ANCESTRAL RINGS

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(Received 28th June 1965)

A ring R is said to be a *P-ancestral* ring if all proper non-zero sub-rings of R have property P. If P is the property that every proper non-zero sub-ring of R is a (two-sided) ideal then the ring $\mathbf{Z}$ of rational integers furnishes an example of a P-ancestral ring.

If S is a sub-ring of R we define the *left-idealizer* of S, written $I(S)$, by $I(S) = \{x \in R\colon\ xs \in S \text{ for } s \in S\}$. Clearly $I(S)$ is the largest sub-ring of R in which S is a left ideal and $I(S) = R$ if and only if S is a left ideal of R. With obvious changes we may consider *right-idealizer* and (two-sided) idealizer. We assume R has a unit denoted by 1.

Our theorems relate conditions of P-ancestral types to conditions of left-idealizers.

Let S and T be sub-rings of a ring R. Then the following results are immediate:

(i) $1 \in I(S)$, (ii) $S \subseteq I(S)$,

(iii) $I(S) \subseteq I(I(S))$, (iv) $I(S) \cap I(T) \subseteq I(S \cap T)$,

(v) $I(T \cup S) \subseteq I(T) \cup I(S)$, (vi) $I(S) \subseteq I(S^2)$.

Let D be the ring of all two by two matrices over $\mathbf{Z}$ and let

$$K = \left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbf{Z}\right\}, \qquad S = \left\{\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} : x,\ y \in \mathbf{Z}\right\},$$

$$T = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,\ y \in \mathbf{Z}\right\}.$$

Then $K \subset S$ and $I(K) \subset I(S)$ properly. Now $S^2 \subseteq S$ always and by (vi) $I(S) \subseteq I(S^2)$. These observations show that knowing the relation between the sub-rings we may still not conclude the direction in which the inclusion relation will go for the left-idealizers. Also in D, $I(T) \cap I(K) \subset I(T \cap K)$ properly and

$$I(T \cup S) \subset I(T) \cup I(S)$$

properly. This shows that (iv) and (v) are the best possible results.

Lemma. *Let S be a non-zero sub-ring of R. Then $I(S) = S$ if and only if $1 \in S$.*

Proof. In general $1 \in I(S)$ and $S \subseteq I(S)$. Thus $I(S) = S$ implies $1 \in S$. Conversely if $1 \in S$ and if $x \in I(S)$ then $x = x1 \in S$ and so $I(S) \subseteq S$, thus $I(S) = S$.

Theorem 1. *The following assertions about a ring R are equivalent.*

(1) *For all non-zero sub-rings S of R, I(S) = S.*

(2) *R and all non-zero sub-rings of R are division rings.*

(3) *R and all non-zero sub-rings of R are division rings and R has prime characteristic.*

(4) *R is a field in which every element has finite order and which is an algebraic extension of the prime field.*

Proof. (1)⇒(2). Let S be a non-zero sub-ring of R. Since $I(S) = S$ it follows from the Lemma that $1 \in S$. Let L be a non-zero left ideal of S. Then $L = I(L) \supseteq S$ and hence $L = S$. Thus S has a unit and no proper left ideals. Thus S is a division ring.

(2)⇒(3). If R has characteristic zero then R has a proper sub-field isomorphic to the rational field Q and thus R has a proper sub-ring isomorphic to Z. Since Z is not a division ring we obtain a contradiction and so R has prime characteristic.

(3)⇒(4). Let S be a non-zero sub-ring of R. Since S is a division sub-ring $1 \in S$. In particular if S is the sub-ring generated by a non-zero element $a \in R$, S consists of polynomials in a over the prime field of R. Since $a^{-1} \in S$, a^{-1} is a polynomial in a. Thus a satisfies an algebraic equation over the prime field of R. Hence S is a finite field. Thus $a^{n(a)} = a$ where $n(a)$ is the number of elements in S and thus by Jacobson [(**1**), theorem 1, p. 217] R is commutative. Hence R is a field and, as shown above, R is an algebraic extension of the prime field.

(4)⇒(1). Let S be a non-zero sub-ring of R. Let $x \in I(S)$ and let $s \in S$, $s \neq 0$. Then $xs = s' \in S$. But s has finite order and so for some integer $\rho > 0$, $s^\rho = 1$. Then $x = x1 = xs^\rho = xss^{\rho-1} = s's^{\rho-1} \in S$.

Hence $I(S) \subseteq S$ and thus $I(S) = S$.

We should remark that if we omit the assumption that R has a unit then R need not be a division ring for (1) to hold. Consider the ring A where

$$A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

matrix addition and multiplication being performed modulo 2. Then A is non-zero and for the only non-zero sub-ring of A, namely A itself, $I(A) = A$ trivially.

We should also observe that even if every proper non-zero sub-ring of a ring R is a division ring R need not be a division ring. A counter-example is provided by the ring

$$B = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$$

with componentwise addition and multiplication modulo 2.

Having dealt with the case of $I(S) = S$ for all S we now consider the opposite situation.

Theorem 2. *The following assertions about a ring R are equivalent.*

(1) *R is a homomorphic image of Z.*

(2) *Every sub-ring S of R is a left ideal.*

(3) *For every proper non-zero sub-ring S of R, $I(S) \neq S$.*

Proof. (1)$\Rightarrow$(2). Every sub-ring of Z is a left ideal and this property is preserved under homomorphism.

(2)$\Rightarrow$(3). This is obvious.

(3)$\Rightarrow$(1). Let $S = \{n1: n \in Z\}$. Then S is a non-zero sub-ring of R. Let $x \in I(S)$. Then $x = x1 \in S$ which implies that $I(S) \subseteq S$ and hence $I(S) = S$. This is only possible if $S = R$ and then R is a homomorphic image of Z.

Theorem 3. *Let R be a ring. Then for every proper non-zero sub-ring S of R there exists an integer n, depending on S, such that $I(S^n) = R$ if and only if for every proper non-zero sub-ring S of R $I(S) \neq S$.*

Proof. Let S be a proper non-zero sub-ring of R such that $I(S^n) = R$ for some integer n. If $I(S) = S$ we should have $1 \in S$ and thus $S^n = S$. Hence $S = I(S) = I(S^n) = R$ which is false. Thus $I(S) \neq S$.

Conversely if $I(S) \neq S$ for every proper non-zero sub-ring S of R, by Theorem 2, every sub-ring is a left ideal and so $I(S) = R$.

The author is grateful to the referee for many helpful suggestions.

REFERENCE

(1) N. Jacobson, *Structure of Rings* (A.M.S. Colloq. Pub. vol. 37, Providence, R.I., 1956).

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