Canad. J. Math. Vol. **63** (1), 2011 pp. 200–221
doi:10.4153/CJM-2010-081-9

An Explicit Polynomial Expression for a q-Analogue of the 9-j Symbols

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Abstract. Using standard transformation and summation formulas for basic hypergeometric series we obtain an explicit polynomial form of the q-analogue of the 9-j symbols, introduced by the author in a recent publication. We also consider a limiting case in which the 9-j symbol factors into two Hahn polynomials. The same factorization occurs in another limit case of the corresponding q-analogue.

1 Introduction

The orthonormality relation for $R_n(x)$, the Racah polynomial, is

$$\sum_{x=0}^{N} \rho(x)\sqrt{h_m h_n} R_m(x) R_n(x) = \delta_{m,n}, \tag{1.1}$$

where

$$\begin{aligned} R_n(x) &\equiv R_n(x; a, b, d, N) \\ &= {}_4F_3\left[\begin{matrix} -n, n+a+d-1, -x, x+a+b-1 \\ a, -N, N+a+b+d-1 \end{matrix}; 1\right], \end{aligned} \tag{1.2}$$

$$\rho(x) = \frac{2x+a+b-1}{a+b-1}\,\frac{(a+b-1, a, N+a+b+d-1, -N)_x}{x!(b, 1-d-N, N+a+b)_x}, \tag{1.3}$$

and

$$h_n = \frac{(b,d)_N}{(a+b, a+d)_N}\,\frac{2n+a+d-1}{a+d-1}\,\frac{(a+d-1, a, N+a+b+d-1, -N)_n}{n!(d, 1-b-N, N+a+d)_n}. \tag{1.4}$$

The ${}_4F_3$ series in (1.2) is a balanced, terminating, hypergeometric series, see for example [7,9,10], and the notations for the shifted factorials used in (1.3) and (1.4) are as introduced in [10].

Received by the editors September 10, 2008.
Published electronically November 23, 2010.
This work was partially supported by the NSERC Grant#A6197
AMS subject classification: **33D45**, 33D50.
Keywords: 6-j and 9-j symbols, q-analogues, balanced and very-well-poised basic hypergeometric series, orthonormal polynomials in one and two variables, Racah and q-Racah polynomials and their extensions.

The orthonormality relation (1.1) is, of course, the same as

$$\sum_x (2x+1)\sqrt{(2m+1)(2n+1)} \begin{Bmatrix} a & b & x \\ c & d & m \end{Bmatrix} \begin{Bmatrix} a & b & x \\ c & d & n \end{Bmatrix} = \delta_{m,n},$$

which is the orthonormality relation for the 6-j symbols, also called the Racah coefficients [18], defined by

(1.5)
$$\begin{Bmatrix} a & b & x \\ c & d & m \end{Bmatrix} := \Delta(abx)\Delta(cdx)\Delta(bcm)\Delta(adm) \\ \times \sum_z \frac{(-1)^z (z+1)!}{(z-a-b-x)!(z-c-d-x)!(z-b-c-m)!(z-a-d-m)!} \\ \times \left((a+b+c+d-z)!(b+d+m+x-z)!(a+c+m+x-z)!\right)^{-1},$$

with the "triangle function"

$$\Delta(abc) := \left\{ \frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!} \right\}^{1/2}.$$

These symbols are familiar in the theory of angular momentum in Quantum Mechanics; see, for example, [8]. The physicists probably did not bother to ask, nor would it matter to them much, whether these objects can also be seen as polynomials in a single variable. Jim Wilson [23], a graduate student at the University of Wisconsin in the late 70's, working towards a doctorate degree under the supervision of Richard Askey, did make the important observation that it is indeed so by relabelling the parameters and variables in an appropriate way so that the 6-j symbols can be written as a polynomial, hitherto unknown in mathematical literature, and that the series in (1.5) is the same as the hypergeometric series in (1.2). Wilson's seminal work on these and other related orthogonal polynomials comprise the bulk of his 1978 thesis [23], as well as the subsequent papers [5, 24]. This work eventually led to the discovery of the Askey–Wilson polynomials (see [6, 10, 13]), which has since become the most attractive and active area of research in the field of Orthogonal Polynomials and Special Functions.

It is quite natural, therefore, that the mathematician's curiosity would then be directed to the next level of objects in Angular Momentum literature known as 9-j symbols, and defined by

$$\begin{Bmatrix} a & b & x \\ c & d & y \\ m & n & e \end{Bmatrix} = \sum_j (2j+1) \begin{Bmatrix} a & c & m \\ n & e & j \end{Bmatrix} \begin{Bmatrix} b & d & n \\ c & j & y \end{Bmatrix} \\ \times \begin{Bmatrix} x & y & e \\ j & a & b \end{Bmatrix}, \tag{1.6}$$

https://doi.org/10.4153/CJM-2010-081-9 Published online by Cambridge University Press

j running through the set of half-integers, such that the sum of any two entries minus the third in every row and column is a non-negative integer; see [8, 15]. They were introduced into the physics literature by Eugene Wigner, and are called Wigner 9-j symbols. From the special functions point of view, an important property of these objects is the orthonormality:

$$\sum_x \sum_y (2x+1)(2y+1)(2m+1)(2n+1) \times \begin{Bmatrix} a & b & x \\ c & d & y \\ m & n & e \end{Bmatrix} \begin{Bmatrix} a & b & x \\ c & d & y \\ m' & n' & e \end{Bmatrix} = \delta_{m,n'}\delta_{n,n'}, \tag{1.7}$$

where the range of double summation is again the sets of half-integers. The obvious question is: is this a polynomial orthonormality in 2 variables? As far as we know the first among a number of curious investigators was Sergei Suslov [21,22], who was able to show that the expression defined in (1.6) are indeed polynomials in certain combinations of the parameters. But he did not succeed in giving an explicit expression for the polynomials, nor did he indicate what the degrees of these polynomials are. It was generally believed, perhaps guided by the experience of some orthogonal polynomial systems in 2 variables, that the 9-j symbols would turn out to be expressible as double hypergeometric series. Alisauskas and Jucys [3] were probably the first who obtained a triple sum expression for the 9-j symbols. Rosengren [20] (see also [1,2]) found a different proof for the triple sum formula. These, plus Zhedanov's [25] results on the 9-j symbols of escillator algebras, and the more recent work by Hoare and the author [11], seem to indicate that the search for a double sum representation of the 9-j symbols may not be fruitful.

Following the lead of Wilson [23], as well as our own experience in [11, 19], we shall first replace $a+b-x$, $c+d-y$, $a+c-m$, $b+d-n$ by x, y, m, n, respectively, set $a+b+c+d-e=N$, all non-negative integers, such that

$$0 \le x+y \le N, \quad 0 \le m+n \le N.$$

For notational simplicity in the main results, we make a further replacement of the parameters a, b, c, d by $-a/2$, $-b/2$, $-c/2$, and $-d/2$, respectively, and, just as we did in [19], we shall introduce the function

$$\begin{aligned} &F_{m,n}(x,y) \\ &= \big[(2x+a+b-1)(2y+c+d-1)(2m+a+c-1)(2n+b+d-1)\big]^{1/2} \\ &\quad \times \begin{Bmatrix} -a/2 & -b/2 & -\frac{a+b+2x}{2} \\ -c/2 & -d/2 & -\frac{c+d+2y}{2} \\ -\frac{a+c+2m}{2} & -\frac{b+d+2n}{2}, & -N-\frac{a+b+c+d}{2} \end{Bmatrix} \end{aligned}$$

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as a normalized version of the 9-j symbols. Using (1.6), (1.5), and (1.2) we find that

(1.8)

$$\begin{aligned}F_{m,n}(x,y) = {}& A_{m,n}(x,y)\sum_{\ell}\frac{2\ell+2y+b+c+d-1}{2y+b+c+d-1}\frac{(2y+b+c+d-1)_\ell}{\ell!}\\ &\times\frac{(N+y+a+b+c+d-1,b,c+y-n,y-N)_\ell}{(1+y-N-a,c+d+2y,b+d+y+n,b+c+d+N+y)_\ell}(-1)^\ell\\ &\times R_\ell(x;b,2y+c+d,a,N-y)\\ &\times R_{y-n+\ell}(m;c,2n+b+d,a,N-n)\\ &\times R_\ell(n;b,2y+c+d,2-2n-b-d,n-y-1),\end{aligned}$$

where

(1.9)

$$\begin{aligned}A_{m,n}(x,y) = {}& \frac{N!(a)_{N-y}(b+c+d)_{2y}(-y)_n}{(a+b+c+d-1)_{N+n}(b+c+d)_{N+y}(c+d)_{2y}(b+d)_{y+n}(-N)_n}\\ &\times\left\{\frac{2x+a+b-1}{a+b-1}\frac{(a+b-1,b,a+b+c+d+N+y-1,y-N)_x}{x!(a,1-N-y-c-d,a+b+N-y)_x}\right.\\ &\times\frac{(a+b+c+d-1,c+d)_{N+y}(c,d)_y}{(N-y)!(a+b)_{N-y}}\frac{2y+c+d-1}{c+d-1}\frac{(c+d-1)_y}{y!}\\ &\times\frac{2m+a+c-1}{a+c-1}\frac{(a+c-1,c,a+b+c+d+N+n-1,n-N)_m}{m!(a,1-N-n-b-d,a+c+N-n)_m}\\ &\times\left.\frac{(a+b+c+d-1,b+d)_{N+n}}{(N-n)!(a+c)_{N-n}}\frac{2n+b+d-1}{b+d-1}\frac{(b+d-1,b)_n}{n!(d)_n}\right\}^{1/2}.\end{aligned}$$

In (1.8) and (1.9) the parameters a, b, c, d are allowed to have complex values, and hence, $F_{m,n}(x,y)$ can be regarded not just as another form of the 9-j symbol but as an analytic continuation thereof. When $n \geq y+1$, one has to consider the product $(-y)_n R_\ell(\dots,n-y-1)$, rather than $R_\ell(\dots,n-y-1)$ alone.

The orthonormality relation (1.7) for the 9-j symbols can be rewritten as

$$\sum_{\substack{0\leq x,y\leq N\\ x+y\leq N}} F_{m,n}(x,y)F_{m',n'}(x,y) = \delta_{m,m'}\delta_{n,n'},$$

The symmetry properties of the 9-j symbols (see, for example, [8]) ensures us that the dual orthonormality

$$\sum_{\substack{0\leq m,n\leq N\\ m+n\leq N}} F_{m,n}(x,y)F_{m,n}(x',y') = \delta_{x,x'}\delta_{y,y'}$$

is also true. So the functions $F_{m,n}(x',y')$ are self-dual just as the $R_n(x)$'s are in x and n; see [5].

The interesting application of $F_{m,n}(x,y)$ that we found in [11] was in solving a long-standing problem of finding the eigenvalues and eigenfunctions of the transition probability kernel of a 2-dimensional version of a "cumulative Bernoulli process", introduced in [12], that can be written as an Appell function:

$$\begin{aligned}(1.10)\qquad K_n(x,y|\xi,\eta) =&(1-\alpha_1)^x(1-\alpha_2)^y\beta_1^{-\xi}\beta_2^{-\eta}b_2(\xi,\eta,N;\beta_1,\beta_2)\\ &\times F_3\Big(-x,-y,-\xi,-\eta,-N;\frac{\alpha_1}{\beta_1(\alpha_1-1)},\frac{\alpha_2}{\beta_2(\alpha_2-1)}\Big),\end{aligned}$$

where

$$b_2(\xi,\eta,N;\beta_1,\beta_2)=\binom{N}{\xi,\eta}\beta_1^{\xi}\beta_2^{\eta}(1-\beta_1-\beta_2)^{N-\xi-\eta}$$

is the trinomial distribution, $0<\alpha_1,\alpha_2,\beta_1,\beta_2<1$, $0<1-\beta_1-\beta_2<1$, and

$$F_3(a,a',b,b',\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a,b)_m(a',b')_n}{m!n!(\gamma)_{m+n}}x^my^n;$$

see [9]. For the sake of completeness we shall just mention here that the eigenfunctions of (1.10) were found to be the 2-variable Krawtchouk polynomials

$$\begin{aligned}(1.11)\qquad P_{m,n}(x,y)=&\sum_i\sum_j\sum_k\sum_\ell\frac{(-m)_{i+j}(-n)_{k+\ell}(-x)_{i+k}(-y)_{j+\ell}}{i!j!k!\ell!(-N)_{i+j+k+\ell}}\\ &\times t^iu^jv^kw^\ell,\end{aligned}$$

where

$$t=\frac{(p_1+p_2)(p_1+p_3)}{p_1(p_1+p_2+p_3+p_4)},\quad u=\frac{(p_1+p_3)(p_4+p_3)}{p_3(p_1+p_2+p_3+p_4)},$$

$$v=\frac{(p_1+p_2)(p_2+p_4)}{p_2(p_1+p_2+p_3+p_4)},\quad w=\frac{(p_2+p_4)(p_3+p_4)}{p_4(p_1+p_2+p_3+p_4)},$$

and the p's are the parameters connected with a,b,c,d in (1.8) as:

$$(1.12)\qquad a=p_1\zeta,\quad b=p_2\zeta,\quad c=p_3\zeta,\quad d=p_4\zeta,\quad \zeta\to\infty.$$

The remarkable thing is that the functions $F_{m,n}(x,y)$ in the above limit, turn out to be

$$\Big\{b_2(x,y,N;\eta_1,\eta_2)\,b_2(m,n,N;\bar\eta_1,\bar\eta_2)\,(1-\eta_1-\eta_2)^{-N}\Big\}^{1/2}P_{m,n}(x,y),$$

where

$$\eta_1=\frac{p_1p_2(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)},\quad \eta_2=\frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_4+p_2)(p_4+p_3)},$$

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$\bar{\eta}_1, \bar{\eta}_2$ being the same as η_1, η_2 with p_2 and p_3 interchanged. To see how t, u, v, w (*i.e.*, the p's) are related to $\alpha_1, \alpha_2, \beta_1, \beta_2$, see [11]. It has recently been pointed out to the author by M. Noumi that a multidimensional version of the 2-variable Krawtchouk polynomials (1.11) was found by Aomoto and Gelfand, see [4], and later by Mizukawa [16], as an orthogonal system with respect to the multinomial distribution.

In Section 2 of this paper we shall establish another limiting result:

$$\begin{aligned} (1.13) \quad & \lim_{a\to\infty} F_{m,n}(x,y) \\ &= \left\{ \frac{(1-N-y-c-d, y-N)_x}{x!(b)_x} \frac{(1-N-n-b-d, n-N)_m}{m!(c)_m} \right. \\ &\times \frac{N!(c,d)_y}{(c+d)_{N+y}(N-y)!} \frac{2y+c+d-1}{c+d-1} \frac{(c+d-1)_y}{y!} \\ &\times \left. \frac{N!}{(b+d)_{N+n}(N-n)!} \frac{2n+b+d-1}{b+d-1} \frac{(b+d-1,b)_n}{n!(d)_n} \right\}^{1/2} \\ &\times \frac{(c, x-N)_m (b)_x}{(-N)_m (1-N-d)_{m+x}} \\ &\times {}_3F_2\left[\begin{matrix} -m, -y, y+c+d-1 \\ c, x-N \end{matrix}; 1 \right] {}_3F_2\left[\begin{matrix} -x, -n, n+b+d-1 \\ b, m-N \end{matrix}; 1 \right]. \end{aligned}$$

Similar factorization occurs when we take the limit $d \to \infty$.

However, the main objective of this paper is to obtain a polynomial form of the q-analogue of $F_{m,n}(x, y)$ in (1.8) that we introduced in [19], namely,

$$\begin{aligned} (1.14) \quad R^{\tau}_{m,n}(x, y|q) :=& R^{\tau}_{m,n}(x, y; a, b, c, d, N|q) \\ =& A_{m,n}(x, y|q) \sum_{\ell} \frac{1-bcdq^{2\ell+2y-1}}{1-bcdq^{2y-1}} \frac{\left(bcdq^{2y-1}, abcdq^{N+y-1}\right)_\ell}{\left(q, q^{1+y-N}/a\right)_\ell} \\ &\times \frac{(b, cq^{y-n}, q^{y-N})_\ell}{(cdq^{2y}, bdq^{y+n}, bcdq^{N+y})_\ell} \left\{ \tau \left(d/bc\right)^{1/2} q^{n+1}/a \right\}^\ell \\ &\times W_\ell(x; b, cdq^{2y}, a, N-y|q) \\ &\times W_{y-n+\ell}(m; c, bdq^{2n}, a, N-n|q) \\ &\times W_\ell(n; b, cdq^{2y}, q^{2-2n}/bd, n-y-1|q), \end{aligned}$$

where $\tau = \pm 1$,

$$W_n(x; a, d, b, N|q) := {}_4\phi_3\left[\begin{matrix} q^{-n}, adq^{n-1}, q^{-x}, abq^{x-1} \\ a, abdq^{N-1}, q^{-N} \end{matrix}; q, q \right]$$

are the q-Racah polynomials; see [5, 6, 10] (note a slight difference in the notation for W_n's compared to the standard one). For the definition of the q-shifted factorials used in (1.14) as elsewhere in this paper, as well as the basic hypergeometric series ${}_4\phi_3$ given in (1.10) and others that are going to be used later, see [10]. In (1.14) the coefficients $A_{m,n}(x,y|q)$ are given by

(1.15)

$$
\begin{aligned}
A_{m,n}(x,y|q) = & \frac{(q)_N(a)_{N-y}(bcd)_{2y}(q^{-y})_n q^{n(y-N)+\binom{n}{2}}}{(abcdq^{-1})_{N+n}(bcd)_{N+y}(cd)_{2y}(bd)_{n+y}(q^{-N})_n} \\
& \times \left\{ \frac{1-abq^{2x-1}}{1-abq^{-1}} \frac{(abq^{-1},b,abcdq^{N+y-1},q^{y-N})_x}{(q,a,q^{1-N-y}/cd,abq^{N-y})_x}(q^{1-2y}/bcd)^x \right. \\
& \times \frac{(abcdq^{-1},cd)_{N+y}(c,d)_y}{(q,ab)_{N-y}} \frac{1-cdq^{2y-1}}{1-cdq^{-1}} \frac{(cdq^{-1})_y}{(q)_y}(bc)^{N-y} \\
& \times \frac{1-acq^{2m-1}}{1-acq^{-1}} \frac{(acq^{-1},c,abcdq^{N+n-1},q^{n-N})_m}{(q,a,q^{1-N-n}/bd,acq^{N-n})_m}(q^{1-2n}/bcd)^m \\
& \times \left. \frac{(abcdq^{-1},bd)_{N+n}}{(q,ac)_{N-n}} \frac{1-bdq^{2n-1}}{1-bdq^{-1}} \frac{(bdq^{-1},b)_n}{(q,d)_n} d^n \right\}^{1/2}
\end{aligned}
$$

One can obtain the $q \to 1$ limit, *i.e.*, $F_{m,n}(x,y)$, by replacing a,b,c,d with q^a, q^b, q^c, q^d, respectively, then set $\tau = -1$, and take the limit. However, as we remarked in [19], the results are equally valid for $\tau = +1$, although the $q \to 1$ limit is going to be different from $F_{m,n}(x,y)$ as given in (1.8). We should like to point out that, as in $F_{m,n}(x,y)$ of (1.11), the parameters a, b, c, d in $R^\tau_{m,n}(x,y|q)$ can also be complex, and therefore the numerical values ± 1 of τ would automatically be implicit in the square root function $(bcd)^{1/2}$. However, for the sake of consistency of notation with our previous work in [19], we prefer to retain the τ symbol, with the understanding that only the principal value of $(bcd)^{1/2}$ is being considered here.

The most important results that we proved in [17] are the orthonormality relation

$$
\sum\sum_{\substack{0\le x,y\le N \\ x+y\le N}} R^\tau_{m,n}(x,y|q) R^\tau_{m',n'}(x,y|q) = \delta_{m,m'}\delta_{n,n'}
$$

and the dual

$$
\sum\sum_{\substack{0\le m,n\le N \\ m+n\le N}} R^\tau_{m,n}(x,y|q) R^\tau_{m,n}(x',y'|q) = \delta_{x,x'}\delta_{y,y'}.
$$

We also derived a polynomial expression for $R^\tau_{m,n}(x,y|q)$ in [19], but it is not the best form, since there are two q-shifted factorials in the denominator that depend on n, so it cannot be claimed that the series is a polynomial in both (x,y) and (m,n) (to be more precise, polynomials in $(q^{-x}+abq^{x-1}, q^{-y}+cdq^{y-1})$ and $(q^{-m}+acq^{m-1}$,

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$q^{-n}+bdq^{n-1}$)). We shall prove in Section 3 that

(1.16)
$$
\begin{aligned}
R_{m,n}(x,y|q) =&(abcdq^{-1})_{2N}(-\tau)^{N-y}q^{\binom{y}{2}+\binom{n}{2}-\binom{N}{2}}\\
&\times\Big\{\frac{1-abq^{2x-1}}{1-abq^{-1}}\frac{(abq^{-1},a,q^{1-N-y}/cd,q^{y-N})_x}{(q,b,abcdq^{N+y-1},abq^{N-y})_x}\Big(bcdq^{2N-1}\Big)^x\\
&\times\frac{(q,d)_N}{(abcdq^{-1},cd)_{N+y}(q,ab)_{N-y}}\frac{1-cdq^{2y-1}}{1-cdq^{-1}}\frac{(cdq^{-1},c)_y}{(q,d)_y}d^{y-N}\\
&\times\frac{1-acq^{2m-1}}{1-acq^{-1}}\frac{(acq^{-1},a,q^{1-N-n}/bd,q^{n-N})_m}{(q,c,abcdq^{N+n-1},acq^{N-n})_m}(bcdq^{2N-1})^m\\
&\times\frac{(q,d)_N}{(abcdq^{-1},bd)_{N+n}(q,ac)_{N-n}}\frac{1-bdq^{2n-1}}{1-bdq^{-1}}\frac{(bdq^{-1},b)_n}{(q,d)_n}d^n\Big\}^{1/2}\\
&\times\sum_i\sum_j\sum_k\sum_\ell\frac{(q^{y-N+k+\ell},q^{1-N-y+k+\ell}/cd)_{i+j}(q^{-x},abq^{x-1})_{k+\ell}}{(q)_i(q)_j(q)_k(q)_\ell(q^{-N},q^{1-N}/d)_{i+j+k+\ell}}\\
&\times\frac{(q^{-m},acq^{m-1})_{j+k}(q^{n-N+j+k},q^{1-N-n+j+k}/bd)_{i+\ell}}{(q^{2-2N}/abcd,q^{1-N}/\tau\sqrt{bcd})_{i+j+k+\ell}}\\
&\times\frac{(q^{1-N}/a\tau\sqrt{bcd})_i(q^{j+k+\ell+1-2N}/bcd)_k}{(a)_{j+k}(a)_{k+\ell}}\Big(\frac{q^{1-N}}{a\tau\sqrt{bcd}}\Big)^{j+k+\ell}\\
&\times(-1)^kq^{i+j+k+\ell-jk-k\ell-\ell j-\binom{k}{2}}.
\end{aligned}
$$

The first significant work on a q-analogue of the 9-j symbols seems to be by Kirillov and Reshetikin [14], in 1988. In 1990, M. Nomura [17] found an analogue that has a biorthogonality property between the q-9j and q^{-1} - 9j symbols. Alisauskas [2] gave one for $u_q(2)$. However, the q-analogue given in [19] and the one above, is quite different from any given before.

It is undoubtedly a formidable expression, but the polynomial character of the series part in (1.16) is quite obvious. Equally importantly, the symmetry under $(x,y,c)\leftrightarrow(m,n,b)$ (which is one of the main properties of the 9-j symbols), is also obvious. However, (1.16) is not always the most suitable form for taking limits, for example, $d\to 0$ or $d\to\infty$. In Section 4 we shall derive an alternate form of $R^\tau_{m,n}(x,y|q)$ that is the right one for taking the limit $d\to 0$ or $d\to\infty$. In Section 5 we shall prove that

(1.17)
$$
\begin{aligned}
\lim_{d\to 0}&R^\tau_{m,n}(x,y|q)\\
&=\Big\{\frac{1-abq^{2x-1}}{1-abq^{-1}}\frac{(abq^{-1},b,q^{y-N})_x}{(q,a,abq^{N-y})_x}(-q^{N-y}/b)^xq^{-\binom{x}{2}}\\
&\times\frac{(c)_y(bc)^{N-y}}{(q)_y(q,ab)_{N-y}}\frac{1-acq^{2m-1}}{1-acq^{-1}}\frac{(acq^{-1},c,q^{n-N})_m}{(q,a,acq^{N-n})_m}(-q^{N-n}/c)^mq^{-\binom{m}{2}}
\end{aligned}
$$

$$\times \left.\frac{(b)_n(bc)^{-n}}{(q)_n(q,ac)_{N-n}}\right\}^{1/2} (a)_{N-y-n}(-1)^{N-y-n}\frac{(q^{y-N})_n}{(q^{-N})_n} q^{\binom{N-y+1}{2}-yn}$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{-x}, abq^{x-1}, q^{-n} \\ b, q^{y-N} \end{matrix}; q, q\right]$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{-m}, acq^{m-1}, q^{-y} \\ c, q^{n-N} \end{matrix}; q, q\right].$$

The $d \to \infty$ limit is essentially the same except for minor differences inside the $\{\quad\}^{1/2}$ part, and in the fact that the ${}_3\phi_2$ series are replaced by ones of type II in their arguments, see [10].

In the Appendix we first list two of the most important transformation formulas that are used in this paper, then derive a more convenient form of a product formula for two balanced and terminating ${}_4\phi_3$ series, that the reader can readily refer to.

2 Proof of (1.13)

In this section we shall use the $q \to 1$ cases of (A.1), (A.2), and (A.5). So, applying the limit case of (A.1) on $R_\ell(x)$ and $R_{y-n+\ell}(m)$ in (1.8) we find that

$$\begin{aligned}
(2.1)\qquad & R_\ell(x; b, 2y+c+d, a, N-y) R_{y-n+\ell}(m; c, 2n+b+d, a, N-n) \\
&= \frac{(a, 1-N-y-c-d)_x}{(b, N+y+a+b+c+d-1)_x} \frac{(a, 1-N-n-b-d)_m}{(c, N+n+a+b+c+d-1)_m} \\
&\quad \times R_{N-y-\ell}(x; a, 2-2N-a-b-c-d, b, N-y) \\
&\quad \times R_{N-y-\ell}(m; a, 2-2N-a-b-c-d, c, N-n),
\end{aligned}$$

and, by the limit case of (A.5),

$$\begin{aligned}
(2.2)\quad & R_{N-y-\ell}(x; a, 2-2N-a-b-c-d, b, N-y) \\
&\quad \times R_{N-y-\ell}(m; a, 2-2N-a-b-c-d, c, N-n) \\
&= \sum_r \frac{(y-N+\ell, 1-N-y-b-c-d-\ell, -m, m+a+c-1)_r}{r!(a, n-N, 1-N-n-b-d)_r} \\
&\quad \times \sum_s \frac{(-r, 1-2N-b-c-d+r, -x, x+a+b-1)_s}{s!(a, y-N, 1-N-y-c-d)_s} \\
&\quad \times {}_4F_3\left[\begin{matrix} -s, N-n+1-r, 1-a-r, N+n+b+d-r \\ m+1-r, 2-m-a-c-r, 2N+b+c+d-r-s \end{matrix}; 1\right].
\end{aligned}$$

Substituting (2.1) and (2.2) into (1.8) we find that the summation part can be written as

$$
\begin{aligned}
G_{m,n}(x,y) := \sum_r & \frac{(y-N, 1-N-y-b-c-d, -m, m+a+c-1)_r}{r!(a, n-N, 1-N-n-b-d)_r} \\
& \times \sum_s \frac{(-r, 1-2N-b-c-d+r, -x, x+a+b-1)_s}{s!(a, y-N, 1-N-y-c-d)_s} \\
& \times {}_4F_3\left[\begin{matrix} -s, N-n+1-r, 1-a-r, N+n+b+d-r \\ m+1-r, 2-m-a-c-r, 2N+b+c+d-r-s \end{matrix}; 1\right] H_r,
\end{aligned}
$$

where

$$
\begin{aligned}
(2.3)\quad H_r = \sum_k & \frac{(-n, 1-d-n, N+y+a+b+c+d-1, y-N+r)_k}{k!(y-n+1, 1+y-N-a, b+c+d+N+y-r, c+d+2y)_k} \\
& \times \frac{(b+c+d+2y)_{2k}}{(b+d+y+n)_k}\, {}_6F_5\left[\begin{matrix} b+c+d+2y+2k-1, y+k+\frac{b+c+d+1}{2}, \\ y+k+\frac{b+c+d-1}{2}, 1-N-a+k, \end{matrix}\right. \\
& \left.\begin{matrix} N+y+a+b+c+d+k-1, b+k, c+y-n+k, y-N+r+k \\ c+d+2y+k, b+d+y+n+k, b+c+d+N+y+k-r \end{matrix}; -1\right]
\end{aligned}
$$

However, by [10, (3.2.12)], ${}_6F_5$ is a multiple of

$$
\begin{aligned}
& {}_3F_2\left[\begin{matrix} y-N+r+k, y+n+d, N+y+a+b+c+d+k-1 \\ c+d+2y+k, b+d+y+n+k \end{matrix}; 1\right] \\
&= \frac{(a+b+r+k)_{N-y-r-k}}{(c+d+2y+k)_{N-y-r-k}}(-1)^{N-y-r-k} \\
&\quad \times {}_3F_2\left[\begin{matrix} y-N+r+k, N+y+a+b+c+d+k-1, b+k \\ b+d+y+n+k, a+b+r+k \end{matrix}; 1\right],
\end{aligned}
$$

by [10, (3.1.1)]. Substituting this reduction of the ${}_6F_5[-1]$ series into (2.3) we obtain

$$
\begin{aligned}
H_r = & \frac{(b+c+d+2y, a+b)_{N-y}}{(c+d+2y, a)_{N-y}} \frac{(a, 1-N-y-c-d)_r}{(a+b, 1-N-y-b-c-d)_r} \\
& \times \sum_k \frac{(-n, 1-d-n, N+y+a+b+c+d-1, y-N+r)_k}{k!(y-n+1, b+d+y+n, a+b+r)_k}(-1)^k \\
& \quad \times {}_3F_2\left[\begin{matrix} y-N+r+k, b+k, N+y+a+b+c+d+k-1 \\ b+d+y+n+k, a+b+r+k \end{matrix}; 1\right].
\end{aligned}
$$

Now,

$$
\begin{aligned}
\lim_{a\to\infty} & {}_3F_2\left[\begin{matrix} y-N+r+k, b+k, N+y+a+b+c+d+k-1 \\ b+d+y+n+k, a+b+r+k \end{matrix}; 1\right] \\
&= {}_2F_1\left[\begin{matrix} y-N+r+k, b+k \\ b+d+y+n+k \end{matrix}; 1\right] \\
&= \frac{(d+y+n)_{N-y-r-k}}{(b+d+y+n+k)_{N-y-r-k}}, \text{ by [10, (1.2.11)]}.
\end{aligned}
$$

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So

(2.4)

$$\lim_{a\to\infty} H_r = \frac{(b+c+d+2y, d+y+n)_{N-y}}{(c+d+2y, b+d+y+n)_{N-y}} \frac{(1-N-y-c-d, 1-N-n-b-d)_r}{(1-N-y-b-c-d, 1-N-n-d)_r} \times {}_3F_2\left[\begin{matrix}-n, 1-d-n, y-N+r\\ y-n+1, 1-N-n-d+r\end{matrix}; 1\right]$$

Note that the ${}_3F_2$ series in (2.4) is balanced, and by the Pfaff–Saalschütz summation formula [10, (1.7.1)], see also [7], its sum is

$$(2.5)\quad \frac{(r-N, 1-N-d+r)_{N-y-r}}{(1-N-n-d+r, r+n-N)_{N-y-r}} = \frac{(-N)_n(d+y)_{N-y}}{(-y)_n(d+n+y)_{N-y}} \frac{(n-N, 1-N-n-d)_r}{(-N, 1-N-d)_r}.$$

Substituting (2.4) and (2.5) in (2.3) we find that

$$(2.6)\quad \lim_{a\to\infty} G_{m,n}(y) = \frac{(-N)_n(b+c+d+2y, d+y)_{N-y}}{(-y)_n(c+d+2yb+d+y+n)_{N-y}} \times \sum_r \frac{(y-N, 1-N-y-c-d, -m)_r}{r!(-N, 1-N-d)_r} \times \sum_s \frac{(-r, 1-2N-b-c-d+r, -x)_s}{s!(y-N, 1-N-y-c-d)_s} \times {}_3F_2\left[\begin{matrix}-s, N-n+1-r, N+n+b+d-r\\ m+1-r, 2N+b+c+d-r-s\end{matrix}; 1\right].$$

By [10, (3.2.8)]

$${}_3F_2\left[\begin{matrix}-s, N-n+1-r, N+n+b+d-r\\ m+1-r, 2N+b+c+d-r-s\end{matrix}; 1\right] = \frac{(1-c-m)_s}{(1-2N-b-c-d+r)_s} {}_3F_2\left[\begin{matrix}-s, m+n-N, m+1-N-n-b-d\\ m+1-r, c+m-s\end{matrix}; 1\right].$$

Using this in (2.6) and simplifying, we get

(2.7)

$$\lim_{a\to\infty} G_{m,n}(x,y) = \frac{(-N)_n(b+c+d+2y, d+y)_{N-y}}{(-y)_n(c+d+2y, b+d+y+n)_{N-y}} \times \sum_i \sum_j \sum_k \frac{(m+n-N, m+1-N-n-b-d)_k(-x)_{j+k}(1-c-m)_j}{i!\,j!k!(y-N, 1-N-y-c-d)_{j+k}} \times \frac{(-m)_{i+j}(y-N, 1-N-y-c-d)_{i+j+k}}{(-N, 1-d-N)_{i+j+k}}(-1)^j.$$

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The sum over i above is

$$
{}_3F_2\left[\begin{matrix} j-m,\, j+k+y-N,\, j+k+1-N-y-c-d \\ j+k-N,\, j+k+1-d-N \end{matrix};1\right] = \frac{(c)_{m-j}}{(j+k+1-d-N)_{m-j}}\, {}_3F_2\left[\begin{matrix} j-m,\, -y,\, y+c+d-1 \\ j+k-N,\, c \end{matrix};1\right],
$$

by [10, (3.2.8)]. The triple sum in (2.7) becomes

$$
\begin{aligned}
&\frac{(c)_m}{(1-d-N)_m}\sum_i\sum_k \frac{(m+n-N,\, 1+m-N-n-b-d,\, -x)_k}{i!k!(1-d-N+m)_k(-N)_{i+k}} \\
&\times \frac{(-m,\, -y,\, y+c+d-1)_i}{(c)_i}\, {}_2F_1\left[\begin{matrix} k-x,\, i-m \\ i+k-N \end{matrix};1\right] \\
=&\frac{(c,\, x-N)_m(b)_x}{(-N)_m(1-d-N)_{m+x}}\, {}_3F_2\left[\begin{matrix} -x,\, -n,\, n+b+d-1 \\ b,\, m-N \end{matrix};1\right] \\
&{}_3F_2\left[\begin{matrix} -m,\, -y,\, y+c+d-1 \\ c,\, x-N \end{matrix};1\right],
\end{aligned}
$$

which is obtained by summing the ${}_2F_1$ series, then using [10, (3.2.8)] once again, followed by some simplification. This completes the proof of (1.13). ■

3 Proof of (1.16)

By (A.1) and (A.5)

$$
\begin{aligned}
&W_\ell(x; b, cdq^{2y}, a, N-y|q)\, W_{y-n+\ell}(m; c, bdq^{2n}, a, N-n|q) \\
&= \frac{(a, q^{1-N-y}/cd)_x}{(b, abcdq^{N+y-1})_x}\frac{(a, q^{1-N-n}/bd)_m}{(c, abcdq^{N+n-1})_m}(bcdq^{N+y-1})^x(bcdq^{N+n-1})^m \\
&\times \sum_r \frac{(q^{y-N+\ell}, q^{1-N-y-\ell}/bcd, q^{-m}, acq^{m-1})_r}{(q, q^{n-N}, a, q^{1-N-n}/bd)_r} q^r \\
&\times \sum_s \frac{(q^{-r}, q^{1-2N+r}/bcd, q^{-x}, abq^{x-1})_s}{(q, q^{y-N}, a, q^{1-N-y}/cd)_s} q^s \\
&\times\ {}_4\phi_3\left[\begin{matrix} q^{-s}, q^{1-r}/a, q^{N-n+1-r}, bdq^{N+n-r} \\ bcdq^{2N-r-s}, q^{m+1-r}, q^{2-m-r}/ac \end{matrix}; q, q\right].
\end{aligned}
\tag{3.1}
$$

Substituting (3.1) into (1.14) and (1.15) we find that

$$
R^\tau_{m,n}(x, y|q) = B_{m,n}(x, y|q) S^\tau_{m,n}(x, y|q), \tag{3.2}
$$

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where

$$(3.3)\quad B_{m,n}(x,y|q)=\frac{(q)_N(a)_{N-y}(bcd)_{2y}(q^{-y})_n q^{n(y-N)+\binom{n}{2}}}{(abcdq^{-1})_{N+n}(bcd)_{N+y}(cd)_{2y}(bd)_{n+y}(q^{-N})_n}$$
$$\times\left\{\frac{1-abq^{2x-1}}{1-abq^{-1}}\frac{(abq^{-1},a,q^{1-N-y}/cd,q^{y-N})_x}{(q,b,abcdq^{N+y-1},abq^{N-y})_x}(bcdq^{2N-1})^x\right.$$
$$\times\frac{(abcdq^{-1},cd)_{N+y}(c,d)_y}{(q,ab)_{N-y}}\frac{1-cdq^{2y-1}}{1-cdq^{-1}}\frac{(cdq^{-1})_y}{(q)_y}(bc)^{N-y}$$
$$\times\frac{1-acq^{2m-1}}{1-acq^{-1}}\frac{(acq^{-1},a,q^{1-N-n}/bd,q^{n-N})_m}{(q,c,abcdq^{N+n-1},acq^{N-n})_m}(bcdq^{2N-1})^m$$
$$\left.\times\frac{(abcdq^{-1},bd)_{N+n}}{(q,ac)_{N-n}}\frac{1-bdq^{2n-1}}{1-bdq^{-1}}\frac{(bdq^{-1},b)_n}{(q,d)_n}d^n\right\}^{1/2},$$

and

$$S^{\tau}_{m,n}(x,y|q)=\sum_r\frac{(q^{-m},acq^{m-1})_r q^r}{(q,q^{n-N},a,q^{1-N-n}/bd)_r}$$
$$\times\sum_s\frac{(q^{-r},q^{r+1-2N}/bcd,q^{-x},abq^{x-1})_s}{(q,q^{y-N},a,q^{1-N-y}/cd)_s}q^s$$
$$\times\ {}_4\phi_3\left[\begin{matrix}q^{-s},q^{N-n+1-r},bdq^{N+n-r},q^{1-r}/a\\ bcdq^{2N-r-s},q^{m+1-r},q^{2-m-r}/ac\end{matrix};q,q\right]$$
$$\times\sum_k\frac{(q^{-n},q^{1-n}/d)_k q^k}{(q,b,q^{y-n+1},cq^{y-n})_k}T_{r,s,k},$$

with

$$T_{r,s,k}=\frac{(bcdq^{2y})_{2k}(abcdq^{N+y-1},b,cq^{y-n},\tau q^y\sqrt{bcd},q^{y-N+r})_k}{(q^{1+y-N}/a,cdq^{2y},bdq^{y+n},\tau^{-1}\sqrt{bcd}q^y,bcdq^{N+y-r})_k}$$
$$\times(q^{y-N},q^{1-N-y}/bcd)_r\left[-(d/bc)^{1/2}q^{n-r}/a\tau\right]^k q^{-\binom{k}{2}}$$
$$\times\ {}_8W_7\Big(bcdq^{2k+2y-1};bq^k,cq^{y-n+k},abcdq^{N+y+k-1},\tau q^{y+k}\sqrt{bcd},$$
$$q^{k+r+y-N};q,(d/bc)^{1/2}q^{n+1-r-k}/a\tau\Big).$$

By using (A.2) and simplifying the coefficients we get

$$(3.4)\quad \sum_k\frac{(q^{-n},q^{1-n}/d)_k q^k}{(q,b,q^{y-n+1},cq^{y-n})_k}T_{r,s,k}$$
$$=\frac{(q^{y-N},q^{1-N-y}/bcd)_r(bcdq^{2y},dq^{y+n},abdq^{N+y-1},\tau q^y\sqrt{bcd})_{N-y-r}}{(cdq^{2y},bdq^{y+n},q^{1+y-N}/a,\tau^{-1}q^y\sqrt{bcd})_{N-y-r}}$$

$$\times \left(\frac{q^{1-N}}{a\tau\sqrt{bcd}}\right)^{N-y-r} \sum_k \frac{(q^{-n}, q^{1-n}/d, q^{r+y-N})_k}{(q, q^{y-n+1}, q^{1-N-n-r}/d)_k} q^k$$

$$\times\ {}_4\phi_3 \left[\begin{matrix} q^{r+k+y-N}, \frac{q^{1-N}}{a\tau\sqrt{bcd}}, q^{1-N-n+r}/bd, q^{1-N-y+r}/cd \\ q^{1-N-n+r+k}/d, q^{2+r-2N}/abcd, q^{1-N+r}/\tau\sqrt{bcd} \end{matrix}; q, q\right].$$

However, since

$$\sum_k \frac{(q^{-n}, q^{1-n}/d, q^{r+y-N})_k q^k}{(q, q^{y-n+1}, q^{1-N-n+r}/d)_k} \frac{(q^{r+k+y-N})_j}{(q^{1-N-n+r+k}/d)_j}$$
$$= \frac{(q^{r+y-N})_j}{(q^{1-N-n+r}/d)_j}\ {}_3\phi_2 \left[\begin{matrix} q^{-n}, q^{1-n}/d, q^{j+r+y-N} \\ q^{y-n+1}, q^{1-N-n+r+j}/d \end{matrix}; q, q\right]$$
$$= \frac{(q^{-N}, q^{1-y-n}/d)_n}{(q^{-y})_n} \frac{(q^{r+y-N}, q^{n-N})_{r+j}}{(q^{-N})_{r+j}(q^{1-N-n+r}/d)_{r+j}},$$

by [10, II.12], the lhs of (3.4) becomes, on simplification,

$$\frac{(dq^y, q^{-N})_n}{(dq^N, q^{-y})_n} q^{(N-y)n} \frac{(bcdq^{2y}, dq^{y+n}, abcdq^{N+y-1})_{N-y}}{(cdq^{2y}, bdq^{y+n}, q^{1+y-N}/a)_{N-y}} \left(\frac{q^{1-N}}{a\tau\sqrt{bcd}}\right)^{N-y}$$
$$\times \frac{(q^{y-N}, q^{1-N-y}/cd, q^{n-N}, q^{1-N-n}/bd, a)_r}{(q^{-N}, q^{1-N}/d, q^{2-2N}/abcd)_r} \left(\frac{q^{2-N}}{a\tau\sqrt{bcd}}\right)^r$$
$$\times\ {}_5\phi_4 \left[\begin{matrix} q^{r+y-N}, q^{1-N-y+r}/cd, q^{n-N+r}, q^{1-N-n+r}/bd, q^{1-N}/a\tau\sqrt{bcd} \\ q^{r-N}, q^{1-N+r}/d, q^{2-2N+r}/abcd, q^{1-N+r}/\tau\sqrt{bcd} \end{matrix}; q, q\right].$$

Therefore,

(3.5)

$$S^{\tau}_{m,n}(x, y|q)$$
$$= \frac{(d)_N (q^{-N})_n}{(d)_y (q^{-y})_n} q^{(N-y)n} \frac{(bcdq^{2y}, abcdq^{N+y+1})_{N-y}}{(cdq^{2y}, bdq^{y+n})_{N-y}(a)_{N-y}} \left(-\frac{q^{1-N}}{\tau\sqrt{bcd}}\right)^{N-y} q^{\binom{N-y}{2}}$$
$$\times \sum_r \frac{(q^{-m}, acq^{m-1}, q^{y-N}, q^{1-N-y}/cd)_r}{(q, q^{-N}, q^{1-N}/d, q^{2-2N}/abcd)_r} \left(\frac{q^{1-N}}{a\tau\sqrt{bcd}}\right)^r$$
$$\times\ {}_5\phi_4 \left[\begin{matrix} q^{r+y-N}, q^{1-N-y+r}/cd, q^{r+n-N}, q^{1-N-n+r}/bd, q^{1-N}/a\tau\sqrt{bcd} \\ q^{r-N}, q^{r+1-N}/d, q^{2-2N+r}/abcd, q^{1-N+r}/\tau\sqrt{bcd} \end{matrix}; q, q\right]$$
$$\times \sum_s \frac{(q^{-r}, q^{r+1-2N}/bcd, q^{-x}, abq^{x-1})_s}{(q, a, q^{y-N}, q^{1-N-y}/cd)_s} q^s$$
$$\times\ {}_4\phi_3 \left[\begin{matrix} q^{-s}, q^{N-n+1-r}, bdq^{N+n-r}, q^{1-r}/a \\ bcdq^{2N-r-s}, q^{m+1-r}, q^{2-m-r}/ac \end{matrix}; q, q\right].$$

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Substituting this into (3.2) and (3.3) followed by some straightforward simplification leads to (1.16).

Note that (x, y) or (m, n) does not appear in any of the denominator factors in the above sum. If we take $\tau = -1$, take the $q \to 1$ limit, then specialize the parameters by using (1.12), we can easily derive (1.11) from above. The question is: can one derive a q-analogue of the Krawtchouk polynomials in (1.11) from (3.5)? It does not seem to be possible to take any sort of limit on the a, b, c, d to get this analogue. However, the only obvious thing would be to replace a, d by $-a$ and $-d$ respectively, then consider the parameters to be positive.

4 Alternate Form of $R^{\tau}_{m,n}(x, y|q)$

As we mentioned earlier, the form of $R^{\tau}_{m,n}(x, y|q)$ in (1.16) is not suitable for taking the limit $d \to 0$ or $d \to \infty$. So we need to proceed in a slightly different way. First we use (A.1) to obtain

$$
\begin{aligned}
W_\ell(x; b, cdq^{2y}, a, N-y|q) &= \frac{(q^{2-2N}/abcd, q^{1+y-N}/b)_{N-x-y}}{(cdq^{2y}, q^{1+y-N}/a)_{N-x-y}} \Big(bcq^{N+y-1} \Big)^{N-x-y} \\
&\times \frac{(cdq^{2y}, q^{1+y-N}/a)_\ell}{(b, abcdq^{N+y-1})_\ell} \Big(abq^{N-y-1} \Big)^{\ell} \\
&\times W_{N-y-\ell} \left(N-x-y; q^{1+y-N}/b, q^{1-N-y}/cd, q^{1+y-N}/a, N-y|q \right).
\end{aligned}
$$

Substituting this into (1.14), then using (A.5) and carrying out the subsequent simplification we find a multiple series that has a part of the form

$$
\begin{aligned}
&\frac{(bcdq^{2y})_{2k}(cq^{y-n}, \tau q^y\sqrt{bcd}, q^{r+y-N})_k}{(bdq^{y+n}, \tau^{-1}q^y\sqrt{bcd}, bcdq^{N+y-r})_k} \left[-\Big(\frac{bd}{c}\Big)^{1/2} \frac{q^{N+n-y-r-1}}{\tau} \right]^k \\
&\times q^{-\binom{k}{2}} {}_6W_5 \left[bcdq^{2y+2k-1}; cq^{y-n+k}, \tau q^{y+k}\sqrt{bcd}, q^{r+k+y-N}; q, \Big(\frac{bd}{c}\Big)^{1/2} \frac{q^{N+n-r-k}}{\tau} \right],
\end{aligned}
$$

which, by virtue of [10, II.21], and simplification, gives

$$
\begin{aligned}
&\frac{(bcdq^{2y}, \tau^{-1}q^n\sqrt{bdc})_{N-y}}{(bdq^{y+n}, \tau^{-1}q^y\sqrt{bcd})_{N-y}} \frac{(cq^{y-n}, \tau q^y\sqrt{bcd}, q^{y+r-N})_k}{(\tau(c/bd)^{1/2}q^{1-N-n+y+r})_k} \\
&\qquad \times \frac{(q^{1-N-n}/bd, \tau q^{1-N}/\sqrt{bcd})_r}{(q^{1-N-y}/bcd, \tau(c/bd)^{1/2}q^{1-N-n+y})_r}.
\end{aligned}
$$

So the series over k in (1.14) becomes

$$
\frac{(bcdq^{2y}, \tau^{-1}q^n\sqrt{bdc})_{N-y}}{(bdq^{y+n}, \tau^{-1}q^y\sqrt{bcd})_{N-y}} \frac{(q^{1-N-n}/bd, \tau q^{1-N}/\sqrt{bcd})_r}{(q^{1-N-y}/bcd, \tau q^{1-N-n+y}\sqrt{c/bd})_r} \times
$$

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$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{r+y-N}, q^{1-n}/d, \tau q^y\sqrt{bcd} \\ b, q^{y-n+1}, \tau q^{1-N-n+y+r}\sqrt{c/bd}\end{matrix}; q, q\right]$$

$$= \frac{(bcdq^{2y}, \tau^{-1}q^n\sqrt{bd/c})_{N-y}}{(bdq^{y+n}, \tau^{-1}q^y\sqrt{bcd})_{N-y}} \frac{(q^{-N}, d, \tau\sqrt{bcd})_n}{(q^{-y}, b, \tau^{-1}q^{N-y}\sqrt{bd/c})_n}(\tau^{-1}\sqrt{b/cd}\, q^{N-y})^n$$

$$\times \frac{(q^{n-N}, q^{1-N-n}/bd, \tau q^{1-N}/\sqrt{bcd})_r}{(q^{-N}, q^{1-N-y}/bcd, \tau q^{1-N+y}\sqrt{c/bd})_r}$$

$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-n}, bdq^{n-1}, q^{-y}, \tau q^{y+r-N}\sqrt{cd/b} \\ q^{r-N}, d, \tau\sqrt{bcd}\end{matrix}; q, q\right],$$

which is obtained by applying (A.1) twice on the ${}_4\phi_3$ series on the lhs. Collecting all contributing terms and series we find an alternate expression for $R^\tau_{m,n}(x, y|q)$:

(4.1)

$$\begin{aligned} R^\tau_{m,n}(x, y|q) = \Bigg\{ & \frac{1-abq^{2x-1}}{1-abq^{-1}} \frac{(abq^{-1}, a, q^{1-N-y}/cd, q^{y-N})_x}{(q, b, abcdq^{N+y-1}, abq^{N-y})_x}(bcdq^{2N-1})^x \\ & \times \frac{(c,d)_y (q)_N (abcdq^{-1})_{2N}}{(abcdq^{-1}, cd)_{N+y}(q, ab)_{N-y}} \frac{1-cdq^{2y-1}}{1-cdq^{-1}} \frac{(cdq^{-1})_y}{(q)_y}(c/b)^{N-y} \\ & \times \frac{1-acq^{2m-1}}{1-acq^{-1}} \frac{(acq^{-1}, a, q^{1-N-n}/bd, q^{n-N})_m}{(q, c, abcdq^{N+n-1}, acq^{N-n})_m}(bcdq^{2N-1})^m \\ & \times \frac{(q)_N (abcdq^{-1})_{2N}}{(abcdq^{-1}, bd)_{N+n}(q, ac)_{N-n}} \frac{1-bdq^{2n-1}}{1-bdq^{-1}} \frac{(bdq^{-1}, d)_n}{(q,q)_n}(b/c)^n \Bigg\}^{1/2} \\ & \times \frac{(b, \tau^{-1}\sqrt{bd/c})_{N-y}}{(\tau^{-1}q^y\sqrt{bcd})_{N-y}}(-q)^{N-y} \frac{(\tau\sqrt{bcd})_n}{(\tau^{-1}\sqrt{bd/c})_n}\tau^{-n}q^{\binom{n}{2}} \\ & \times \sum_r \frac{(q^{x+y-N}, q^{1-N-x+y}/ab, q^{n-N}, q^{1-N-n}/bd, \tau q^{1-N}/\sqrt{bcd})_r}{(q, q^{-N}, q^{1+y-N}/b, q^{2-2N}/abcd, \tau\sqrt{c/bd}q^{1-N+y})_r} q^r \\ & \times \sum_s \frac{(q^{-r}, q^{r+1-2N}/bcd, q^{-m}, acq^{m-1})_s}{(q, q^{n-N}, a, q^{1-N-n}/bd)_s} q^s \\ & \times\ {}_4\phi_3\left[\begin{matrix} q^{-s}, q^{N-y+1-r}, bq^{N-y-r}, abcdq^{2N-r-1} \\ bcdq^{2N-r-s}, q^{N+1-x-y-r}, abq^{N+x-y-r}\end{matrix}; q, q\right] \\ & \times\ {}_4\phi_3\left[\begin{matrix} q^{-n}, bdq^{n-1}, q^{-y}, \tau q^{y+r-N}\sqrt{cd/b} \\ q^{r-N}, d, \tau\sqrt{bcd}\end{matrix}; q, q\right]. \end{aligned}$$

Because of the term $\left(\tau\sqrt{c/bd}q^{1-N+y}\right)_r$ in the denominator, the quadruple series above is not clearly a polynomial in y, but it has the right form for the limits $d \to 0$ or $d \to \infty$.

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5 Limit of $R^{\tau}_{m,n}(x,y|q)$ as $d \to 0$ or $d \to \infty$

We need to consider only one of the limits, since the other is going to be almost the same. From (4.1) it follows that

(5.1)
$$\lim_{d\to 0} R^{\tau}_{m,n}(x,y|q) = C_{m,n}(x,y|q) \sum_r \frac{(q^{x+y-N}, q^{1-x-N+y}/ab, q^{y+n-N})_r}{(q, q^{y-N}, q^{1+y-N}/b)_r} (aq^{N-n-y})^r$$
$$\times \sum_s \frac{(q^{-r}, q^{-m}, acq^{m-1})_s}{(q, a, q^{n-N})_s} (q^{1-N+n+r}/c)^s$$
$$\times\ {}_3\phi_2 \left[\begin{matrix} q^{-s}, q^{N-y+1-r}, bq^{N-y-r} \\ abq^{N+x-y-r}, q^{N+1-x-y-r} \end{matrix}; q, q\right],$$

with

(5.2)
$$C_{m,n}(x,y|q) = \left\{ \frac{1-abq^{2x-1}}{1-abq^{-1}} \frac{(abq^{-1}, a, q^{y-N})_x}{(q, b, abq^{N-y})_x} (-bq^{N-y})^x q^{\binom{x}{2}}\right.$$
$$\times \frac{(q)_N (c)_y (c/b)^{N-y}}{(q, ab)_{N-y} (q)_y} \frac{1-acq^{2m-1}}{1-acq^{-1}} \frac{(acq^{-1}, a, q^{n-N})_m}{(q, c, acq^{N-n})_m} (-cq^{N-n})^m q^{\binom{m}{2}}$$
$$\left.\times \frac{(q)_N (b/c)^n}{(q, ac)_{N-n} (q,q)_n} \right\}^{1/2} \frac{(b)_{N-y} (q^{y-N})_n}{(q^{-N})_n} (-q)^{N-y} q^{\binom{n}{2}-yn},$$

since

$$\lim_{d\to 0} {}_4\phi_3 \left[\begin{matrix} q^{-n}, bdq^{n-1}, q^{-y}, \tau q^{y+r-N}\sqrt{cd/b} \\ q^{r-N}, d, \tau\sqrt{bcd} \end{matrix}; q, q\right] = {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{-y} \\ q^{r-N} \end{matrix}; q, q\right]$$
$$= \frac{(q^{y+r-N})_n}{(q^{r-N})_n} q^{-yn}, \text{ by [10, II.6]}.$$

By [10, (3.2.2) and (3.2.6)],

$${}_3\phi_2 \left[\begin{matrix} q^{-s}, q^{N-y+1-r}, bq^{N-y-r} \\ abq^{N+x-y-r}, q^{N+1-x-y-r} \end{matrix}; q, q\right] = \frac{(a, q^{N-y+1-r})_s}{(q^{N+1-x-y-r}, abq^{N+x-y-r})_s} (bq^{N-y-r})^s$$
$$\times\ {}_3\phi_2 \left[\begin{matrix} q^{-s}, q^{-x}, abq^{x-1} \\ a, q^{y-N+r-s} \end{matrix}; q, q^{1+y-N+r}/b\right].$$

So the triple sum in (5.1) can be written as

$$\text{(5.3)} \qquad \sum_s \sum_t \frac{(q^{-m}, acq^{m-1}, q^{y+n-N})_s (q^{-s}, q^{-x}, abq^{x-1})_t}{(q, q^{n-N}, q^{1+y-N}/b)_s (q, a, q^{y-N})_t} (q^{1-y}/c)^s$$

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$$\times\,(q^{1-N+y+s}/b)^t\, {}_3\phi_2\left[\begin{matrix} q^{s+y+n-N}, q^{x+y-N}q^{1-x-N+y}/ab \\ q^{t+y-N}, q^{1+y-N+s}/b \end{matrix}; q, aq^{N-n-y+t}\right].$$

By [10, (2.2.7)] the ${}_3\phi_2$ series in (5.3) transforms to

$$\frac{(a)_{N-y-n-s}}{(q^{1+y-N+s}/b)_{N-y-n-s}}\, {}_3\phi_2\left[\begin{matrix} q^{s+y+n-N}, q^{t-x}, abq^{t+x-1} \\ q^{t+y-N}, aq^{s+t} \end{matrix}; q, q^{1-n}/b\right] \tag{5.4}$$

Substituting (5.4) in (5.3) gives

$$\begin{aligned}
&\frac{(a)_{N-y-n}}{(q^{1+y-N}/b)_{N-y-n}} \sum_s \sum_s \sum_t \frac{(q^{n+y-N})_{r+s}(q^{-x}, abq^{x-1})_{r+t}(q^{-m}, acq^{m-1})_s}{(q)_r(q, q^{n-N})_s(a)_{r+s+t}(q^{y-N})_{r+t}} \\
&\quad\times \frac{(q^{-s}, aq^{N-y-n})_t}{(q, a)_t}(q^{1-n}/b)^r(q^{1-y}/c)^s(q^{1-N+y+s}/b)^t \\
&= \frac{(a)_{N-y-n}(b)_n}{(b)_{N-y}}(-b)^{N-y-n}q^{\binom{N-y}{2}-\binom{n}{2}} \\
&\quad\times \sum_s \sum_j \frac{(q^{n+y-N})_{s+j}(q^{-x}, abq^{x-1})_j(q^{-m}, acq^{m-1})_s}{(q, q^{y-N})_j(a)_{s+j}(q, q^{n-N})_s} \\
&\quad\times (q^{1-n}/b)^j(q^{1-y}/c)^s\, {}_3\phi_2\left[\begin{matrix} q^{-s}, q^{-j}, aq^{N-n-y} \\ , q^{N+1-y-n-s-j} \end{matrix}; q, q\right],
\end{aligned} \tag{5.5}$$

which we obtain by setting $r + t = j$. Note that the ${}_3\phi_2$ series here is balanced and so, by [10, II.12], has the sum

$$\frac{(aq^s, q^{y+n-N})_j}{(a, q^{n+y-N+s})_j},$$

which, on substitution, leads to the following product for (5.5)

$$\begin{aligned}
&\frac{(b)_n(a)_{N-y-n}}{(b)_{N-y}}(-b)^{N-y-n}q^{\binom{N-y}{2}-\binom{n}{2}} \\
&\times\, {}_3\phi_2\left[\begin{matrix} q^{-x}, abq^{x-1}, q^{n+y-N} \\ a, q^{y-N} \end{matrix}; q, q^{1-n}/b\right] {}_3\phi_2\left[\begin{matrix} q^{-m}, acq^{m-1}, q^{n+y-N} \\ a, q^{n-N} \end{matrix}; q, q^{1-y}/c\right] \\
&= \frac{(b)_n(c)_m(b)_x(a)_{N-y-n}}{(b)_{N-y}(a)_m(a)_x}(-b)^{N-x-y-n}(-c)^{-m}q^{\binom{N-y}{2}-\binom{x}{2}-\binom{m}{2}-\binom{n}{2}} \\
&\times\, {}_3\phi_2\left[\begin{matrix} q^{-x}, abq^{x-1}, q^{-n} \\ b, q^{y-N} \end{matrix}; q, q\right] {}_3\phi_2\left[\begin{matrix} q^{-m}, acq^{m-1}, q^{-y} \\ c, q^{n-N} \end{matrix}; q, q\right]
\end{aligned}$$

by [10, (3.2.5)]. This, combined with (5.2), yields (1.17).

6 Remarks

Unlike the $q \to 1$ case dealt with in Section 2, there does not seem to be any factorization possible in the limit $a \to \infty$ or 0 in the $q \neq 1$ case. One can see why it is not likely to be so by setting $m = n = 0$ in (4.1). The quadruple series collapses to

$$ {}_4\phi_3 \left[\begin{matrix} q^{x+y-N}, q^{1-N-x+y}/ab, q^{1-N}/bd, \tau q^{1-N}/\sqrt{bcd} \\ q^{1+y-N}/b, q^{2-2N}/abcd, \tau\sqrt{c/bd}q^{1-N+y} \end{matrix} ; q, q \right] $$

This is balanced and terminating, but in the limit $a \to 0$ or ∞ the resulting ${}_3\phi_2$ series is no longer balanced and hence cannot be summed. In the $q \to 1$, however, with $\tau = -1$, it becomes a ${}_2F_1$ series with argument 1, and hence summable. Note that in both $d \to 0$ and $d \to \infty$ cases, the above ${}_4\phi_3$ series becomes a summable ${}_2\phi_1$ series.

One may ask: how about the limits b and/or $c \to 0$ or ∞? In either case the limit of the weight function does not exist, so these limits are not permissible.

A Appendix

Two of the most important formulas in the theory of basic hypergeometric series that are frequently used in this paper are Sears' transformation formula

$$ \text{(A.1)} \quad {}_4\phi_3 = \left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix} ; q, q \right] = \frac{(e/a, f/a)_n}{(e, f)_n} a^n \, {}_4\phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix} ; q, q \right], $$

$defq^{n-1} = abc$, that transforms one terminating balanced ${}_4\phi_3$ series into another, and the terminating form of Watson's transformation formula

$$ \text{(A.2)} \quad {}_8W_7\left(a; b, c, d, e, q^{-n}; q, a^2q^{n+2}/bcde\right) = \frac{(aq/aq/de)_n}{(aq/d, aq/e)_n} \, {}_4\phi_3 \left[\begin{matrix} q^{-n}, d, e, aq/bc \\ aq/b, aq/c, deq^{-n}/a \end{matrix} ; q, q \right], $$

that connects a very-well-poised ${}_8\phi_7$ series to a balanced ${}_4\phi_3$ series. The W-notation used in (A.2) for an ${}_8\phi_7$ series is in keeping with the one adopted in [10] for notational economy. See [10] for these notations and formulas. In the two formulas above as well as the others throughout this paper we have used the abbreviated symbol $(a)_n$ to mean the q-shifted factorial $(a; q)_n$. The same symbol is also used for the ordinary shifted factorials in Section 2 where we consider the $q \to 1$ cases.

For the sake of quick reference we lift the following product formula for 2 balanced and terminating ${}_4\phi_3$ series from [10, (8.2.5.)];

$$ \text{(A.3)} \qquad {}_4\phi_3 \left[\begin{matrix} q^{-n}, aq^n, b_1, b_2 \\ b, b_3, qab_1b_2/bb_3 \end{matrix} ; q, q \right] \, {}_4\phi_3 \left[\begin{matrix} q^{-n}, aq^n, c_1, c_2 \\ c, c_3, qac_1c_2/cc_3 \end{matrix} ; q, q \right] = $$

$$= \sum_{k=0}^{n} \frac{(q^{-n}, aq^n, c_1, c_2, qab_1/bb_3, qab_2/bb_3)_k}{(q, c, c_3, qa/bb_3, qac_1c_2/cc_3, qab_1b_2/bb_3)_k} q^k$$
$$\times \sum_{j=0}^{k} \frac{1 - bb_3q^{2j-k-1}/a}{1 - bb_3q^{-k-1}/a} \frac{(bb_3q^{-k-1}/a, b_1, b_2, bq^{-k}/a, b_3q^{-k}/a, q^{-k})_j}{(q, bb_3q^{-k}/ab_1, bb_3q^{-k}/ab_2, b_3, b, bb_3/a)_j} \left(\frac{bb_3q^k}{b_1b_2}\right)^j$$
$$\times \ {}_5\phi_4 \left[\begin{matrix} q^{-j}, bb_3q^{j-k-1}/a, q^{1-k}/c, q^{1-k}/c_3, cc_3q^{-k}/ac_1c_2 \\ bq^{-k}/a, b_3q^{-k}/a, q^{1-k}/c_1, q^{1-k}/c_2 \end{matrix}; q, q\right].$$

Unfortunately, the symmetry of the lhs in (b_1, b_2, b, b_3) and (c_1, c_2, c, c_3) is not obvious in the rhs. Also, there is no known transformation formula between a general balanced ${}_5\phi_4$ series and another single series. So (A.3) is not very useful. Note, however, that the inner series in j on the rhs can be reversed and simplified as

$$\text{(A.4)} \quad \sum_{j=0}^{k} \frac{(q^{-k}, q^{1-k}/c, q^{1-k}/c_3, cc_3q^{-k}/ac_1c_2, b_1, b_2)_j}{(q, b, b_3, bb_3/a, q^{1-k}/c_1, q^{1-k}/c_2, bb_3q^{-k}/ab_1, bb_3q^{-k}/ab_2)_j}$$
$$\times (bb_3q^{-k}/a)_{2j} \Big(-\frac{bb_3q^{k-1}}{b_1b_2}\Big)^j q^{-\binom{j}{2}}$$
$$\times \ {}_8W_7 \left(bb_3q^{2j-k-1}/a; b_1q^j, b_2q^j, bq^{j-k}/a, b_3q^{j-k}/a, q^{j-k}; q, \frac{bb_3q^{k-j}}{b_1b_2}\right).$$

By using (A.2) we find that this ${}_8W_7$ series transforms to

$$\frac{(bb_3q^{2j-k}/a, bb_3q^{-k}/ab_1b_2)_{k-j}}{(bb_3q^{j-k}/ab_1, bb_3q^{j-k}/ab_2)_{k-j}} \times \ {}_4\phi_3 \left[\begin{matrix} q^{j-k}, aq^k, b_1q^j, b_2q^j \\ bq^j, b_3q^j, ab_1b_2q^{j+1}/bb_3 \end{matrix}; q, q\right],$$

which, upon substitution in (A.4), leads to the following double sum

$$\frac{(qa/bb_3, qab_1b_2/bb_3)_k}{(qab_1/bb_3, qab_2/bb_3)_k}$$
$$\times \sum_j \sum_\ell \frac{(q^{-k}, b_1, b_2)_{j+\ell}(aq^k)_j(q^{1-k}/c, q^{1-k}/c_3, cc_3q^{-k}/ac_1c_2)_\ell}{(b, b_3, qab_1b_2/bb_3)_{j+\ell}(q)_j(q, q^{1-k}/c_1, q^{1-k}/c_2)_\ell}$$
$$\times q^j(aq^{k+1})^\ell$$
$$= \frac{(qa/bb_3, qab_1b_2/bb_3)_k}{(qab_1/bb_3, qab_2/bb_3)_k} \sum_j \frac{(q^{-k}, aq^k, b_1, b_2)_j}{(q, b, b_3, qab_1b_2/bb_3)_j} q^j$$
$$\times \ {}_4\phi_3 \left[\begin{matrix} q^{-j}, q^{1-k}/c, q^{1-k}/c_3, cc_3q^{-k}/ac_1c_2 \\ q^{1-k}/c_1, q^{1-k}/c_2, q^{1-k-j}/a \end{matrix}; q, q\right].$$

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Hence we obtain a much more useful formula:

(A.5)
$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b_1, b_2 \\ b, b_3, qab_1b_2/bb_3 \end{matrix}; q, q\right] {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, c_1, c_2 \\ c, c_3, qac_1c_2/cc_3 \end{matrix}; q, q\right] \\
&= \sum_{k=0}^{n} \frac{(q^{-n}, aq^n, b_1, b_2)_k}{(q, b, b_3, qab_1b_2/bb_3)_k} q^k \\
&\times \sum_{j=0}^{k} \frac{(q^{-k}, aq^k, c_1, c_2)_j}{(q, c, c_3, qac_1c_2/cc_3)_j} q^j \\
&\times {}_4\phi_3\left[\begin{matrix} q^{-j}, q^{1-k}/b, q^{1-k}/b_3, bb_3q^{-k}/ab_1b_2 \\ q^{1-k}/b_1, q^{1-k}/b_2, q^{1-k-j}/a \end{matrix}; q, q\right].
\end{aligned}
$$

Acknowledgment I am grateful to Hjalmar Rosengren for bringing to my attention the important references [1–3]. Gratitude is also due to the anonymous referee whose very valuable remarks and suggestions have helped improve the quality of the paper. Thanks to him/her also for pointing out the references [14, 17].

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