Fifth Meeting, 11th March 1904.

Mr Charles Tweedie, President, in the Chair.

Some Points in Diophantine Analysis.

By Alexander Holm, M.A.

What I intend to put before you first is a graphical view of the solution of indeterminate equations of the third degree.

1. Suppose that a rational solution of an equation of the third degree in x and y is known, and that it is required to find other rational solutions.

For example, given $x=1$, $y=2$ a rational solution of

$$2x^3 - x^2y - 2xy^2 + y^3 + 7x^2 - xy - 2y^2 + 4x - 2y + 3 = 0,$$

to find other rational solutions.

Let the equation be represented by a graph (Fig. 5).

A straight line can be drawn through the given point (1, 2) to cut the graph in two more points; for there will arise a cubic equation to find the abscissae of the points of intersection.

Suppose now that the straight line is so drawn that one of these two points coincides with the given point, then the straight line becomes the tangent at the given point; and, the coordinates of the two coincident points being given rational, it follows that those of the other point of intersection are also rational.

Thus the tangent at the given point will cut the graph at a point whose coordinates are rational.

No new point will be obtained when the given point on the graph happens to be a point of inflexion.

Example: the above equation can be put into the form

$$(x-y+2)(x+y+1)(2x-y+1) - 3x - y + 1 = 0.$$

Transfer the origin to the given point (1, 2) on the graph by putting

$$x = \mathrm{X} + 1 \text{ and } y = \mathrm{Y} + 2;$$

$$\therefore \quad (\mathrm{X} - \mathrm{Y} + 1)(\mathrm{X} + \mathrm{Y} + 4)(2\mathrm{X} - \mathrm{Y} + 1) - 3\mathrm{X} - \mathrm{Y} - 4 = 0.$$

https://doi.org/10.1017/S0013091500001942 Published online by Cambridge University Press

The constant term is now zero, and the tangent at the new origin is

$$4(X-Y)+(X+Y)+4(2X-Y)-3X-Y=0,$$

that is, $Y=\frac{5}{4}X$, and it cuts the graph where

$$\left(-\frac{X}{4}+1\right)\left(\frac{9X}{4}+4\right)\left(\frac{3X}{4}+1\right)-3X-\frac{5X}{4}-4=0,$$

$$i.e., \quad -\frac{27}{64}X^3+\frac{3}{8}X^2=0,$$

$$\therefore \quad X=0 \text{ or } 0 \text{ or } \frac{8}{9}.$$

Taking $X=\frac{8}{9}$, $Y=\frac{10}{9}$,

$$\therefore \quad x=\frac{17}{9}, \quad y=\frac{28}{9}.$$

The coordinates of the point where the tangent at $\left(\frac{17}{9}, \frac{28}{9}\right)$ cuts the graph could now be found, and so on. In this way an infinite number of rational values of x and y could be found to satisfy the given equation.

Diophantine problems of the third degree generally present themselves under one of the following two more limited forms.

To find a value of x which will make

$$(1) \quad ax^3+bx^2+cx+d=\text{a square}=y^2,$$

or

$$(2) \quad ax^3+bx^2+cx+d=\text{a cube} \quad =y^3,$$

having given one value of x which does so.
The usual Diophantine method of solution is equivalent to the above tangent-method.

2. *When the graph happens to have one or three rational asymptotes.*

If a straight line drawn through a given point on the graph rotates until it becomes parallel to an asymptote, one point of intersection with the graph recedes to infinity. Hence the cubic equation, which determines the abscissae of the points of intersection, becomes a quadratic equation ; and, the abscissa of the given point being rational, it follows that the abscissa of the other point of intersection is also rational.

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Example: an asymptote to the above graph is $x+y+1=0$; a straight line through the point (1, 2) parallel to it is $y=-x+3$; this line cuts the graph where

$$(2x-1)4(3x-2)-3x+x-3+1=0,$$

$$i.e.,\quad 4x^2-5x+1=0,$$

$$\therefore \left.\begin{array}{l} x=1 \\ y=2 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=\frac{1}{4} \\ y=\frac{11}{4} \end{array}\right\};$$

$\therefore\quad x=\frac{1}{4},\ y=\frac{11}{4}$ is a new solution.

Similarly a straight line through the point (1, 2) parallel to the second asymptote $x-y+2=0$ cuts the graph at

$$x=0, \quad y=1.$$

A straight line through the point (1, 2) parallel to the third asymptote $2x-y+1=0$ cuts the graph at

$$x=-1, \quad y=-2.$$

3. If a straight line moves parallel to an asymptote until it coincides with the asymptote, a second point of intersection with the graph recedes to infinity. Hence the cubic equation, which determines the abscissae of the points of intersection, becomes a simple equation giving a rational value for the abscissa of the point where the asymptote cuts the graph.

e.g., the asymptote $x+y+1=0$ cuts the graph at $x=1$, $y=-2$.

4. *When two rational solutions are known.*

A straight line joining two points with rational coordinates will cut the graph again in a third point, whose coordinates are rational. This third point will be different from either of the two given points, except when the straight line joining these two points happens to be the tangent at one of them. No third point of intersection will be obtained when the straight line joining the two given points happens to be parallel to an asymptote.

Thus (0, 1) and (−1, −2) are points on the graph.

The straight line joining them is $y=3x+1$, and it cuts the graph where $\quad (-2x+1)(4x+2)(-x)-6x=0,$

$$i.e.,\quad 8x^3-8x=0;$$

$$\therefore \left.\begin{array}{l} x=0 \\ y=1 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=-1 \\ y=-2 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=1 \\ y=4 \end{array}\right\};$$

$\therefore\quad x=1,\ y=4$ is a new solution.

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I have not noticed any Diophantine methods corresponding to §§ 2, 3, 4.

5. An example of the tangent-method.

To find two cubes whose difference is equal to the sum of two given cubes.

Let a^3 and b^3 be the given cubes, x^3 and y^3 the required cubes,

then $$x^3 - y^3 = a^3 + b^3\ ;$$

$$x = a, \qquad y = -b \text{ is a particular solution.}$$

Let $$x = X + a,\ y = Y - b\ ;$$

$$\therefore\quad X^3 - Y^3 + 3aX^2 + 3bY^2 + 3a^2X - 3b^2Y = 0.$$

Choose $$3a^2X - 3b^2Y = 0 \qquad \text{or} \qquad Y = \frac{a^2}{b^2}X.$$

$$\therefore\quad \left(1 - \frac{a^6}{b^6}\right)X^3 + 3a\left(1 + \frac{a^3}{b^3}\right)X^2 = 0.$$

$$\therefore\quad X = 0 \text{ or } 0 \text{ or } \frac{3a\,b^3}{a^3 - b^3}.$$

Taking $$X = \frac{3a\,b^3}{a^3 - b^3}, \qquad Y = \frac{3\,a^3b}{a^3 - b^3},$$

$$\therefore\qquad x = \frac{a^3 + 2b^3}{a^3 - b^3}.a, \quad y = \frac{2a^3 + b^3}{a^3 - b^3}.b\,.$$

Thus $$a^3 + b^3 = \left(\frac{a^3 + 2b^3}{a^3 - b^3}.a\right)^3 - \left(\frac{2a^3 + b^3}{a^3 - b^3}.b\right)^3;$$

e.g., if $a = 2$, $b = 1$ then $2^3 + 1^3 = \left(\frac{20}{7}\right)^3 - \left(\frac{17}{7}\right)^3$.

The problem of this article is the converse of Diophantus' third porism, which, along with the analogous problems added by Vieta and Bachet, can be resolved in a similar manner.*

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* Cf. Heath's *Diophantus*, Camb. 1885, pp. 123–124.
Bachet's *Diophantus*, Paris 1621, pp. 178–182.
Oeuvres de Fermat, Paris 1891, t. I., pp. 297–299.

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6. The sides of a rational right-angled triangle can be represented in the most general way possible by

$$(m^2+1)n, \ (m^2-1)n \text{ and } 2mn,$$

where m and n are positive rational numbers, m being >1.

If the sides are x, y, z, then $x^2=y^2+z^2$. (*Euc.* I., 47.)

$$\therefore \quad x^2-y^2=z^2;$$

$$\therefore \quad (x+y)(x-y)=z^2;$$

$$\therefore \quad \frac{x+y}{z}=\frac{z}{x-y}=\text{a rational number}=m;$$

$$\therefore \quad x+y-mz=0$$

$$\text{and} \quad -mx+my+z=0;$$

$$\therefore \quad \frac{x}{m^2+1}=\frac{y}{m^2-1}=\frac{z}{2m}=n;$$

$$\therefore \quad x=(m^2+1)n, \ y=(m^2-1)n, \ z=2mn.$$

The factor m^2+1 in the hypotenuse $(m^2+1)n$ is always positive; therefore, in order that the hypotenuse may be positive, n must be positive; then, in order that the side $2mn$ may be positive, m must be positive; and in order that the side $(m^2-1)n$ may be positive, m must be >1. The species of the triangle depends solely upon the value of m; if m is kept constant, and n varied, a series of similar triangles will be obtained.

It is often more convenient to take $\frac{1}{n}$ instead of n, and thus the sides would be $\frac{m^2+1}{n}, \ \frac{m^2-1}{n}, \ \frac{2m}{n}$.

I have found the results of this article very useful in solving the problems of Diophantus' Sixth Book.

7. The following general problem includes Diophantus VI., 6, 7, 8, 9, 10, 11, and the similar problems which have been added by Bachet and Fermat.*

* Cf. Heath's *Diophantus*, pp. 227–229.
Bachet's *Diophantus*, pp. 383–388.
Oeuvres de Fermat, t. I., pp 329–331.

https://doi.org/10.1017/S0013091500001942 Published online by Cambridge University Press

To find a rational right-angled triangle such that the sum of given multiples of the area and of the three sides may be equal to a given number.

Let a, b, c, d be the given multiples, e the given number, and $\frac{x^2+1}{y}$, $\frac{x^2-1}{y}$, $\frac{2x}{y}$ the sides of the required triangle;

then $$a \cdot \frac{(x^2-1)x}{y^2} + b \cdot \frac{x^2+1}{y} + c \cdot \frac{x^2-1}{y} + d \cdot \frac{2x}{y} = e;$$

$$\therefore \quad ey^2 - \{(b+c)x^2 + 2dx + (b-c)\}y - a(x^2-1)x = 0.$$

In order that this quadratic equation may give a rational value for y, we must have

$$\{(b+c)x^2 + 2dx + (b-c)\}^2 + 4ae(x^2-1)x = \text{a square},$$

which can in general be resolved for x in six ways, since the term in x^4 and the constant are both squares.*

The corresponding values for y can then be found from the quadratic equation. If x' is any one of the six values found for x, by putting $x = X + x'$, six new values may be obtained for x, and so on.

Hence the problem admits of an infinite number of solutions.

8. An example of the use of the method of § 4 and of § 6 is furnished by the following problem, which is necessary for the completion of the solution of Diophantus V., 24 and 25.†

To find two rational right-angled triangles such that their areas may be in a given ratio.

Let the given ratio be $r : s$,

the sides of the first triangle $m(x^2+1)$, $m(x^2-1)$, $2mx$,

and those of the second $n(y^2+1)$, $n(y^2-1)$, $2ny$.

Then $$\frac{m^2(x^2-1)x}{n^2(y^2-1)y} = \frac{r}{s};$$

$$\therefore \quad m^2s(x^2-1)x - n^2r(y^2-1)y = 0.$$

Suppose the equation to be represented by a graph;

when $x = -1$ or 0 or 1 then $y = -1$ or 0 or 1.

* Cf. Euler's *Algebra*, English translation, Part II., chap. IX., § 134.

† Cf. *Oeuvres de Fermat*, t. I., pp. 318–325.

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Thus

$(-1,-1), (-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0), (1,1)$

are points on the graph.

The straight lines joining twelve pairs of these points will cut the graph again in a third point, which is different from any of the above points, and which necessarily has rational coordinates.

(i) The straight line joining $(0, -1)$ to $(1, 0)$ is $y = x - 1$, and it cuts the graph where

$$m^2s(x^2-1)x - n^2r\{(x-1)^2-1\}(x-1)=0\,;$$

$$i.e.,\quad (n^2r - m^2s)x^3 - 3n^2rx^2 + (2n^2r + m^2s)x = 0\,;$$

$$\therefore\quad x = 0 \text{ or } 1 \text{ or } \frac{2n^2r + m^2s}{n^2r - m^2s}\,;$$

$$\therefore\quad x = \frac{2n^2r + m^2s}{n^2r - m^2s},\quad y = \frac{n^2r + 2m^2s}{n^2r - m^2s},$$

where m and n can be taken arbitrarily.

e.g., if $m = 1$, $n = 1$, $x = \dfrac{2r+s}{r-s}$, $y = \dfrac{r+2s}{r-s}$.

Hence the sides of the first triangle are

$$\left\{\frac{(2r+s)^2}{(r-s)^2}+1\right\},\quad \left\{\frac{(2r+s)^2}{(r-s)^2}-1\right\},\quad \frac{2(2r+s)}{r-s}$$

and those of the second

$$\left\{\frac{(r+2s)^2}{(r-s)^2}+1\right\},\quad \left\{\frac{(r+2s)^2}{(r-s)^2}-1\right\},\quad \frac{2(r+2s)}{r-s}\,;$$

or, multiplying both sets by $(r-s)^2$, we may take the sides of the first triangle to be

$$\{(2r+s)^2+(r-s)^2\},\quad \{(2r+s)^2-(r-s)^2\},\quad 2(2r+s)(r-s)$$

and those of the second

$$\{(r+2s)^2+(r-s)^2\},\quad \{(r+2s)^2-(r-s)^2\},\quad 2(r+2s)(r-s)\,;$$

that is, the first triangle may be formed from $(2r+s, r-s)$ and the second from $(r+2s, r-s)$. This is Fermat's first method, which he merely states, without showing how it was obtained.*

It is evident that it is only one of an infinite group of methods.

* Cf. *Oeuvres de Fermat*, t. I., p. 320.

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If $r=3$, $s=1$, the first triangle, formed from (7, 2), is (53, 45, 28) and the second, formed from (5, 2), is (29, 21, 20).

(ii) The straight line joining (0, 1) to (1, 0) cuts the graph again at

$$x=\frac{2n^2r-m^2s}{n^2r+m^2s},\quad y=\frac{2m^2s-n^2r}{n^2r+m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=1$, $$x=\frac{2r-s}{r+s},\quad y=\frac{2s-r}{r+s}.$$

Thus the first triangle may be formed from $(2r-s,\ r+s)$ and the second from $(r+s,\ r-2s)$, which is Fermat's second method.*

If $r=3$, $s=1$, the first triangle, formed from (5, 4), is (41, 9, 40) and the second, formed from (4, 1), is (17, 15, 8).

(iii) The straight line joining $(-1,\ -1)$ to (1, 0) cuts the graph again at

$$x=\frac{3n^2r}{n^2r-8m^2s},\quad y=\frac{n^2r+4m^2s}{n^2r-8m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=4$, $$x=\frac{6r}{2r-s},\quad y=\frac{4r+s}{4r-2s}.$$

Hence the first triangle may be formed from $(6r,\ 2r-s)$ and the second from $(4r+s,\ 4r-2s)$. This is Fermat's third method.*

If $r=3$, $s=1$, the first triangle, formed from (18, 5), is (349, 299, 180) and the second, formed from (13, 10), is (269, 69, 260).

(iv) The straight line joining $(-1,\ 1)$ to (1, 0) cuts the graph again at

$$x=\frac{3n^2r}{n^2r+8m^2s},\quad y=\frac{4m^2s-n^2r}{n^2r+8m^2s},$$

where m and n are arbitrary.

If $m=1$, $n=4$, $$x=\frac{6r}{2r+s},\quad y=\frac{s-4r}{4r+2s}.$$

* Cf. *Oeuvres de Fermat*, t. I., p. 320.

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Thus the first triangle may be formed from $(6r,\ 2r+s)$ and the second from $(4r+2s,\ 4r-s)$, which is Fermat's fourth method.*

If $r=3, s=1$, the first triangle, formed from (18, 7), is (373, 275, 252) and the second, formed from (14, 11), is (317, 75, 308).

(v) The straight line joining $(0,\ -1)$ to $(1,\ 1)$ cuts the graph again at

$$x=\frac{4n^2r+m^2s}{8n^2r-m^2s},\quad y=\frac{3m^2s}{8n^2r-m^2s},$$

where m and n are arbitrary.

If $m=4, n=1$, $\qquad x=\dfrac{r+4s}{2r-4s},\quad y=\dfrac{6s}{r-2s}.$

Therefore the first triangle may be formed from $(r+4s,\ 2r-4s)$ and the second from $(6s,\ r-2s)$.

If $r=3,\ s=1$, the first triangle, formed from (7, 2), is (53, 45, 28) and the second, formed from (6, 1), is (37, 35, 12).

(vi) The straight line joining $(0,\ 1)$ to $(1,\ -1)$ cuts the graph again at

$$x=\frac{4n^2r-m^2s}{8n^2r+m^2s},\quad y=\frac{3m^2s}{8n^2r+m^2s},$$

where m and n are arbitrary.

If $m=4, n=1$, $\qquad x=\dfrac{r-4s}{2r+4s},\quad y=\dfrac{6s}{r+2s}.$

Thus the first triangle may be formed from $(2r+4s,\ 4s-r)$ and the second from $(6s,\ r+2s)$.

If $r=3, s=1$, the first triangle, formed from (10, 1), is (101, 99, 20) and the second, formed from (6, 5), is (61, 11, 60).

Fermat does not give a particular case of v and vi.

No results essentially different from the above are obtained by joining the remaining six pairs of points.

Since m and n can be taken arbitrarily in any of the above, it is clear that the two triangles can be formed in an infinite number of ways. Also, starting from any of the above solutions we could deduce an infinite number of others by the tangent-method; but these solutions would be of a much more complex nature.

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* *Oeuvres de Fermat*, t. I., p. 320.

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