On the Summation of $1^r + 2^r + 3^r + \dots + n^r$.

By J. A. DONALDSON.

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We shall use the notation $\Sigma n^r \equiv 1^r + 2^r + \dots + n^r$. Mr A. J. Gray suggested to me that $n(n+1)$ is always a factor of Σn^r, and that, in addition, $2n+1$ is a factor when r is even.

§1. That $n(n+1)$ is a factor of Σn^r is proved in Chrystal's Algebra I., chap. XX., §9, but the following proof seems somewhat simpler:—

Let $\Sigma n^r = f(n)$.

We know that $f(n)$ is an integral algebraic function of n.

When $n = 0$, $\Sigma n^r = 0$; *i.e.* $f(n) = 0$; therefore $f(n)$ contains the factor n.

Hence $f(n+1)$ contains the factor $(n+1)$.

Now $\Sigma(n+1)^r = f(n) + (n+1)^r = f(n+1)$,

and $f(n+1)$ and $(n+1)^r$ both contain the factor $(n+1)$;

$\therefore$ $f(n)$ contains the factor $(n+1)$;

i.e. Σn^r contains the factor $n(n+1)$.

§2. To prove that $(2n+1)$ is a factor when r is even, we must attempt to calculate the even sums without making use of the odd ones. In the identity

$$(x+1)^r - (x-1)^r \equiv 2rx^{r-1} + 2{}_rC_3x^{r-3} + \dots + 2{}_rC_2x^2 + 2,$$

which holds *when r is odd*, put successively $x = n$, $x = n-1$, $\dots x = 2$, $x = 1$, and we have

$$\begin{aligned}
(n+1)^r - (n-1)^r &= 2rn^{r-1} + \dots\dots\dots + 2{}_rC_2n^2 + 2 \\
n^r - (n-2)^r &= 2r(n-1)^{r-1} + \dots + 2{}_rC_2(n-1)^2 + 2 \\
(n-1)^r - (n-3)^r &= 2r(n-2)^{r-1} + \dots + 2{}_rC_2(n-2)^2 + 2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
3^r - 1^r &= 2r2^{r-1} + \dots\dots\dots\dots\dots + 2{}_rC_2 2^2 + 2 \\
2^r - 0 &= 2r1^{r-1} + \dots\dots\dots\dots\dots + 2{}_rC_2 1^2 + 2.
\end{aligned}$$

If we add these equations together, we have

$$(n+1)^r + n^r - 1 = 2r\Sigma n^{r-1} + \ldots + 2{}_r\text{C}_2\Sigma n^2 + 2n\,;$$

i.e. $$(n+1)^r + n^r - (2n+1) = 2r\Sigma n^{r-1} + \ldots + 2{}_r\text{C}_2\Sigma n^2.$$

When we put $n = -\frac{1}{2}$, the left-hand side of this last equation vanishes; hence, since Σn^2 is divisible by $2n+1$, we can prove successively that Σn^4, Σn^6, .. Σn^{2s} are all divisible by $2n+1$.

§3. When examining the above, I worked out the sums as far as Σn^7 and factorised them. The expressions for Σn^3, Σn^5, and Σn^7 contained the factor $n^2(n+1)^2$ and suggested $n^2(n+1)^2$ as a factor of Σn^r when r is odd, except in the case of $r=1$.

To prove this we have the identity

$$(x+1)^r - (x-1)^r \equiv 2rx^{r-1} + 2{}_r\text{C}_3 x^{r-3} + \ldots + 2{}_r\text{C}_3 x^3 + 2rx,$$

which holds *when r is even.*

When we give x the values $n, n-1, \ldots 2, 1$, and add the resulting identities, we have

$$(n+1)^r + n^r - 1 = 2r\Sigma n^{r-1} + \ldots\ldots + 2{}_r\text{C}_3\Sigma n^3 + 2r\Sigma n\,;$$

i.e. $$(n+1)^r + n^r - 1 - rn(n+1) = 2r\Sigma n^{r-1} + \ldots + 2{}_r\text{C}_3\Sigma n^3.$$

When we expand the left-hand side in powers of n, the terms below n^2 are absent; therefore n^2 is a factor of the left-hand side. Also when we put $n+1=m$ in the left side and expand in powers of m, the terms below m^2 are absent; *i.e.* m^2 or $(n+1)^2$ is a factor of the left side.

Now $n^2(n+1)^2$ is a factor of Σn^3; hence we can show successively that it is a factor of Σn^5, Σn^7 .. Σn^{2s+1}.

Hence we have proved that :—

Σn^r contains the factor $n(n+1)(2n+1)$ when r is even, and *the factor $n^2(n+1)^2$ when r is odd and greater than* 1.

§4. *Expression of Σn^r in powers of n.*

From the section in Chrystal's Algebra, referred to above, we learn that Σn^r is an integral function of n of the $(r+1)^{\text{th}}$ degree, say

$$\Sigma n^r = {}_ra_0 n^{r+1} + {}_ra_1 n^r + {}_ra_2 n^{r-1} + \ldots + {}_ra_{r-1} n^2 + {}_ra_r n. \qquad \text{I.}$$

Hence we have

$${}_ra_0 n^{r+1} + {}_ra_1 n^r + \ldots + {}_ra_r n + (n+1)^r = \Sigma(n+1)^r$$
$$= {}_ra_0(n+1)^{r+1} + {}_ra_1(n+1)^r + \ldots + {}_ra_r(n+1). \qquad \text{II.}$$

II. is an identity and true for all positive integral values of n, and so we may equate coefficients of like powers of n giving :—

n^r $\quad {}_ra_0 \times {}_{r+1}C_1 = 1, \ldots\ldots(1)$

n^{r-1} $\quad {}_ra_0 \times {}_{r+1}C_2 + {}_ra_1 \times {}_rC_1 = {}_rC_1, \ldots\ldots(2)$

n^{r-2} $\quad {}_ra_0 \times {}_{r+1}C_3 + {}_ra_1 \times {}_rC_2 + {}_ra_2 \times {}_{r-1}C_1 = {}_rC_2, \ldots\ldots(3)$

....

n^{r-s} $\quad {}_ra_0 \times {}_{r+1}C_{s+1} + {}_ra_1 \times {}_rC_s + \ldots + {}_ra_s \times {}_{r-s+1}C_1 = {}_rC_s, \ldots\ldots (s+1)$

....

n $\quad (r+1){}_ra_0 + r{}_ra_1 + (r-1){}_ra_2 + \ldots + 2{}_ra_{r-1} = r, \ldots\ldots(r)$

constant $\; {}_ra_0 + {}_ra_1 + {}_ra_2 + \ldots + {}_ra_{r-1} + {}_ra_r = 1. \ldots\ldots(r+1)$

The $(r+1)^{\text{th}}$ equation is also the expression of the fact that the identity I. holds when $n = 1$.

These equations enable us to calculate ${}_ra_0, {}_ra_1, \ldots {}_ra_r$ successively in terms of r.

Evidently ${}_ra_{r+1}$ is always equal to zero, since Σn^r is always divisible by n. We shall see that the equations $(1)\ldots(r+1)$ give us no information as to ${}_ra_{r+1}$.

From (1), ${}_ra_0 = \dfrac{1}{r+1}$.

From (2), $\dfrac{1}{r+1} \dfrac{(r+1)r}{2\,!} + r{}_ra_1 = r,$

giving $\qquad {}_ra_1 = \frac{1}{2}$.

From (3), $\dfrac{1}{r+1} \dfrac{(r+1)r(r-1)}{3\,!} + \frac{1}{2}\dfrac{r(r-1)}{2\,!} + (r-1){}_ra_2 = \dfrac{r(r-1)}{2\,!},$

which, *except when* $r = 1$, gives ${}_ra_2 = \frac{1}{6}\dfrac{r}{2\,!}$. When $r = 1$, ${}_ra_2$ is the term ${}_ra_{r+1}$, and this shows how these equations give us no information about ${}_ra_{r+1}$.

From (4), $\dfrac{1}{r+1} \dfrac{(r+1)\ldots(r-2)}{4\,!} + \frac{1}{2}\dfrac{r\ldots(r-2)}{3\,!} + \dfrac{r}{12}\dfrac{(r-1)(r-2)}{2\,!}$

$$+ (r-2){}_ra_3 = \frac{r(r-1)(r-2)}{3\,!},$$

which reduces to ${}_ra_3 = 0 \times \dfrac{r(r-1)}{3\,!}$.

From (5), $\frac{1}{r+1}\frac{(r+1)\dots(r-3)}{5\,!}+\frac{1}{2}\frac{r\dots(r-3)}{4\,!}+\frac{r}{12}\frac{(r-1)\dots(r-3)}{3\,!}$

$$+(r-3)_r a_4=\frac{r\dots(r-3)}{4\,!},$$

which, *except when* $r=3$, gives

$$_r a_4=-\tfrac{1}{30}\frac{r(r-1)(r-2)}{4\,!}.$$

From the above sample of calculation, and the form of equations, (1)...$(r+1)$, it is obvious that $_r a_s$ is of the form

$$a_s\times\frac{r(r-1)(r-2)\dots(r-s+2)}{s\,!}$$

where the constant a_s depends on s only.

If we go back now to equation I., when $r=3$, $_3a_3=0$ since Σn^3 contains the factor n^2.

Now $_r a_3=a_3\times\frac{r(r-1)}{3\,!}$, and when $r=3$, $\frac{r(r-1)}{3\,!}$ does not vanish; therefore $a_3=0$, as we saw above by calculation.

For the same reason, viz., that Σn^r is divisible by n^2 when r is odd and greater than 1, $a_5=a_7=a_9=\dots=0$.

Values of a are given below up to a_{18}.

s	a_s
2	1/6
4	-1/30
6	1/42
8	-1/30
10	5/66
12	-691/2730
14	7/6
16	-3617/510
18	43867/798

The values of a_s given above were not all calculated from the equations (1)...$(r+1)$ as before, but by giving r a particular value in a way which the following example will make clear :—

$$\Sigma n^9 = {}_9a_0n^{10} + {}_9a_1n^9 + {}_9a_2n^8 + {}_9a_4n^6 + {}_9a_6n^4 + {}_9a_8n^2. \qquad \text{III.}$$

Our object is to calculate a_8, knowing the others up to a_6.

Equation $(r+1)$ shows, when we put $r=9$, that

$${}_9a_0 + {}_9a_1 + {}_9a_2 + {}_9a_4 + {}_9a_6 + {}_9a_8 = 1.$$

Now $\quad {}_9a_0 + {}_9a_1 + \ldots + {}_9a_6 = \frac{1}{10} + \frac{1}{2} + \frac{3}{4} - \frac{7}{10} + \frac{1}{2} = \frac{23}{20}$;

$\therefore \quad {}_9a_8 = -\frac{3}{20}$.

Also $\quad {}_ra_8 = a_8\dfrac{r(r-1)\ldots(r-6)}{8\,!}$;

$\therefore \quad {}_9a_8 = a_8\dfrac{9\,.\,8\ldots3}{8\,!} = -\frac{3}{20}$;

whence $\quad a_8 = -\frac{1}{30}$.

We might also get a_8 by starting with Σn^8 for, putting $r=8$,

$$\Sigma n^8 = {}_8a_0n^9 + {}_8a_1n^8 + {}_8a_2n^7 + {}_8a_4n^5 + {}_8a_6n^3 + {}_8a_8n,$$

and

$${}_8a_0 + {}_8a_1 + {}_8a_2 + {}_8a_4 + {}_8a_6 + {}_8a_8 = 1.$$

In general to get a_{2s}, make r equal to $2s$ or $2s+1$, and it sometimes happens that one value will give simpler calculations than the other.

§5. As we have not used the fact that Σn^r is divisible by $(n+1)$ in §4, we may use these coefficients to show that Σn^r is divisible by $(n+1)$ always, and by $(n+1)^2$ when r is odd and greater than 1.

Leaving out left-hand suffixes,

$$\Sigma n^r = a_0n^{r+1} + a_1n^r + a_2n^{r-1} + a_4n^{r-3} + \ldots + a_rn \qquad (r \text{ even}),$$

$$\Sigma n^r = a_0n^{r+1} + a_1n^r + a_2n^{r-1} + a_4n^{r-3} + \ldots + a_{r-1}n^2 \quad (r \text{ odd}).$$

(1) *r even.* By $(r+1)$, $a_0 + a_1 + a_2 + \ldots a_r = 1$;

$$\therefore \quad a_0 + a_2 + \ldots a_r = 1 - a_1 = \tfrac{1}{2}.$$

Now $\underset{n=-1}{\mathrm{L}} \Sigma n^r = a_1 - (a_0 + a_2 + \ldots a_r) = 0$;

$\therefore \quad \Sigma n^r$ is divisible by $(n+1)$ when r is even.

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(2) *r odd.* Just as in the case of r even, we show that Σn^r is divisible by $n+1$.

Equation (r) is

$$(r+1)a_0+ra_1+(r-1)a_2+\ldots 2a_{r-1}=r\,;$$

i.e. $$(r+1)a_0+(r-1)a_2+\ldots+2a_{r-1}=r-a_1r=\tfrac{1}{2}r\,;$$

i.e. $$(r+1)a_0-ra_1+(r-1)a_2+\ldots+2a_{r-1}=0\,;$$

i.e. $$\underset{n=-1}{\mathrm{L}}\left(\frac{d}{dn}\Sigma n^r\right)=0 \quad (r \text{ odd})\,;$$

i.e. Σn^r contains the factor $(n+1)^2$ when r is odd and greater than 1 (for equation (r) does not involve a_1 when $r=1$).

§ 6. *Proof of Bernoulli's Theorem.*

I had reached this stage when Professor Whittaker pointed out to me Bernoulli's expansion (see Chrystal's Algebra II., §7, Chap. XXVIII.). We can demonstrate this expansion from the above as follows :—

The tabulated values of a_s are simply Bernoulli's numbers with alternating signs.

From what we have proved in § 4, we can assume

$$\Sigma n^r=\frac{n^{r+1}}{r+1}+\tfrac{1}{2}n^r+\frac{\beta_1}{2}rn^{r-1}-\frac{\beta_2}{4}{}_rC_3n^{r-3}+\frac{\beta_3}{6}{}_rC_5n^{r-5}-\frac{\beta_4}{8}{}_rC_7n^{r-7}\ldots\ldots\ldots$$

where $\beta_s=|\,a_{2s}\,|$;

i.e. $${}_ra_{2s}=(-)^{s-1}\frac{\beta_s}{2s}{}_rC_{2s-1}.$$

When we substitute for the a's in $(r+1)$, we have

(1) when r is even, $=2p$ say,

$${}_ra_0+{}_ra_1+{}_ra_2+{}_ra_4+\ldots+{}_ra_r=1\,;$$

i.e. $$\frac{1}{r+1}+\tfrac{1}{2}+\frac{\beta_1}{2}{}_rC_1-\frac{\beta_2}{4}{}_rC_3+\ldots=1\,;$$

i.e. $${}_{2p}C_1\frac{\beta_1}{2}-{}_{2p}C_3\frac{\beta_2}{4}+\ldots+(-)^{p-1}{}_{2p}C_{2p-1}\frac{\beta_p}{2p}=\tfrac{1}{2}-\frac{1}{2p+1}=\frac{2p-1}{2(2p+1)}. \qquad \text{IV.}$$

(2) When r is odd, $=2p+1$ say,

$${}_r a_0 + {}_r a_1 + {}_r a_2 + \ldots + {}_r a_{r-1} = 1\ ;$$

i.e. $$\frac{1}{r+1} + \tfrac{1}{2} + \frac{\beta_1}{2}{}_r C_1 - \frac{\beta_2}{4}{}_r C_3 + \ldots = 1\ ;$$

i.e. $${}_{2p+1}C_1\frac{\beta_1}{2} - {}_{2p+1}C_3\frac{\beta_2}{4} + \ldots + (-)^{p-1}\,{}_{2p+1}C_{2p-1}\frac{\beta_p}{2p}$$

$$= \tfrac{1}{2} - \frac{1}{2p+2} = \frac{p}{2p+2}. \qquad \text{V.}$$

When we multiply IV. by $(-)^{p-1}(2p+1)$, we get, since ${}_{2p}C_{2s-1} \times \frac{2p+1}{2s} = {}_{2p+1}C_{2s}$,

$${}_{2p+1}C_{2p}\,\beta_p - {}_{2p+1}C_{2p-2}\,\beta_{p-1} + \ldots + (-)^{p-1}\,{}_{2p+1}C_2\beta_1 = (-)^{p-1}(p-\tfrac{1}{2}). \qquad \text{IV.}^1$$

Similarly, when we multiply V. by $(-)^{p-1}(2p+2)$, we get

$${}_{2p+2}C_{2p}\,\beta_p - {}_{2p+2}C_{2p-2}\,\beta_{p-1} + \ldots + (-)^{p-1}\,{}_{2p+2}C_2\,\beta_1 = (-)^{p-1}p. \qquad \text{V.}^1$$

We can calculate the β's from IV.[1] or V.[1] by giving p the values 1, 2, 3... successively. *Cf.* Calculation of a_8, in end of § 4, by putting $r=8$ or 9 in equation $(r+1)$.

These alternative recurrence formulae for β are the same as those for Bernoulli's numbers (see Chrystal's Algebra II., Equations (10′) and (11′), § 6, Chap. XXVIII.).

Hence the numbers β_1, β_2, etc., are Bernoulli's numbers B_1, B_2, etc.

Hence

$$\Sigma n^r = \frac{n^{r+1}}{r+1} + \tfrac{1}{2}n^r + \frac{B_1}{2}rn^{r-1} - \frac{B_2}{4}{}_r C_3 n^{r-3} + \frac{B_3}{6}{}_r C_5 n^{r-5} - \frac{B_4}{8}{}_r C_7 n^{r-7} + \ldots\ldots,$$

the last term being $(-)^{\frac{1}{2}(r-2)}B_{\frac{1}{2}r}n$ or $\frac{1}{2}(-)^{\frac{1}{2}(r-3)}r\,B_{\frac{1}{2}(r-1)}n^2$ according as r is even or odd (Bernoulli's Theorem).

The proof seems rather complicated, as I have taken trouble to make every step clear, but it depends on very simple principles, and does not involve even an infinite series.

§ 7. To show how intimately connected the simple factors of Σn^r are with the Bernoullian expansions given in that section of

Chrystal referred to in § 6 of this paper, we will give a formula for calculating the Bernoullian numbers, deduced from the fact that $2n+1$ is a factor of Σn^{2p}.

Since Σn^{2p} contains the factor $2n+1$, we have, putting $n=-\frac{1}{2}$ in the expression for Σn^{2p},

$$\frac{1}{2p+1}-1+2^2\frac{B_1}{2}2p \quad 2^4\frac{B_2}{4}{}_{2p}C_3+\ldots+(-)^{p-1}2^{2p}\frac{B_p}{2p}{}_{2p}C_{2p-1}=0.$$

i.e. taking $\frac{1}{2p+1}-1$ to the right-hand side of the equation and multiplying up by $2p+1$,

$$2^2B_1\,{}_{2p+1}C_2-2^4B_2\,{}_{2p+1}C_4+2^6B_3\,{}_{2p+1}C_6\ldots$$
$$+(-)^{p-1}2^{2p}B_p\,{}_{2p+1}C_{2p}=2p. \qquad \text{VI.}$$

Now $$x\frac{e^x+e^{-x}}{e^x-e^{-x}}=1+\frac{B_1}{2\,!}2^2x^2-\frac{B_2}{4\,!}2^4x^4+\frac{B_3}{6\,!}2^6x^6\ldots.$$

Multiply up by e^x-e^{-x}, expand both sides, and write down the condition that the coefficient of x^{2p+1} is the same on both sides, and we get equation VI.