A FURTHER EXTENSION OF CAYLEY'S PARAMETERIZATION

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1. Introduction. In a recent paper **(3)*** the following theorem was proved for real matrices.

THEOREM 1. *If A is a symmetric matrix and Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then*

$$P = (A + Q)^{-1}(A - Q) \tag{1}$$

is a cogredient automorph (c.a.) *of A whose determinant is* $+1$ *and having the property that A and $I + P$ span the same row space.*

Conversely, if P is a c.a. *of A whose determinant is* $+1$ *and if P has the property that $I + P$ and A span the same row space, then there exists a skew-symmetric matrix Q such that P is given by equation* (1).

Theorem 1 reduces to the well-known Cayley parameterization in the case where A is non-singular. A similar and somewhat simpler result (Theorem 4) was given for the case when the underlying field is the complex field. It was also shown that the second part of the theorem (in either form) is false when the characteristic of the underlying field is 2. The purpose of this paper is to simplify the proof of Theorem 1 and at the same time, to extend these results to matrices over an arbitrary field of characteristic $\neq 2$.

2. Matrices Over Arbitrary Fields. Let F be a field whose characteristic is not 2 and let $\lambda: a \to \bar{a}$ be an involutory automorphism on F. We will designate by F_0 the set of elements of F which can be expressed as $a/\bar{a}$ for some $a \in F$. If λ is not the identity mapping, we may also characterize F_0 as the set of elements x for which $x\bar{x} = 1$. If $x \neq -1$, set $a = x + 1$; if $x = -1$, set $a = c - \bar{c}$ where $c \neq \bar{c}$. Then F_0 is a subgroup of the multiplicative group of F.

We define the *conjugate transpose* A^* **(1, 2)** of the matrix $A = [a_{ij}]$, $a_{ij} \in F$ by $A^* = [b_{ij}]$, $b_{ij} = \bar{a}_{ji}$. Then A is *symmetric* if $A^* = A$ and is skew-symmetric if $A^* = -A$.

THEOREM 1′. *If A is a symmetric matrix and Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then*

Received March 13, 1958. This work was completed under a National Research Council-National Bureau of Standards Postdoctoral Research Associateship. See **(3)** for all undefined terms.

https://doi.org/10.4153/CJM-1959-005-1 Published online by Cambridge University Press

$$P = (A + Q)^{-1}(A - Q)$$

is a c.a. *of* A, $|P| \in F_0$ *and* $I + P$ *spans the same row space as* A.

Conversely, if P *is a* c.a. *of* A *such that* $I + P$ *spans the same row space as* A *and if* $|P| \in F_0$, *then there exists a skew-symmetric matrix* Q *such that* P *is given by equation* (1).

Theorem 1 is a special case of Theorem 1′ since F_0 contains just the element $+1$ when F is the real field.

The proof of Theorem 1′ is in three parts. As in the proof of Theorem 1, the first half is immediate. The first part of the proof of the converse consists of repeated simplifications of the forms of A and Q and is analogous to the first part of the proof of the converse in Theorem 1. However, in **(3)** orthogonal transformations $A \to U'AU$ were used to simplify the forms of all of the matrices involved in the construction of the real skew-symmetric matrix Q (for example, the reduction of A to diagonal form). Since the orthogonal reductions possible in the real field are, in general, not possible in the arbitrary field now under consideration, it is necessary that orthogonality conditions be avoided. Hence, Lemma 1 must be replaced by

LEMMA 1′. *For any non-singular matrix* U, $U^{-1}P\,U$ *is a* c.a. *of* $U^*A\,U$ *if and only if* P *is a* c.a. *of* A. *Equation* (1) *holds if and only if*

$$U^{-1}PU = (U^*AU + U^*QU)^{-1}(U^*AU - U^*QU).$$

Moreover, $|P| = |U^{-1}PU|$, *and* $I + P$ *spans the same row space as* A *if and only if* $I + U^{-1}PU$ *spans the same row space as* U^*AU.

Using Lemma 1′, A may be expressed as the direct sum of a non-singular diaongal matrix and a zero matrix (**2**, Theorem 36.2).

The last part of the proof consists of the verification of two conditions (analogous to conditions (6′) and (6″) of **(3)**.)

Condition 1. A non-singular skew-symmetric matrix Z of order $n - 2r + s$ can be constructed. If λ is the identity mapping, it follows from Lemma 2 that $n - 2r + s$ is even and hence a non-singular, skew-symmetric matrix Z exists. If λ is not the identity mapping, then for some $a \in F$, $a \neq \bar{a}$. Hence $(a - \bar{a})/(\bar{a} - a) = -1 \in F_0$ and thus $(a - \bar{a})I$ is non-singular and skew-symmetric.

Condition 2. If B is a c.a. of a non-singular, symmetric, diagonal matrix d, then $I + B$ is a Pr matrix. First we prove

LEMMA. *If the rank of* B *is* r, *then the* r *by* r *matrix* C *formed by the intersection of a set of* r *linearly independent rows of* B *and of* r *linearly independent columns of* B *is non-singular.*

Proof. For convenience, we shall assume that C is in the upper-left-hand corner of B. Let

$$B = \begin{bmatrix} C & D \\ E & F \end{bmatrix}.$$

Since the rows $[CD]$ of B are linearly independent and are r in number, the rows $[EF]$ are linear combinations of them. Thus, there exists a matrix M such that $E = MC$, $F = MD$. Similarly, $D = CN$, $F = EN = MCN$ for some N. Thus,

$$B = \begin{bmatrix} I & O \\ M & I \end{bmatrix} \quad \begin{bmatrix} C & O \\ O & O \end{bmatrix} \quad \begin{bmatrix} I & N \\ O & I \end{bmatrix}$$

and hence C is non-singular.

Proof of Condition 2. Since B is a c.a. of d, $I + B = d^{-1} (I + (B^*)^{-1})d$ and hence, if any set of rows of $I + B$ are linearly independent, the same set of rows of $I + (B^*)^{-1}$ are linearly independent and hence the corresponding set of columns of $I + \bar{B}^{-1} = \bar{B}^{-1}(I + \bar{B})$ are linearly independent, where $\bar{B}$ is the conjugate of B, that is, if $B = (b_{ij})$ then $\bar{B} = (\bar{b}_{ij})$. Consequently, the corresponding set of columns of $I + B$ are linearly independent. By the above lemma, the principal submatrix of $I + B$ determined by these rows is non-singular.

References

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