INEQUALITIES CONCERNING THE INVERSES OF POSITIVE DEFINITE MATRICES

by W. W. MUIR

(Received 4th October 1972)

0. Introduction

Much has been written on inequalities concerning positive definite matrices, but a new insight may be gained by examining inequalities from the standpoint of the inverse matrix. The standard inequality of Hölder can then be used in a more fruitful manner. This leads to some new results and a rediscovery of some known results.

1. Compound matrices

The following theory requires the use of the Binet-Cauchy theory of compound matrices which is described here. M is a given $m \times n$ matrix and k is an integer less than the smaller of m and n. α is a subset of k integers from the set $(1, 2, \ldots, m)$ and β is a subset of k integers from the set $(1, 2, \ldots, n)$. Suppose we delete all rows of M whose indices do not belong to α and also all columns whose indices do not belong to β. The determinant of the remaining $k \times k$ matrix is denoted by $[M]_{\alpha\beta}$ or $m_{\alpha\beta}$. The matrix whose elements are $m_{\alpha\beta}$ is denoted by $M^{(k)}$. The priority of the elements in rows or columns is in lexicographical order of the elements of either the set α or the set β respectively. $M^{(k)}$ therefore is a matrix of order $m_{(k)} \times n_{(k)}$ where $m_{(k)} = m!/(k!(m-k)!)$. It can be proved (Aitken (**1**), p. 94) that

$$(MN)^{(k)} = M^{(k)}N^{(k)}. \tag{1.1}$$

2. The inverse log-convex property

In the text that follows it is assumed that A and B are positive definite real symmetric matrices each of order $n \times n$ and λ and μ are real non-negative numbers such that $\lambda + \mu = 1$.

Let $f(M)$ be a scalar function of the elements of a matrix M. Then if

$$f((\lambda A + \mu B)^{-1}) \leqq \{f(A^{-1})\}^{\lambda}\{f(B^{-1})\}^{\mu} \leqq \lambda f(A^{-1}) + \mu f(B^{-1})$$

we say that the function f possesses the inverse logconvex property or ILC property for short. We note that if $f(A^{-1})$ and $f(B^{-1})$ are real non-negative numbers then the right-hand inequality follows (Bellman (**2**), p. 129).

3. A basic theorem

Let X be any real matrix and let $g(M) = [X'MX]_{\alpha\alpha}$. Then the function g has the ILC property.

https://doi.org/10.1017/S001309150001021X Published online by Cambridge University Press

Since A and B are positive definite real symmetric matrices of order $n \times n$, we can find a matrix P such that

$$P'AP = I,\ P'BP = F = \text{diag}\,(\gamma_1, \gamma_2, \ldots, \gamma_n)$$

and

$$\gamma_i > 0 \text{ for all } i. \tag{3.1}$$

It follows that

$$(\lambda A + \mu B)^{-1} = P(\lambda I + \mu F)^{-1} P'.$$

We use also two inequalities:

(i) If x and y are real positive or zero numbers then (Bellman (3), p. 129),

$$\lambda x + \mu y \geqq x^{\lambda} y^{\mu}. \tag{3.2}$$

(ii) If u_i and v_i are non-negative for $i = 1, 2, \ldots, n$ then (Hölder's Inequality)

$$\sum_{i=1}^{n} u_i^{\lambda} v_i^{\mu} \leqq \left(\sum_{i=1}^{n} u_i\right)^{\lambda} \left(\sum_{i=1}^{n} v_i\right)^{\mu}. \tag{3.3}$$

From (1.1) and (3.1) it follows that

$$(P')^{(k)} A^{(k)} P^{(k)} = I \text{ and } P'^{(k)} B^{(k)} P^{(k)} = F^{(k)},$$

where for instance if $k = 3$, from the matrix $F^{(k)}$ we obtain

$$f_{11} = \gamma_1 \gamma_2 \gamma_3, f_{22} = \gamma_1 \gamma_2 \gamma_4, \ldots, \text{etc.}$$

Now

$$\begin{aligned}
[X'(\lambda A + \mu B)^{-1} X]_{\alpha\alpha} &= [X' P(\lambda I + \mu F)^{-1} P' X]_{\alpha\alpha} \\
&= [U'(\lambda I + \mu F)^{-1} U]_{\alpha\alpha} \text{ (where } U = P'X) \\
&= \sum_{\beta} \sum_{\gamma} u'_{\alpha\beta} [(\lambda I + \mu F)^{-1}]_{\beta\gamma} u_{\gamma\alpha} \\
&= \sum_{\beta} u'_{\alpha\beta} [(\lambda I + \mu F)^{-1}]_{\beta\beta} u_{\beta\alpha} \text{ (as } \lambda I + \mu F \text{ is diagonal)} \\
&= \sum_{\beta} \left\{ \frac{u_{\beta\alpha}^2}{(\lambda + \mu\gamma_i)(\lambda + \mu\gamma_j)\ldots(\lambda + \mu\gamma_l)} \right\}
\end{aligned} \tag{3.4}$$

and β represents the subset $(i, j, \ldots, l)$ from the numbers $(1, 2, \ldots, n)$. From (3.2) therefore

$$\begin{aligned}
[X'(\lambda A + \mu B)^{-1} X]_{\alpha\alpha} &\leqq \sum_{\beta} \left\{ \frac{u_{\beta\alpha}^2}{(\gamma_i \gamma_j \ldots \gamma_l)^{\mu}} \right\} \\
&= \sum_{\beta} u_{\beta\alpha}^{2\lambda} \left\{ \frac{u_{\beta\alpha}^2}{f_{\beta\beta}} \right\}^{\mu} \\
&\leqq \left\{ \sum_{\beta} u_{\beta\alpha}^2 \right\}^{\lambda} \left\{ \sum_{\beta} \frac{u_{\beta\alpha}^2}{f_{\beta\beta}} \right\}^{\mu}
\end{aligned} \tag{3.5}$$

by Hölder's inequality.

https://doi.org/10.1017/S001309150001021X Published online by Cambridge University Press

By choosing appropriate values of λ and μ in (3.4) we obtain

$$[X'A^{-1}X]_{\alpha\alpha} = \sum_{\beta} u_{\beta\alpha}^2; \; [X'B^{-1}X]_{\alpha\alpha} = \sum_{\beta} \left(\frac{u_{\beta\alpha}^2}{f_{\beta\beta}}\right).$$

Hence (3.5) gives

$$[X'(\lambda A+\mu B)^{-1}X]_{\alpha\alpha} \leqq \{[X'A^{-1}X]_{\alpha\alpha}\}^{\lambda}\{[X'B^{-1}X]_{\alpha\alpha}\}^{\mu}$$

and the result is proved.

4. Deductions from the basic theorem

(*a*) It follows that $g(M) = [M]_{\alpha\alpha}$ is a matrix function that has the ILC property: (4.1)

in particular if $k = 1$ then

$$[(\lambda A+\mu B)^{-1}]_{ii} \leqq \{[A^{-1}]_{ii}\}^{\lambda}\{[B^{-1}]_{ii}\}^{\mu},$$

which is given as Bergström's Inequality in Bellman (**2**, p. 131). Also if $k = n$ we obtain

$$|\lambda A+\mu B| \geqq |A|^{\lambda}|B|^{\mu},$$

(Bellman (**2**), p. 128).

(b) Let $\phi_k(M) = \sum_{\alpha} m_{\alpha\alpha}$. Thus $\phi_k(M)$ is the sum of the products of the eigenvalues of M taken k at a time.

Then ϕ_k is a function with the ILC property.

Proof.

$$\begin{aligned}\phi_k((\lambda A+\mu B)^{-1}) &= \sum_{\alpha}[(\lambda A+\mu B)^{-1}]_{\alpha\alpha} \\ &\leqq \sum_{\alpha}\{[A^{-1}]_{\alpha\alpha}\}^{\lambda}\{[B^{-1}]_{\alpha\alpha}\}^{\mu} && \text{(from (4.1))} \\ &\leqq \{\sum_{\alpha}[A^{-1}]_{\alpha\alpha}\}^{\lambda}\{\sum_{\alpha}[B^{-1}]_{\alpha\alpha}\}^{\mu} && \text{(from (3.3))} \\ &= \{\phi_k(A^{-1})\}^{\lambda}\{\phi_k(B^{-1})\}^{\mu}. && (4.2)\end{aligned}$$

If $k = 1$ then (4.2) yields

$$\operatorname{tr}\{(\lambda A+\mu B)^{-1}\} \leqq \{\operatorname{tr}(A^{-1})\}^{\lambda}\{\operatorname{tr}(B^{-1})\}^{\mu},$$

where tr stands for trace. Hence the trace of a matrix possesses the ILC property.

(*c*) Let M be a positive definite symmetric matrix and let the eigenvalues be

$$l_1, l_2, \ldots, l_n, \text{ where } l_1 \geqq l_2 \geqq \ldots, \geqq l_n > 0.$$

Let $L_k(M) = l_1 l_2 \ldots l_k$.
Then L_k is a function that follows the ILC property. (4.3)

The proof from the basic theorem is omitted since this result has already been given in equivalent form in Bellman (**2**, p. 130).

https://doi.org/10.1017/S001309150001021X Published online by Cambridge University Press

5. Certain types of matrices A and B

Suppose we partition A as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is of order $p \times p$ say. Let

$$B=\begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}.$$

Let $\lambda = \mu = \frac{1}{2}$ and hence

$$\lambda A+\mu B=\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

A, B, λ and μ are defined as above from now on in this paper.

Let $D = A^{-1}$ and let D be partitioned similarly to A with

$$D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Then it is easy to prove that

$$B^{-1}=\begin{bmatrix} D_{11} & -D_{12} \\ -D_{21} & D_{22} \end{bmatrix} \tag{5.1}$$

and that the eigenvalues of A and B are the same.

The determinant of a principal submatrix of A is the same as the determinant of the corresponding principal submatrix of B and the same applies to their inverses. In our previous notation therefore we obtain

$$[A^{-1}]_{\alpha\alpha}=[B^{-1}]_{\alpha\alpha}. \tag{5.2}$$

Hence from (4.1)

$$[(\lambda A+\mu B)^{-1}]_{\alpha\alpha} \leqq [A^{-1}]^{\lambda}_{\alpha\alpha}[B^{-1}]^{\mu}_{\alpha\alpha}$$

and inserting the appropriate values of λ, μ, A and B and letting $\alpha = 1, 2, \ldots, p$ we obtain

$$|A_{11}^{-1}| \leqq |D_{11}| \tag{5.3}$$

i.e. $|D_{11}| \geqq 1/|A_{11}|$ (De Bruijn (3), page 28.)

6. Further inequalities concerning products of largest eigenvalues

Following the definition L_k of the product of largest eigenvalues and using (4.3) and (5.1) we obtain

$$L_k(A^{-1}) \geqq L_k\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

Thus $L_k(A^{-1}) \geqq L_k(A_{11}^{-1})$, or, if we take r of the eigenvalues from A_{11}^{-1} and the rest from A_{22}^{-1}, we obtain

$$L_k(A^{-1}) \geqq L_r(A_{11}^{-1}).L_{k-r}(A_{22}^{-1}). \tag{6.1}$$

https://doi.org/10.1017/S001309150001021X Published online by Cambridge University Press

Suppose the eigenvalues of A are $\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \ldots \geqq \alpha_n > 0$; then it follows from (6.1) that

$$\frac{1}{\alpha_n} \cdot \frac{1}{\alpha_{n-1}} \cdots \frac{1}{\alpha_{n-k+1}} \geqq \frac{1}{a_{11}} \frac{1}{a_{22}} \cdots \frac{1}{a_{kk}}$$

or $\alpha_n \alpha_{n-1} \ldots \alpha_{n-k+1} \leqq a_{11} a_{22} \ldots a_{kk}$. (See Bellman (2), page 134.)

7. Further inequalities concerning sums of products of eigenvalues

As $\phi_k(A)$ is the sum of principal minors of order k and using the definitions of A and B given in Section 5 and the relationship (5.2) we see that

$$\phi_k(A^{-1}) = \phi_k(B^{-1}).$$

Hence from (4.2)

$$\phi_k(A^{-1}) \geqq \phi_k \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \tag{7.1}$$

$$= \phi_k(A_{11}^{-1}) + \phi_{k-1}(A_{11}^{-1})\phi_1(A_{22}^{-1}) + \phi_{k-2}(A_{11}^{-1})\phi_2(A_{22}^{-1})$$
$$+ \ldots + \phi_1(A_{11}^{-1})\phi_{k-1}(A_{22}^{-1}) + \phi_k(A_{22}^{-1})$$

or in short, if $\phi_0(A) = 1$, then

$$\phi_k(A^{-1}) \geqq \sum_{r=0}^{k} \phi_r(A_{11}^{-1})\phi_{k-r}(A_{22}^{-1}).$$

If $k = n$ then from (7.1)

$$|A| \leqq |A_{11}|\,|A_{22}| \tag{7.2}$$

the well-known Hadamard-Fischer theorem.

If $k = 1$, from (7.1) it follows that

$$\operatorname{tr}(A^{-1}) \geqq \operatorname{tr}(A_{11}^{-1}) + \operatorname{tr}(A_{22}^{-1}),$$

a sort of dual to the Hadamard-Fischer theorem.

Also we see from (5.3) and (7.2) that

$$\phi_k(A^{-1}) = \sum_{\alpha} [A^{-1}]_{\alpha\alpha} \geqq \sum_{\alpha} \frac{1}{[A]_{\alpha\alpha}} \geqq \sum \frac{1}{a_{11} a_{22} \ldots a_{kk}}.$$

REFERENCES

(1) A. C. Aitken, *Determinants and Matrices* (Oliver and Boyd, 1964).

(2) R. Bellman, *Introduction to Matrix Analysis*, 2nd Edition (McGraw-Hill, 1970).

(3) N. G. de Bruijn, Inequalities concerning minors and eigenvalues, *Nieuw Arch. Wisk.* 3 (1956), 18-35.

https://doi.org/10.1017/S001309150001021X Published online by Cambridge University Press