such that $\{p_n\} \in M$ and $\alpha_n > 0$ for all n. Then $\sum a_n = s(\mathrm{N}, p, \alpha)$ implies $s_n = s + o\left((p * \alpha)_n/\alpha_n\right)$ as*

$n \to \infty$.

For the $|N, p, \alpha|$ method, we have

THEOREM 2. *Let* $\{p_n\}$ *and* $\{\alpha_n\}$ *be two positive sequences and suppose*

(i) $\Delta(p * \alpha)_n \le 0$ *for all* n ,

(ii) $\sum |c_n| < \infty$,

(iii) $\{\alpha_n/(p * \alpha)_n\}$ *is of bounded variation.*

Then, for every series $\sum a_n \in |N, p, \alpha|$ *with partial sum* s_n , *the sequence* $\{s_n\alpha_n/(p * \alpha)_n\}$ *is of bounded variation.*

When $\alpha_n = 1$ for all n , the conditions (i) and (iii) are always satisfied, and we obtain a result of Kishore [4]. Also when $p_n = 1$ for all n , the conditions (i) and (ii) hold and we get a result of Mohanty ([5], Lemma 3).

THEOREM 3. *Let* $\{p_n\}$ *and* $\{\alpha_n\}$ *be such that*

(i) $\sum |c_n| < \infty$,

(ii) $\sum_{\mu=0}^{n} |\nabla(p * \alpha)_\mu| \le K|(p * \alpha)_n|$,

(iii) $\sum_{n=\nu+1}^{\infty} \left|1-\left(\alpha_n/\alpha_{n-1}\right)\right| \sum_{\mu=0}^{\nu} |c_{n-\mu}| \le K$ *for every* $\nu \ge 0$.

Then a necessary and sufficient condition for $\sum \varepsilon_n a_n$ *to be absolutely convergent whenever* $\sum a_n \in |N, p, \alpha|$ *is*

$$\varepsilon_n = O\left(\alpha_n/(p * \alpha)_n\right) . \tag{2.1}$$

When $\alpha_n = 1$ for all n , condition (iii) is always satisfied and we obtain a theorem of Das ([1], Theorem 1). On the other hand when $p_n = 1$ for all n , condition (i) is satisfied and (iii) is equivalent to *(iii)'* $\alpha_n/\alpha_{n-1} = O(1)$,

so we have

COROLLARY. *Let* $\{\alpha_n\}$ *be such that (iii)' holds and* $\sum_{\nu=0}^{n} |\alpha_\nu| = O\left((1 * \alpha)_n\right)$. *Then a necessary and sufficient condition for* $\sum \varepsilon_n a_n$ *to be absolutely convergent whenever* $\sum a_n \in |\overline{N}, \alpha|$ *is* $\varepsilon_n = O\left(\alpha_n/(1 * \alpha)_n\right)$.

3. Proof of the theorems

We need the following lemmas.

LEMMA 1 (Das [2]). *Let* $\alpha_n \neq 0$ *for all* n . *If* $\left\{t_n^{p,\alpha}\right\}$ *is defined by* (1.1), *then*

$$s_n = \left(1/\alpha_n\right) \sum_{\nu=0}^{n} c_{n-\nu}(p * \alpha)_\nu t_\nu^{p,\alpha} \quad \textit{for all} \quad n .$$

LEMMA 2 (Kulza; see [3], Theorem 22). *If* $\{p_n\} \in M$, *then* $c_0 > 0$, $c_n \leq 0$ $(n \geq 1)$ *and* $\sum_{n=0}^{\infty} c_n \geq 0$.

LEMMA 3 (see Peyerimhoff [7], Theorem II, 14). *Let* $A = \left(a_{n\nu}\right)$ *be normal and regular, and let* $\sigma_n = \sum_{\nu=0}^{n} a_{n\nu}s_\nu$. *Suppose that* $M_K(A)$ *hold:*

$$\left|\sum_{\nu=0}^{m} a_{n\nu}s_\nu\right| \leq K \cdot \sup_{\mu \leq m} |\sigma_\mu| \quad \textit{for} \quad m \leq n .$$

Then $\sum a_n = s(A)$ *implies* $s_n = s + o\left(1/a_{nn}\right)$.

LEMMA 4 (Das [1], Lemma 2). *If* $y_n = \sum_{\nu=0}^{\infty} d_{n\nu}x_\nu$ *for all* n *where* $\{d_{n\nu}\}$ *is a double sequence, then a necessary and sufficient condition that the series* $\sum |y_n|$ *is convergent whenever* $\sum |x_n|$ *is convergent is that*

$$\sum_{n=0}^{\infty} |d_{n\nu}| \le K \quad \textit{for each} \quad \nu \ge 0 \ .$$

3.1. Proof of Theorem 1. By Lemma 3 it is sufficient to show that

$$\left|\sum_{\nu=0}^{m} \left(p_{n-\nu}\alpha_\nu/(p * \alpha)_n\right)s_\nu\right| \le \sup_{\mu\le m} \left|t_\mu^{p,\alpha}\right| \quad \text{for} \quad m \le n \ .$$

Now by Lemma 2 we see $c_0 > 0$, $c_n \le 0$ for $n \ge 1$. So we have

$$\begin{aligned}\sum_{\nu=\mu}^{m} p_{n-\nu}c_{\nu-\mu} &= \sum_{\nu=0}^{m-\mu} p_{n-\nu-\mu}c_\nu \\ &= \sum_{\nu=0}^{n-\mu} p_{n-\nu-\mu}c_\nu - \sum_{\nu=m-\mu+1}^{n-\mu} p_{n-\nu-\mu}c_\nu \\ &= \delta_{n-\mu,0} - \sum_{\nu=m-\mu+1}^{n-\mu} p_{n-\nu-\mu}c_\nu \\ &\ge 0 \ ,\end{aligned}$$

since $m - \mu + 1 \ge 1$. Hence we get

$$\begin{aligned}&\sum_{\mu=0}^{m} \left|\sum_{\nu=\mu}^{m} \left(p_{n-\nu}\alpha_\nu/(p * \alpha)_n\right)\left(c_{\nu-\mu}/\alpha_\nu\right)(p * \alpha)_\mu\right| \\ &\qquad = \sum_{\mu=0}^{m} \left((p * \alpha)_\mu/(p * \alpha)_n\right) \sum_{\nu=\mu}^{m} p_{n-\nu}c_{\nu-\mu} \\ &\qquad = \left(1/(p * \alpha)_n\right) \sum_{\nu=0}^{m} p_{n-\nu} \sum_{\mu=0}^{\nu} (p * \alpha)_\mu c_{\nu-\mu} \\ &\qquad = \left(1/(p * \alpha)_n\right) \sum_{\nu=0}^{m} p_{n-\nu}\alpha_\nu \\ &\qquad \le 1 \quad \text{for} \quad m \le n \ .\end{aligned}$$

But this result is a necessary and sufficient condition for $M_1\left((\mathrm{N}, p, \alpha)\right)$ since by Lemma 1 the inverse matrix of (N, p, α) is $\left(a'_{n\nu}\right)$ where $a'_{n\nu} = c_{n-\nu}(p * \alpha)_\nu/\alpha_n$ $(n \ge \nu)$, $= 0$ $(n < \gamma)$ (see Peyerimhoff [7], p. 31).

Therefore we have the conclusion.

3.2. Proof of Theorem 2. By Abel's transformation it follows from Lemma 1 that

$$s_n\alpha_n = \sum_{\nu=0}^{n-1} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu + t_n \sum_{\mu=0}^{n} c_{n-\mu}(p * \alpha)_\mu$$
$$= \sum_{\nu=0}^{n-1} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu + t_n\alpha_n ,$$

and also

$$\Delta(s_n\alpha_n) = \sum_{\nu=0}^{n+1} (c_{n-\nu}-c_{n+1-\nu})(p * \alpha)_\nu t_\nu$$
$$= \sum_{\nu=0}^{n} (\Delta t_\nu)\left\{c_{n-\nu}(p * \alpha)_{\nu+1} - \sum_{\mu=0}^{\nu+1} c_{n+1-\mu}\nabla(p * \alpha)_\mu\right\} + t_{n+1}(\alpha_n-\alpha_{n+1}) .$$

Hence we have

$$\Delta\left(s_n\alpha_n/(p * \alpha)_n\right)$$
$$= \left(\Delta\left(1/(p * \alpha)_n\right)\right)s_n\alpha_n + \left(1/(p * \alpha)_{n+1}\right)\Delta(s_n\alpha_n)$$
$$= \left(\Delta\left(1/(p * \alpha)_n\right)\right)\left\{\sum_{\nu=0}^{n-1} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu + t_n\alpha_n\right\} + \left(1/(p * \alpha)_{n+1}\right)$$
$$\times \left\{\sum_{\nu=0}^{n} (\Delta t_\nu)c_{n-\nu}(p * \alpha)_{\nu+1} - \sum_{\nu=0}^{n} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n+1-\mu}\nabla(p * \alpha)_\mu + t_{n+1}(\alpha_n-\alpha_{n+1})\right\}$$
$$= \left(\Delta\left(1/(p * \alpha)_n\right)\right) \sum_{\nu=0}^{n-1} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu$$
$$+ \left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} (\Delta t_\nu)c_{n-\nu}(p * \alpha)_{\nu+1}$$
$$- \left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n+1-\mu}\nabla(p * \alpha)_\mu$$
$$+ \Delta\left(\alpha_n t_n/(p * \alpha)_n\right) - \left(\alpha_n/(p * \alpha)_{n+1}\right)(\Delta t_n) .$$

Therefore we get

$$\sum_{n=0}^{\infty} |\Delta(s_n\alpha_n/(p * \alpha)_n)|$$

$$\leq \sum_{n=0}^{\infty} \left|\left(\Delta\left(1/(p * \alpha)_n\right)\right) \sum_{\nu=0}^{n-1} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu\right|$$

$$+ \sum_{n=0}^{\infty} \left|\left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} (\Delta t_\nu) c_{n-\nu}(p * \alpha)_{\nu+1}\right|$$

$$+ \sum_{n=0}^{\infty} \left|\left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} (\Delta t_\nu) \sum_{\mu=0}^{\nu} c_{n+1-\mu}\nabla(p * \alpha)_\mu\right|$$

$$+ \sum_{n=0}^{\infty} |\Delta(\alpha_n t_n/(p * \alpha)_n)| + \sum_{n=0}^{\infty} |(\alpha_n/(p * \alpha)_{n+1})(\Delta t_n)|$$

$$= J_1 + J_2 + J_3 + J_4 + J_5 \text{ , say.}$$

Then by *(i)* and *(ii)*,

$$J_1 \leq \sum_{n=0}^{\infty} \left(\Delta\left(1/(p * \alpha)_n\right)\right) \sum_{\nu=0}^{n-1} |\Delta t_\nu| \sum_{\mu=0}^{\nu} |c_{n-\mu}|(p * \alpha)_\mu$$

$$= \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} \left(\Delta\left(1/(p * \alpha)_n\right)\right) \sum_{\mu=0}^{\nu} |c_{n-\mu}|(p * \alpha)_\mu$$

$$\leq \sum_{\nu=0}^{\infty} |\Delta t_\nu|(p * \alpha)_\nu \left(\sum_{\mu=0}^{\infty} |c_\mu|\right) \sum_{n=\nu+1}^{\infty} \Delta\left(1/(p * \alpha)_n\right)$$

$$\leq K \sum_{\nu=0}^{\infty} |\Delta t_\nu| < \infty ,$$

$$J_2 \leq \sum_{n=0}^{\infty} \left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} |\Delta t_\nu||c_{n-\nu}|(p * \alpha)_{\nu+1}$$

$$= \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu}^{\infty} \left(1/(p * \alpha)_{n+1}\right)|c_{n-\nu}|(p * \alpha)_{\nu+1}$$

$$\leq \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=0}^{\infty} |c_n| < \infty ,$$

$$\begin{aligned}
J_3 &\le \sum_{n=0}^{\infty} \left(1/(p * \alpha)_{n+1}\right) \sum_{\nu=0}^{n} |\Delta t_\nu| \sum_{\mu=0}^{\nu} |c_{n+1-\mu}| \nabla(p * \alpha)_\mu \\
&= \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu}^{\infty} \left(1/(p * \alpha)_{n+1}\right) \sum_{\mu=0}^{\nu} |c_{n+1-\mu}| \nabla(p * \alpha)_\mu \\
&\le K \cdot \sum_{\nu=0}^{\infty} |\Delta t_\nu| \left(1/(p * \alpha)_{\nu+1}\right) \sum_{\mu=0}^{\nu} \nabla(p * \alpha)_\mu \\
&\le K \cdot \sum_{\nu=0}^{\infty} |\Delta t_\nu| < \infty .
\end{aligned}$$

Also we have by *(iii)*, and by our assumption,

$$J_4 = \sum_{n=0}^{\infty} \left|\Delta\left(\alpha_n t_n/(p * \alpha)_n\right)\right| < \infty ,$$

$$J_5 \le K \cdot \sum_{n=0}^{\infty} |\Delta t_n| < \infty .$$

Therefore it follows that

$$\sum_{n=0}^{\infty} \left|\Delta\left(s_n \alpha_n/(p * \alpha)_n\right)\right| < \infty .$$

Thus the proof of our theorem is completed.

3.3. Proof of Theorem 3 . Sufficiency. By Lemma 1 and by Abel's transformation we have, for $n \ge 1$,

$$\text{(3.1)} \quad \begin{aligned}
a_n &= \sum_{\nu=0}^{n-1} \Delta t_\nu \sum_{\mu=0}^{\nu} \left(\Delta_n\left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu \\
&\qquad + t_n \sum_{\mu=0}^{n} \left(\nabla_n\left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu \\
&= \sum_{\nu=0}^{n-1} \Delta t_\nu \sum_{\mu=0}^{\nu} \left(\nabla_n\left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu ,
\end{aligned}$$

since (1.2) implies

$$\begin{aligned}
\sum_{\mu=0}^{n} \left(\nabla_n\left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu &= \left(1/\alpha_n\right)(c * p * \alpha)_n - \left(1/\alpha_{n-1}\right)(c * p * \alpha)_{n-1} \\
&= 0 .
\end{aligned}$$

Moreover using Abel's transformation again,

$$\sum_{\mu=0}^{\nu} \left(\nabla_n\left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu = \left(\nabla\left(1/\alpha_n\right)\right) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu$$

$$+ \left(1/\alpha_{n-1}\right) \sum_{\mu=0}^{\nu} c_{n-\mu}\nabla(p * \alpha)_\mu - \left(1/\alpha_{n-1}\right)c_{n-1-\nu}(p * \alpha)_\nu ,$$

and it follows that

$$\sum_{n=1}^{\infty} |\varepsilon_n a_n| = \sum_{n=1}^{\infty} \left|\varepsilon_n \sum_{\nu=0}^{n-1} \Delta t_\nu\{\ \}\right|$$

$$\leq \sum_{n=1}^{\infty} \left|\varepsilon_n \sum_{\nu=0}^{n-1} \Delta t_\nu\left(\nabla\left(1/\alpha_n\right)\right) \sum_{\mu=0}^{\nu} c_{n-\mu}(p * \alpha)_\mu\right|$$

$$+ \sum_{n=1}^{\infty} \left|\varepsilon_n \sum_{\nu=0}^{n-1} \Delta t_\nu\left(1/\alpha_{n-1}\right) \sum_{\mu=0}^{\nu} c_{n-\mu}\nabla(p * \alpha)_\mu\right|$$

$$+ \sum_{n=1}^{\infty} \left|\varepsilon_n \sum_{\nu=0}^{n-1} \Delta t_\nu\left(1/\alpha_{n-1}\right)c_{n-1-\nu}(p * \alpha)_\nu\right|$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 \text{ , say.}$$

Now, by (2.1) with *(ii)*, we get

$$|\varepsilon_n| \leq K|\alpha_n|\left(|(p * \alpha)_n|\right)^{-1} \leq K|\alpha_n|\left(\sum_{\mu=0}^{n} |\nabla(p * \alpha)_\mu|\right)^{-1}$$

$$\leq K|\alpha_n|\left(\sum_{\mu=0}^{\nu} |\nabla(p * \alpha)_\mu|\right)^{-1} \quad \text{for} \quad n \geq \nu \text{ .}$$

Hence we have by *(iii)*,

$$\begin{aligned}
\Sigma_1 &\le \sum_{n=1}^{\infty} |\varepsilon_n| \sum_{\nu=0}^{n-1} |\Delta t_\nu| |\nabla(1/\alpha_n)| \sum_{\mu=0}^{\nu} |c_{n-\mu}| |(p * \alpha)_\mu| \\
&= \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} |\varepsilon_n| |\nabla(1/\alpha_n)| \sum_{\mu=0}^{\nu} |c_{n-\mu}| |(p * \alpha)_\mu| \\
&\le \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} |\varepsilon_n| |\nabla(1/\alpha_n)| \left(\sum_{i=0}^{\nu} |\nabla(p * \alpha)_i|\right) \sum_{\mu=0}^{n} |c_{n-\mu}| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} |1-(\alpha_n/\alpha_{n-1})| \sum_{\mu=0}^{\nu} |c_{n-\mu}| \\
&\le K \cdot \sum_{\nu=0}^{\infty} |\Delta t_\nu| < \infty .
\end{aligned}$$

Also by *(i)* and since *(iii)* implies $\alpha_n/\alpha_{n-1} = O(1)$, we get

$$\begin{aligned}
\Sigma_2 &\le \sum_{n=1}^{\infty} |\varepsilon_n| |1/\alpha_{n-1}| \sum_{\nu=0}^{n-1} |\Delta t_\nu| \sum_{\mu=0}^{\nu} |c_{n-\mu}| |\nabla(p * \alpha)_\mu| \\
&= \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} |\varepsilon_n| |1/\alpha_{n-1}| \sum_{\mu=0}^{\nu} |c_{n-\mu}| |\nabla(p * \alpha)_\mu| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{\mu=0}^{\nu} |\nabla(p * \alpha)_\mu| \left(\sum_{\mu=0}^{\nu} |\nabla(p * \alpha)_\mu|\right)^{-1} \sum_{n=\nu+1}^{\infty} |\alpha_n/\alpha_{n-1}| |c_{n-\mu}| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| \sum_{n=\nu+1}^{\infty} |c_{n-\nu-1}| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| < \infty .
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\Sigma_3 &\le \sum_{n=1}^{\infty} |\varepsilon_n| \sum_{\nu=0}^{n-1} |\Delta t_\nu| |1/\alpha_{n-1}| |c_{n-1-\mu}| |(p * \alpha)_\nu| \\
&= \sum_{\nu=0}^{\infty} |\Delta t_\nu| |(p * \alpha)_\nu| \sum_{n=\nu+1}^{\infty} |\varepsilon_n| |1/\alpha_{n-1}| |c_{n-1-\nu}| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| |(p * \alpha)_\nu| \left(\sum_{\mu=0}^{\nu} |\nabla(p * \alpha)_\mu|\right)^{-1} \sum_{n=\nu+1}^{\infty} |\alpha_n/\alpha_{n-1}| |c_{n-1-\nu}| \\
&\le K \sum_{\nu=0}^{\infty} |\Delta t_\nu| < \infty .
\end{aligned}$$

Hence it follows that $\sum_{n=0}^{\infty} |\varepsilon_n a_n| < \infty$, and the proof of the sufficiency part is completed.

Necessity. From (3.1) we have, for $n \geq 1$,

$$\varepsilon_n a_n = \sum_{\nu=0}^{n} \Delta t_\nu d_{n,\nu}$$

where

$$d_{n,\nu} = \begin{cases} \varepsilon_n \sum_{\mu=0}^{\nu} \left(\nabla_n \left(c_{n-\mu}/\alpha_n\right)\right)(p * \alpha)_\mu & (\nu \leq n) , \\ 0 & (\nu > n) . \end{cases}$$

Now, by Lemma 4, a necessary condition for $\sum |\varepsilon_n a_n|$ to be convergent whenever $\sum a_n$ is summable $|N, p, \alpha|$ is that $\sum_{n=\nu+1}^{\infty} |d_{n,\nu}| \leq K$. Hence it is necessary that $d_{\nu+1,\nu} = O(1)$ as $\nu \to \infty$. But

$$\begin{aligned} d_{\nu+1,\nu} &= \varepsilon_{\nu+1} \sum_{\mu=0}^{\nu} \left(\nabla_\nu \left(c_{\nu+1-\mu}/\alpha_{\nu+1}\right)\right)(p * \alpha)_\mu \\ &= -\varepsilon_{\nu+1} \left(c_0/\alpha_{\nu+1}\right)(p * \alpha)_{\nu+1} . \end{aligned}$$

Therefore the condition (2.1) is necessary.

This completes the proof of Theorem 3.

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