Canad. J. Math. Vol. **53** (3), 2001 pp. 631–672

K-Theory of Non-Commutative Spheres Arising from the Fourier Automorphism

Samuel G. Walters

Abstract. For a dense G_δ set of real parameters θ in $[0,1]$ (containing the rationals) it is shown that the group $K_0(A_\theta \rtimes_\sigma \mathbb{Z}_4)$ is isomorphic to $\mathbb{Z}^9$, where A_θ is the rotation C*-algebra generated by unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$ and σ is the Fourier automorphism of A_θ defined by $\sigma(U) = V$, $\sigma(V) = U^{-1}$. More precisely, an explicit basis for K_0 consisting of nine canonical modules is given. (A slight generalization of this result is also obtained for certain separable continuous fields of unital C*-algebras over $[0,1]$.) The Connes Chern character $\mathrm{ch}\colon K_0(A_\theta \rtimes_\sigma \mathbb{Z}_4) \to H^{\mathrm{ev}}(A_\theta \rtimes_\sigma \mathbb{Z}_4)^*$ is shown to be injective for a dense G_δ set of parameters θ. The main computational tool in this paper is a group homomorphism $\mathbf{T}\colon K_0(A_\theta \rtimes_\sigma \mathbb{Z}_4) \to \mathbb{R}^8 \times \mathbb{Z}$ obtained from the Connes Chern character by restricting the functionals in its codomain to a certain nine-dimensional subspace of $H^{\mathrm{ev}}(A_\theta \rtimes_\sigma \mathbb{Z}_4)$. The range of $\mathbf{T}$ is fully determined for each θ. (We conjecture that this subspace is all of H^{ev}.)

1 Introduction

For $0 < \theta < 1$ let A_θ denote the rotation C*-algebra generated by unitaries U, V satisfying $VU = \lambda UV$, where $\lambda := e^{2\pi i\theta}$. Denote by σ the order-four automorphism of A_θ defined by

$$\sigma(U) = V, \quad \sigma(V) = U^{-1}.$$

We shall call it the *Fourier* automorphism because of its close connection with the Fourier transform of classical analysis (already used in [15] in the construction of the Fourier module). Throughout, we shall denote the associated crossed product by $B_\theta := A_\theta \rtimes_\sigma \mathbb{Z}_4$, where $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$.

The basic problem here is to compute the K-groups of B_θ, particularly $K_0(B_\theta)$, for any θ, to find a canonical basis for it, and to compute (as much as possible) the associated Connes Chern character. The difficulty of this problem is due to the fact that there are no known tools for calculating the K-groups of crossed products by finite cyclic groups (analogous to the Pimsner-Voiculescu sequence for crossed products by the integers [10] and by the free group on a finite number of generators [11]). A second problem is whether B_θ is approximately finite dimensional when θ is irrational, as is the case for the flip automorphism [3], [14], and whether the Fourier automorphism is an inductive limit of type I automorphisms as is true for the flip [14]. (See the Addendum at the end of the paper.) These and other questions related to the Fourier automorphism were raised by George Elliott in private communication with the author and have been of interest to him.

Received by the editors November 6, 1999; revised June 14, 2000.
Research partly supported by NSERC grant OGP0169928.
AMS subject classification: 46L80, 46L40, 19K14.
Keywords: C*-algebras, K-theory, automorphisms, rotation algebras, unbounded traces, Chern characters.

We shall use A_θ to also denote its canonical smooth dense $*$-subalgebra under the canonical toral action, and by B_θ the dense $*$-subalgebra of elements of the form $\sum_j a_j W^j$ where a_j are smooth elements in A_θ, $j = 0, 1, 2, 3$, and W is the canonical (order four) unitary of the crossed product implementing σ by $\sigma(a) = WaW^{-1}$. (This identification is justified since both the C*-algebra and its smooth $*$-subalgebra have the same K-theory, since the dense $*$-subalgebras are closed under the holomorphic functional calculus, and since it will be clear from the context which algebra is intended.)

In [15], the author constructed nine canonical modules over B_θ and showed (using theta functions) that they give rise to nine independent positive classes in $K_0(B_\theta)$ for each θ (rational or irrational). This was done by examination of the Connes Chern character

$$\mathrm{ch}\colon K_0(B_\theta) \to H^{\mathrm{ev}}(B_\theta)^*$$

where $H^{\mathrm{ev}}(B_\theta)$ is Connes' even periodic cyclic cohomology group and $H^{\mathrm{ev}}(B_\theta)^*$ is its vector space dual [5, III]. (We prefer to view the codomain of ch as above instead of the usual cyclic homology group so as to readily use Connes' canonical pairing between K_0 and cyclic cohomology.) From ch a group homomorphism $\mathbf{T}\colon K_0(B_\theta) \to \mathbb{R}^8 \times \mathbb{Z}$ can be defined by taking the Connes Chern character $\mathrm{ch}(x)$ of each element x in $K_0(B_\theta)$ and restricting it to a certain (nine-dimensional) subspace of $H^{\mathrm{ev}}(B_\theta)$ spanned by the traces on the (smooth) algebra B_θ (as in [15]) and by Connes' canonical cyclic 2-cocycle (as in [4] or [5, III.2.β]). It was shown in [15] that $\mathbf{T}$ is injective when θ is rational. This suggests, presumably, that the subspace in question is all of $H^{\mathrm{ev}}(B_\theta)$ and that ch will in fact turn out to be, after tensoring with the complex plane, an isomorphism. (In view of this, we shall sometimes refer to $\mathbf{T}$ as the Connes Chern character.)

The first result of the present paper is to show that the nine modules under consideration generate (and so form a basis for) $K_0(B_\theta)$ when θ is rational (Corollary 6-C). Together with Corollary 7.3-E (Section 7.3), this result yields a suitable parametrization of $K_0(B_\theta)$ which is independent of θ. These results culminate with the following main theorem:

Theorem *There is a dense G_δ set of parameters θ (containing the rationals) such that $K_0(B_\theta)$ is isomorphic to $\mathbb{Z}^9$. In addition, for such parameters,*

(i) *the nine canonical modules form a basis for $K_0(B_\theta)$,*
(ii) *the Connes Chern character $\mathrm{ch}\colon K_0(B_\theta) \to H^{\mathrm{ev}}(B_\theta)^*$ is injective,*
(iii) *the range of $\mathbf{T}\colon K_0(B_\theta) \to \mathbb{R}^8 \times \mathbb{Z}$ is the integral span of the rows in the Character Table (in Section 2.1) for all θ,*
(iv) $K_1(B_\theta) = 0$.

In particular, these conclusions hold for many irrationals. The fact that $K_0(B_\theta) \cong \mathbb{Z}^9$ for rational θ is a result of [7]. One of the results used in the proofs below (especially in Section 3) is a realization, in the rational case, of B_θ as a 2-sphere with singularities due to Farsi and Watling [7, Theorem 6.2.1]. (Some corrections to the latter paper in this connection, to be used here, are noted in the Appendix below (Section 8).)

In Section 7, the context of the present situation (the existence of a finite number of generating modules) is generalized slightly by imposing two hypotheses on a separable continuous field of C*-algebras $\{C_t : 0 \le t \le 1\}$ so as to obtain the same conclusion—namely, that the K-groups are the same on a dense G_δ set of the parameter t if they are the same on a dense set of parameters t. (See Corollary 7.3-E and hypotheses (H1) and (H2) of Section 7.1.) Also obtained are K-group short exact sequences involving the C*-algebra of the field Γ and each fiber C_t (Corollary 7.3-E(c)). In fact, it is shown (under (H1)) that there is a canonical surjection $K_0(\Gamma) \to K_0(C_t)$ for each t (induced by the evaluation map at t); see Corollary 7.3-E(b).

2 Nine Modules and the Connes Chern Character

Throughout, we shall assume that $0 < \theta < 1$ and adopt the notation

$$e(t) := \exp(2\pi i t).$$

Thus $\lambda = e(\theta)$. When considering the case that θ is rational, we shall tacitly assume throughout that $\theta = p/q$ where $p < q$ are positive relatively prime integers.

2.1 The Nine Modules

As in [15], one has the following six projections in B_θ

$$\begin{aligned}
P_1(\theta) &= \tfrac{1}{2}(1 + W^2)\\
P_2(\theta) &= \tfrac{1}{2} + \left(\tfrac{1+i}{4}\right) W + \left(\tfrac{1-i}{4}\right) W^3\\
P_3(\theta) &= \tfrac{1}{4}(1 + W + W^2 + W^3)\\
P_4(\theta) &= \tfrac{1}{2}(1 + \lambda^{1/2} UVW^2)\\
P_5(\theta) &= \tfrac{1}{2} + \left(\tfrac{1+i}{4}\right) \lambda^{1/4} UW + \left(\tfrac{1-i}{4}\right) \lambda^{-1/4} VW^3\\
P_6(\theta) &= \tfrac{1}{4}(1 + \lambda^{1/4} UW + \lambda^{1/2} UVW^2 + \lambda^{-1/4} VW^3).
\end{aligned} \tag{2.1.1}$$

Note that the last three are obtained from the first three by replacing W by the (order four) unitary $\lambda^{1/4}UW$. One further has the Fourier module $\mathcal{F}_\theta$ over B_θ ($0 < \theta < 1$) obtained by equipping the Heisenberg module (see [4]) over A_θ with the action of W represented by a suitable scaling of the Fourier transform on the Schwartz space $S(\mathbb{R})$ (see [15, Section 3]). Using the dual automorphism $\hat{\sigma}$ of B_θ (which fixes U and V and maps W to iW), one obtains two other modules denoted in [15] by $\mathcal{F}_\theta(i)$ and $\mathcal{F}_\theta(-1)$, where the action of W is multiplied by i and -1, respectively. For simplicity, we shall write (taking the module picture)

$$P_7(\theta) = \mathcal{F}_\theta, \quad P_8(\theta) = \mathcal{F}_\theta(i), \quad P_9(\theta) = \mathcal{F}_\theta(-1).$$

The algebra B_θ has the canonical (bounded) trace τ given by

$$\tau\Big(\sum_{j=0}^{3} a_j W^j\Big) = \tau(a_0)$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

for $a_j \in A_\theta$, where $\tau(a_0)$ is the canonical trace of a_0 in A_θ (relative to the unitaries U, V). In [15] it was shown that one has the following unbounded traces on B_θ (the smooth *-subalgebra) given by

$$\begin{aligned} &T_{20}(U^mV^nW^2) = \lambda^{-mn/2}\delta_{\overline{m},0}\delta_{\overline{n},0} && T_{10}(U^mV^nW^3) = \lambda^{(m-n)^2/4}\delta_{\overline{m},\overline{n}} \\ &T_{21}(U^mV^nW^2) = \lambda^{-mn/2}\delta_{\overline{m},1}\delta_{\overline{n},1} && T_{11}(U^mV^nW^3) = \lambda^{(m-n)^2/4}\delta_{\overline{m},\overline{n+1}} \\ &T_{22}(U^mV^nW^2) = \lambda^{-mn/2}\delta_{\overline{m},\overline{n+1}} \end{aligned} \tag{2.1.2}$$

where at other generic elements $U^mV^nW^k$ they vanish, and $\delta_{r,s}$ is the usual δ-function and $\overline{m}$ is m reduced modulo 2. (Note that the T_{11} of [15] has here been multiplied by $\lambda^{1/4}$ for normalization.)

Observe that the maps T_{2j} are self-adjoint trace functionals, but that T_{1j} are not self-adjoint. This unfortunate choice (made in [15] and [16]), while not incorrect, can now be mended by looking at the real and imaginary parts of T_{1j}. Let

$$\phi_0 = \tfrac{1}{2}(T_{10} + T_{10}^*), \quad \phi_0' = -\tfrac{i}{2}(T_{10} - T_{10}^*)$$

be the real and imaginary parts of T_{10}, respectively, and

$$\phi_1 = \tfrac{1}{2}(T_{11} + T_{11}^*), \quad \phi_1' = -\tfrac{i}{2}(T_{11} - T_{11}^*)$$

be those of T_{11} (where $T^*(x) := \overline{T(x^*)}$).

The last invariant we need to recall is Connes' canonical cyclic 2-cocycle given on the rotation algebra A_θ by

$$\varphi(x^0, x^1, x^2) = \frac{1}{2\pi i}\tau\Big(x^0[\delta_1(x^1)\delta_2(x^2) - \delta_2(x^1)\delta_1(x^2)]\Big) \tag{2.1.3}$$

(see [5, III.2.β]) where δ_j, $j = 1, 2$, are the canonical derivations of A_θ under the canonical action of the 2-torus $\mathbb{T}^2$ (relative to U, V). The Chern character invariant that φ induces is the group homomorphism $c_1 \colon K_0(A_\theta) \to \mathbb{Z}$ given by the cup product

$$c_1[E] := (\varphi \# \mathrm{Tr}_n)(E, E, E) \tag{2.1.4}$$

for E any smooth projection in $M_n(A_\theta)$. In [15, Section 2], this invariant was extended to B_θ by taking the composition

$$C_1 := c_1 \circ \Psi_* \colon K_0(B_\theta) \to \mathbb{Z} \tag{2.1.5}$$

where $\Psi\colon B_\theta \to M_4(A_\theta)$ is the canonical injection given by, for $a = \sum_j a_jW^j \in B_\theta$,

$$\Psi(a) = [\sigma^{-i}(a_{i-j})]_{i,j=0}^3 = \begin{bmatrix} a_0 & a_3 & a_2 & a_1 \\ \sigma^3(a_1) & \sigma^3(a_0) & \sigma^3(a_3) & \sigma^3(a_2) \\ \sigma^2(a_2) & \sigma^2(a_1) & \sigma^2(a_0) & \sigma^2(a_3) \\ \sigma(a_3) & \sigma(a_2) & \sigma(a_1) & \sigma(a_0) \end{bmatrix} \tag{2.1.6}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

where $i - j$ is reduced mod 4 and where $a_j \in A_\theta$. (To clarify Ψ_*, if E is a projection in some matrix algebra over B_θ, then $\Psi(E)$ is a projection in some matrix algebra over $M_4(A_\theta)$, hence in a matrix algebra over A_θ, and thus gives a class in $K_0(A_\theta)$—*e.g.* $\Psi_*[1] = 4[1]_{K_0(A_\theta)}$.) For example (and we shall need this later), if e_θ is a smooth Powers-Rieffel projection in A_θ with trace θ ($0 < \theta < 1$ rational or irrational) then, viewing e_θ as an element of B_θ via the canonical inclusion $A_\theta \hookrightarrow B_\theta$, one has

$$C_1[e_\theta] = -4.$$

This follows since $c_1[e_\theta] = -1$, $\Psi(e_\theta) = \operatorname{diag}\big(e_\theta, \sigma^3(e_\theta), \sigma^2(e_\theta), \sigma(e_\theta)\big)$, and $[\sigma(e_\theta)] = [e_\theta]$ in $K_0(A_\theta)$, so that $\Psi_*[e_\theta]_{K_0(B_\theta)} = 4[e_\theta]_{K_0(A_\theta)}$, where $\Psi_* \colon K_0(B_\theta) \to K_0(A_\theta)$ is the induced map.

Consider the Connes Chern character

$$\mathrm{ch}\colon K_0(B_\theta) \to HC^{\mathrm{ev}}(B_\theta)^*$$

where $HC^{\mathrm{ev}}(B_\theta)^*$ is the complex vector space dual of the even periodic cyclic cohomology group [5, III.1.α]. From this, one defines the map $\mathbf{T}\colon K_0(B_\theta) \to \mathbb{R}^8 \times \mathbb{Z}$ by the pairing

$$\begin{aligned}\mathbf{T}(x) &= \langle(\tau; \phi_0, \phi_0', \phi_1, \phi_1'; T_{20}, T_{21}, T_{22}; C_1), \mathrm{ch}(x)\rangle \\ &:= \big(\tau(x); \phi_0(x), \phi_0'(x), \phi_1(x), \phi_1'(x); T_{20}(x), T_{21}(x), T_{22}(x); C_1(x)\big).\end{aligned}$$

All computations below will be done in terms of this map (as was done in [15]), and there is some justification for calling $\mathbf{T}$ the Connes Chern character, since there is evidence that after tensoring with $\mathbb{C}$, one eventually has an isomorphism

$$\mathrm{ch}_{\mathbb{C}}\colon K_0(B_\theta) \otimes \mathbb{C} \to HC^{\mathrm{ev}}(B_\theta)^*$$

between vector spaces of dimension nine. The evidence for this comes from the fact proved in [15, Theorem 2.3] that for irrational θ one has $HC^0(B_\theta) \cong \mathbb{C}^8$ and has as basis $\{\tau, \phi_0, \phi_0', \phi_1, \phi_1', T_{20}, T_{21}, T_{22}\}$. These, together with the class associated to Connes' cyclic 2-cocycle would presumably constitute a basis for $HC^{\mathrm{ev}}(B_\theta)$, which the author suspects is $HC^0(B_\theta) \oplus HC^2(B_\theta)$ modulo identifications given by the periodicity operator S (in Connes' notation) after tensoring with the complex plane over the ring $HC^*(\mathbb{C})$. This further suggests that the Hochschild dimension of B_θ is two, as Connes showed to be the case for the rotation algebra. (Of course, for rational θ, the group $HC^0(B_\theta)$ is infinite dimensional, but one would still expect that the periodic cohomology group $HC^{\mathrm{ev}}(B_\theta)$ to be finite dimensional—in fact, nine-dimensional.)

For the identity element and the Powers-Rieffel projection one clearly has

$$\mathbf{T}(1) = (1; 0, 0, 0, 0; 0, 0, 0; 0), \qquad \mathbf{T}(e_\theta) = (\theta; 0, 0, 0, 0; 0, 0, 0; -4).$$

The main result of [15] is the data of Connes Chern character values for the above nine modules for any θ shown in Table 1.

Table 1 yields the following.

Theorem 2.1 **([15, Theorem 2.4])** *For $0 < \theta < 1$, the nine modules $\{P_1(\theta), \dots, P_9(\theta)\}$ give rise to independent classes in $K_0(B_\theta)$. When θ is rational, the map $\mathbf{T}$ is injective on $K_0(B_\theta)$, and hence so is the Connes Chern character* $\mathrm{ch}\colon K_0(B_\theta) \to HC^{\mathrm{ev}}(B_\theta)^*$.

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Projection	τ	ϕ_0	ϕ_0'	ϕ_1	ϕ_1'	T_{20}	T_{21}	T_{22}	C_1
$P_1(\theta)$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	0	0	0
$P_2(\theta)$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	0
$P_3(\theta)$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	0	0
$P_4(\theta)$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	0	0
$P_5(\theta)$	$\frac{1}{2}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0
$P_6(\theta)$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0
$P_7(\theta)$	$\frac{\theta}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	-1
$P_8(\theta)$	$\frac{\theta}{4}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	-1
$P_9(\theta)$	$\frac{\theta}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	-1

Table 1: Character Table

Notation We shall denote by $\mathcal{R}_\theta$ the subgroup of $K_0(B_\theta)$ generated by the classes $\{P_j(\theta)\}_{j=1}^9$.

Consider the element of $K_0(B_{p/q})$ defined by (for relatively prime integers p, q)

(2.1.7) $$\begin{aligned}\kappa_{p,q} &= (p+q)([P_1] - 2[P_3] + [P_4] - 2[P_6]) \\ &\quad + p([P_2] + [P_5] + [P_7]) - 2q[P_8] - (2q+p)[P_9].\end{aligned}$$

(Here, each P_j is evaluated at $\theta = p/q$.) It is easy to check that $\mathbf{T}(\kappa_{p,q}) = (0; 0, 0, 0, 0; 0, 0, 0; 4q)$ from Table 1. Since $\mathbf{T}(p[1] - q[e_\theta]) = (0; 0, 0, 0, 0; 0, 0, 0; 4q) = \mathbf{T}(\kappa_{p,q})$, the injectivity of $\mathbf{T}$ (in the rational case, Theorem 2.1) gives the equality

$$p[1] - q[e_\theta] = \kappa_{p,q}$$

in $K_0(B_\theta)$. In fact, in the same manner one easily checks that the Powers-Rieffel projection e_θ is related to the nine modules as follows for rational θ

$$[e_\theta] = -[P_1] + 2[P_3] - [P_4] + 2[P_6] + 2[P_8] + 2[P_9]$$

in $K_0(B_\theta)$ (the right side evaluated at θ). This shows that $[e_\theta] \in \mathcal{R}_\theta$ for rational θ.

Define the *reduced* character $\mathbf{T}'\colon K_0(B_\theta) \to \mathbb{R}^8$ to be the degree zero part of the Connes Chern character $\mathbf{T}$, namely,

$$\mathbf{T}' = (\tau; \phi_0, \phi_0', \phi_1, \phi_1'; T_{20}, T_{21}, T_{22}).$$

Sometimes, especially in Sections 4 and 5, we shall collapse ϕ_j, ϕ'_j back to T_{1j} and simply write $\mathbf{T}' = (\tau; T_{10}, T_{11}; T_{20}, T_{21}, T_{22})$. This will help simplify matters later on and so will be more convenient to do.

Note that $\kappa_{p,q}$ is in $\operatorname{Ker}(\mathbf{T}')$ from above. Two key steps in the proofs below is to show that in fact $\kappa_{p,q}$ generates $\operatorname{Ker}(\mathbf{T}')$ (Corollary 5-D) and that the range of $\mathbf{T}'$ on $K_0(B_\theta)$ is equal to its range on $\mathcal{R}_\theta$ for θ in a special dense set of rationals $\mathbb{Q}'$ described in Section 2.3 (Proposition 4-D). These steps lead one to the equality

$$K_0(B_{p/q}) = \mathcal{R}_{p/q}$$

from which it follows that the modules $P_j(p/q)$ form a basis for $K_0(B_{p/q})$.

2.2 Realization of $A_{p/q}$ as a Dimension-Drop Algebra

Begin with the following realization of the rational rotation algebra as the subalgebra of $C([0,1] \times [0,1], M_q)$ given in [2, p. 64], by

$$\begin{aligned} A_{p/q} = \big\{ f \in C([0,1] \times [0,1], M_q) : f(x,1) = \alpha_1\big(f(x,0)\big), \\ f(1,y) = \alpha_2\big(f(0,y)\big) \big\} \end{aligned} \tag{2.2.1}$$

where $M_q := M_q(\mathbb{C})$ is generated by the unitaries

$$U_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda^{q-1} \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

satisfying $V_0U_0 = \lambda U_0V_0$, where $\lambda = e(p/q)$, and α_1, α_2 are the automorphisms of M_q given by

$$\alpha_1(U_0) = U_0, \quad \alpha_2(U_0) = wU_0$$
$$\alpha_1(V_0) = wV_0, \quad \alpha_2(V_0) = V_0$$

where $w = e(1/q)$. With this realization, the canonical generators U, V of A_θ are given by the functions

$$U(x,y) = e(x/q)U_0, \quad V(x,y) = e(y/q)V_0$$

and the Fourier automorphism is given by

$$\sigma(f)(x,y) = \sigma_0\big(f(y, 1-x)\big)$$

where $\sigma_0 \in \operatorname{Aut}(M_q)$ is given by

$$\sigma_0(U_0) = V_0, \quad \sigma_0(V_0) = \overline{w}U_0^{-1}.$$

In addition, the following $q \times q$ matrices were introduced in [2]:

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$W_1 = U_0^{-p'} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & w & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & w^{q-1} \end{bmatrix}, \quad W_2 = V_0^{-p''} = \begin{bmatrix} \mathbf{0} & I_{p''} \\ I_{q-p''} & \mathbf{0} \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix, and p', p'' are the unique integers in $[1, q-1]$ such that

(2.2.2) $$pp' \equiv -1 \mod q, \quad pp'' \equiv 1 \mod q.$$

In [7] the following $q \times q$ matrix was introduced (in addition to the above)

$$W_0 = \frac{1}{\sqrt{q}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^{q-1} \\ 1 & \lambda^2 & \lambda^4 & \cdots & \lambda^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda^{q-1} & \lambda^{2(q-1)} & \cdots & \lambda^{(q-1)^2} \end{bmatrix}.$$

One has the following relations that will be used below

(2.2.3) $$W_0^2 = \Gamma_0, \quad W_0 V_0 = U_0^{-1} W_0, \quad W_0 U_0 = V_0 W_0,$$
$$W_0 W_1 = W_2^{-1} W_0, \quad W_0 W_2 = W_1 W_0.$$

It is easy to see that $\alpha_1(x) = W_1^{-1} x W_1$ and $\alpha_2(x) = W_2^{-1} x W_2$. Using the inner automorphisms $\alpha_0(x) = W_0^{-1} x W_0$ and $\gamma_0(x) = \Gamma_0^{-1} x \Gamma_0$ (as in [7]) the relations (2.2.3) yield

(2.2.4) $$\alpha_0(U_0) = V_0^{-1}, \quad \alpha_1(U_0) = U_0, \quad \alpha_2(U_0) = wU_0, \quad \gamma_0(U_0) = U_0^{-1}$$
$$\alpha_0(V_0) = U_0, \quad \alpha_1(V_0) = wV_0 \quad \alpha_2(V_0) = V_0, \quad \gamma_0(V_0) = V_0^{-1}.$$

Further, if

(2.2.5) $$W_0' = \lambda^{-p'p''/4} W_0 W_1$$

(which is denoted by $\widetilde{W_0 W_1}$ in [7]), then $W_0'^4 = 1$ and one easily checks that $\sigma_0(x) = W_0' x W_0'^{-1}$.

These matrices are used in [2] and [7] in their realizations of the crossed products $A_\theta \rtimes \mathbb{Z}_2$ (under the flip) and B_θ, respectively, as spheres with singularities. In Section 3 below, the basic facts related to B_θ (when θ is rational) from [7] are recalled for use in this paper (with some corrections made in Section 8).

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

2.3 The Subset of Rationals $\mathbb{Q}'$

Given positive relatively prime integers p, q, let p', p'' be the integers as in (2.2.2) and write $pp' = -1 + q\tilde{p}$, $pp'' = 1 + q\tilde{q}$ for some integers $\tilde{p}$ and $\tilde{q}$. One easily checks that $p = \tilde{p} + \tilde{q}$ and $q = p' + p''$. In the present paper we shall be interested in the following dense set of rational numbers in $(0, 1)$

$$\mathbb{Q}' := \left\{ \tfrac{2^d(2k)+1}{4^d} : k = 1, \dots, 2^{d-1} - 1,\ d \geq 2 \right\}. \tag{2.3.1}$$

For such rationals, $p = 2^d(2k) + 1$, $q = 4^d$, and one can verify directly that

$$p' = 2^d(2k) - 1, \quad p'' = 2^d(2^d - 2k) + 1, \quad \tilde{p} = 4k^2, \quad \tilde{q} = 2k(2^d - 2k) + 1. \tag{2.3.2}$$

This choice will facilitate the computation of the Gaussian sums that arise below. In this case the Gaussian sum $G(p, q) := \sum_{j=0}^{q-1} e^{2\pi i p j^2/q}$ takes the simpler form (see [8])

$$G(p, 4^d) = 2^d(1 + i^p). \tag{2.3.3}$$

2.4 The Connes Chern Character on A_θ (For Rational θ)

Realizing A_θ as M_q-valued functions on the unit square, as in (2.2.1), where $\theta = p/q$, the canonical trace is given by

$$\tau(F) = \frac{1}{q} \int_0^1 \int_0^1 \mathrm{Tr}_q\big(F(x, y)\big)\, dx\, dy$$

for $F \in A_\theta$, where Tr_q is the usual trace on $M_q(\mathbb{C})$. Also, the canonical derivations of A_θ are given by

$$\delta_1 = q\frac{\partial}{\partial x}, \quad \delta_2 = q\frac{\partial}{\partial y}.$$

They are defined by

$$\delta_1(U^m V^n) = 2\pi i m U^m V^n, \quad \delta_2(U^m V^n) = 2\pi i n U^m V^n.$$

Connes' canonical cyclic 2-cocycle is given by (see [5, III.2.β])

$$\begin{aligned}
\varphi_q(F^0, F^1, F^2) &= \frac{1}{2\pi i}\tau\big(F^0[\delta_1(F^1)\delta_2(F^2) - \delta_2(F^1)\delta_1(F^2)]\big) \\
&= \frac{1}{2\pi i}\frac{1}{q}\int_0^1\int_0^1 \mathrm{Tr}_q\big(F^0[\delta_1(F^1)\delta_2(F^2) - \delta_2(F^1)\delta_1(F^2)]\big)\, dx\, dy \\
&= \frac{q}{2\pi i}\int_0^1\int_0^1 \mathrm{Tr}_q\Big(F^0\Big[\frac{\partial F^1}{\partial x}\frac{\partial F^2}{\partial y} - \frac{\partial F^1}{\partial y}\frac{\partial F^2}{\partial x}\Big]\Big)\, dx\, dy
\end{aligned}$$

where $F^j \in A_\theta$ (are smooth elements). The extension of φ_q to $M_n(A_\theta)$ is given by the cup product

$$(\varphi_q \# \mathrm{Tr}_n)(F^0 \otimes a^0, F^1 \otimes a^1, F^2 \otimes a^2) = \varphi_q(F^0, F^1, F^2) \cdot \mathrm{Tr}_n(a^0 a^1 a^2)$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

where $F^j \in A_\theta$ and $a^j \in M_n(\mathbb{C})$. The Chern character invariant of Connes is then given by: $c_1 : K_0(A_\theta) \to \mathbb{Z}$,

$$c_1[Q] := \langle [Q], \varphi_q \rangle = (\varphi_q \# \mathrm{Tr}_n)(Q, Q, Q),$$

where Q is a projection in $M_n(A_\theta)$. For $0 < \theta < 1$ the Powers-Rieffel projection e_θ has $c_1(e_\theta) = \varphi_q(e_\theta, e_\theta, e_\theta) = -1$ (as was shown by Connes). For $\theta = 1$, one can show that c_1 of the Bott projection is ± 1, depending on the choices made for it (see Section 5 below for details, and the footnote there).

2.5 Gaussian Sums

Recall the classical quadratic Gauss sum defined by

$$G(p, q) := \sum_{j=0}^{q-1} \exp(2\pi i j^2 p/q) = \sum_{j=0}^{q-1} \lambda^{j^2}$$

where p, q are relatively prime positive integers and $\lambda = e^{2\pi i p/q}$. For relatively prime positive integers p, q and any $m \in \mathbb{Z}$ define the following variant of the Gaussian sum

$$F(p, q; m) := \sum_{j=0}^{q-1} \lambda^{j^2+mj} = \sum_{j=0}^{q-1} \exp\big(2\pi i(j^2 + mj)p/q\big),$$

$$F(p, q) := F(p, q; 1) = \sum_{j=0}^{q-1} \exp\big(2\pi i(j^2 + j)p/q\big).$$

(These sums arise in our trace computations below.) Suppose first that $m = 2n$. Then

$$\begin{aligned} F(p, q; m) &= \exp(-2\pi i p n^2/q) \cdot \sum_{j=0}^{q-1} \exp\big(2\pi i (j + n)^2 p/q\big) \\ &= \exp(-\pi i p m^2/2q) \cdot G(p, q) \end{aligned}$$

since the terms in the preceding summation are just a cyclic permutation of the terms comprising $G(p, q)$. Now suppose $m = 2n + 1$ is odd. Then by the same reasoning one has

$$\begin{aligned} F(p, q; m) &= \exp\big(-2\pi i p(n^2 + n)/q\big) \cdot \sum_{j=0}^{q-1} \exp\big(2\pi i[(j + n)^2 + (j + n)]p/q\big) \\ &= \exp\big(-\pi i p(m^2 - 1)/2q\big) \cdot F(p, q). \end{aligned}$$

Therefore, in either case one has

$$F(p, q; m) = \lambda^{-m^2/4} G(p, q)\delta_{\overline{m},0} + \lambda^{-(m^2-1)/4} F(p, q)\delta_{\overline{m},1}. \tag{2.5.1}$$

Lemma 2.5-A *Let $\lambda = e^{2\pi i p/q}$, where $p < q$ are relatively prime positive integers. Then for $q = 4^d$, where d is a positive integer, one has*

$$F(p, 4^d) = 0, \quad \text{and} \quad F(p, 4^d; m) = 2^d(1 + i^p)\lambda^{-m^2/4}\delta_{\overline{m},0}.$$

Proof Once $F(p, 4^d) = 0$ has been proven, one uses the fact that $G(p, 4^d) = 2^d(1 + i^p)$ to get, from (2.5.1),

$$F(p, q; m) = \lambda^{-m^2/4}G(p, q)\delta_{\overline{m},0} = 2^d(1 + i^p)\lambda^{-m^2/4}\delta_{\overline{m},0}$$

giving the second equality. To see the first equality, we will show more generally that if 4 divides q (so that p is odd), then $F(p, q) = 0$. Dividing the sum as follows

$$F(p, q) = \sum_{j=0}^{q-1} \lambda^{j^2+j} = \sum_{j=0}^{\frac{q}{2}-1} \lambda^{j^2+j} + \sum_{j=\frac{q}{2}}^{q-1} \lambda^{j^2+j}$$

set $k = j - \frac{q}{2}$ in the second sum, and since $q/4$ is an integer and $\lambda^{q/2} = -1$, one obtains

$$F(p, q) = \sum_{j=0}^{\frac{q}{2}-1} \lambda^{j^2+j} + \sum_{k=0}^{\frac{q}{2}-1} \lambda^{k^2+k}\lambda^{q(q/4)}\lambda^{q/2} = \sum_{j=0}^{\frac{q}{2}-1} \lambda^{j^2+j} - \sum_{k=0}^{\frac{q}{2}-1} \lambda^{k^2+k} = 0. \qquad \blacksquare$$

Lemma 2.5-B *For relatively prime p, q one has*

$$\operatorname{Tr}(U_0^m V_0^n W_0^2) = \lambda^{-mn/2}\left(\delta_{\overline{n},0} + (-1)^{pm}\delta_{\overline{n},\overline{q}}\right),$$

$$\operatorname{Tr}(U_0^m V_0^n W_0^3) = \tfrac{1}{\sqrt{q}}\overline{F(p, q, m - n)}$$

$$\operatorname{Tr}(U_0^m V_0^n W_0'^3) = \tfrac{1}{\sqrt{q}}\lambda^{p'p''/4}\lambda^{np'}\overline{F(p, q, m + p' - n)}.$$

In particular, when $q = 4^d$ (where d is a positive integer) these become

$$\operatorname{Tr}(U_0^m V_0^n W_0^2) = 2T_{20}(U^m V^n W^2),$$

$$\operatorname{Tr}(U_0^m V_0^n W_0^3) = (1 - i^p)T_{10}(U^m V^n W^3),$$

$$\operatorname{Tr}(U_0^m V_0^n W_0'^3) = -i(1 - i^p)\lambda^{p'(m+n)/2}T_{11}(U^m V^n W^3).$$

Proof Since

$$V_0^n = \begin{bmatrix} \mathbf{O} & I_{q-n} \\ I_n & \mathbf{O} \end{bmatrix}$$

one decomposes W_0 into the following block form

$$W_0 = \begin{bmatrix} n \times (q - n) & n \times n \\ (q - n) \times (q - n) & (q - n) \times n \end{bmatrix} = \tfrac{1}{\sqrt{q}} \begin{bmatrix} * & X \\ Y & * \end{bmatrix}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

where, writing out only the (relevant) diagonal entries,

$$X = \begin{bmatrix} 1 & & & & & \\ & \lambda^{q-n+1} & & & & \\ & & \ddots & & & \\ & & & \lambda^{j(q-n+j)} & & \\ & & & & \ddots & \\ & & & & & \lambda^{(q-1)(n-1)} \end{bmatrix}$$

where $j = 0, 1, \ldots, n-1$, and

$$Y = \begin{bmatrix} 1 & & & & & \\ & \lambda^{n+1} & & & & \\ & & \ddots & & & \\ & & & \lambda^{j(n+j)} & & \\ & & & & \ddots & \\ & & & & & \lambda^{(q-1)(q-n-1)} \end{bmatrix}$$

where $j = 0, 1, \ldots, q-n-1$ (the non diagonal entries here have been left blank). From this one has

$$U_0^m V_0^n W_0 = \tfrac{1}{\sqrt{q}} U_0^m \begin{bmatrix} \mathbf{O} & I_{q-n} \\ I_n & \mathbf{O} \end{bmatrix} \begin{bmatrix} * & X \\ Y & * \end{bmatrix} = \tfrac{1}{\sqrt{q}} U_0^m \begin{bmatrix} Y & * \\ * & X \end{bmatrix}$$

and since

$$U_0^m = \operatorname{diag}(1, \lambda^m, \cdots, \lambda^{m(q-n-1)} \mid \lambda^{m(q-n)}, \cdots, \lambda^{m(q-1)})$$

one obtains

$$\sqrt{q}\,\operatorname{Tr}(U_0^m V_0^n W_0) = \sum_{j=0}^{q-n-1} \lambda^{mj} \cdot \lambda^{j(n+j)} + \sum_{j=0}^{n-1} \lambda^{m(q-n+j)} \cdot \lambda^{j(q-n+j)}.$$

Making the translation $k = j + n$ in the first sum it becomes $\sum_{k=n}^{q-1} \lambda^{m(k-n)} \cdot \lambda^{k(k-n)}$ which has the same type of terms as the second sum. Thus (and using $\lambda^q = 1$),

$$\sqrt{q}\,\operatorname{Tr}(U_0^m V_0^n W_0) = \lambda^{-mn} \sum_{k=0}^{q-1} \lambda^{k^2+(m-n)k} = \lambda^{-mn} F(p, q, m-n)$$

so that one gets

$$\operatorname{Tr}(U_0^m V_0^n W_0) = \tfrac{1}{\sqrt{q}} \lambda^{-mn} F(p, q, m-n).$$

From this one easily gets

$$\operatorname{Tr}(U_0^m V_0^n W_0^3) = \lambda^{-mn} \overline{\operatorname{Tr}(U_0^{-m} V_0^{-n} W_0)} = \tfrac{1}{\sqrt{q}} \overline{F(p, q, m-n)}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

as required. From this one gets, after recalling that $W_0' = \lambda^{-p'p''/4}W_0W_1 = \lambda^{-p'p''/4}W_0U_0^{-p'}$,

$$\begin{aligned}\operatorname{Tr}(U_0^mV_0^nW_0'^3) &= \operatorname{Tr}(U_0^mV_0^nW_0'^{-1})\\ &= \lambda^{p'p''/4}\operatorname{Tr}(U_0^mV_0^nU_0^{p'}W_0^3)\\ &= \lambda^{p'p''/4}\lambda^{np'}\operatorname{Tr}(U_0^{m+p'}V_0^nW_0^3)\\ &= \lambda^{p'p''/4}\lambda^{np'}\tfrac{1}{\sqrt{q}}\overline{F(p,q,m+p'-n)}\,.\end{aligned}$$

Next we compute $\operatorname{Tr}(U_0^mV_0^nW_0^2)$. Since $W_0^2 = \Gamma_0$,

$$V_0^nW_0^2 = \begin{bmatrix}\mathbf{O} & I_{q-n}\\ I_n & \mathbf{O}\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & ⋰ & \vdots & \\ 0 & 1 & \cdots & 0 & 0\end{bmatrix} = \begin{bmatrix}S_{q-n+1} & \mathbf{O}\\ \mathbf{O} & S_{n-1}\end{bmatrix}$$

where S_k the $k\times k$ symmetry matrix

$$S_k = \begin{bmatrix}0 & \cdots & 0 & 1\\ 0 & \cdots & 1 & 0\\ \vdots & ⋰ & \vdots & \vdots\\ 1 & \cdots & 0 & 0\end{bmatrix}.$$

So

$$U_0^mV_0^nW_0^2 = \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \lambda^m & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & \lambda^{m(q-1)}\end{bmatrix}\cdot\begin{bmatrix}S_{q-n+1} & \mathbf{O}\\ \mathbf{O} & S_{n-1}\end{bmatrix} = \begin{bmatrix}X_{q-n+1} & \mathbf{O}\\ \mathbf{O} & Y_{n-1}\end{bmatrix}$$

where

$$X_{q-n+1} = \begin{bmatrix}0 & \cdots & 0 & 1\\ 0 & \cdots & \lambda^m & 0\\ \vdots & ⋰ & \vdots & \vdots\\ \lambda^{m(q-n)} & \cdots & 0 & 0\end{bmatrix},$$

$$Y_{n-1} = \begin{bmatrix}0 & \cdots & 0 & \lambda^{m(q-n+1)}\\ 0 & \cdots & \lambda^{m(q-n+2)} & 0\\ \vdots & ⋰ & \vdots & \vdots\\ \lambda^{m(q-1)} & \cdots & 0 & 0\end{bmatrix}.$$

Therefore,

$$\mathrm{Tr}(U_0^m V_0^n W_0^2) = \lambda^{m(q-n)/2}\delta_{\overline{q-n},0} + \lambda^{-mn/2}\delta_{\overline{n},0} = \lambda^{-mn/2}\big(\delta_{\overline{n},0} + (-1)^{pm}\delta_{\overline{n},\overline{q}}\big).$$

Specializing to the case $q = 4^d$ (so p is odd) and using Lemma 2.5-A one has

$$\mathrm{Tr}(U_0^m V_0^n W_0^3) = \tfrac{2^d}{\sqrt{q}}(1+i^p)\lambda^{(m-n)^2/4}\delta_{\overline{m},\overline{n}} = (1+i^p)T_{10}(U^m V^n W^3),$$

$$\mathrm{Tr}(U_0^m V_0^n W_0^2) = \lambda^{-mn/2}\big(1+(-1)^m\big)\,\delta_{\overline{n},0} = 2\lambda^{-mn/2}\delta_{\overline{m},0}\delta_{\overline{n},0} = 2T_{20}(U^m V^n W^2),$$

and (as p, p' are odd and $q = p' + p''$, from Section 2.3)

$$\begin{aligned}\mathrm{Tr}(U_0^m V_0^n {W_0'}^3) &= \tfrac{1}{\sqrt{q}}\lambda^{p'p''/4}\lambda^{np'}\overline{F(p,4^d,m+p'-n)}\\ &= \tfrac{1}{\sqrt{q}}\lambda^{p'p''/4}\lambda^{np'}\cdot 2^d(1+i^{-p})\lambda^{(m+p'-n)^2/4}\delta_{\overline{m+p'},\overline{n}}\\ &= (1-i^p)\lambda^{(p'p''+p'^2)/4}\lambda^{p'(m+n)/2}\cdot\lambda^{(m-n)^2/4}\delta_{\overline{m+1},\overline{n}}\\ &= -i(1-i^p)\lambda^{p'(m+n)/2}T_{11}(U^m V^n W^3)\end{aligned}$$

since $\lambda^{(p'p''+p'^2)/4} = \lambda^{p'q/4} = e(pp'/4) = e\big((-1+q\tilde{p})/4\big) = -i$, as required. ∎

3 Relation Between Two Sets of Unbounded Traces

In [7] it is proved that the algebra B_θ, for rational $\theta = p/q$ (with $(p,q) = 1$), is isomorphic to a subalgebra of $C(\mathbb{S}^2, M_{4q})$ of functions that commute with certain projections at three special points (which we refer to as "singularities"). As they do, we shall identify the 2-sphere with the triangle T (shown in the figures below) with the appropriate edges identified. We shall use T to denote the triangle without identifying its edges and write $\mathbb{S}^2$ to denote the triangle with its edges identified. For convenience, we shall view this subalgebra as the set of all functions that commute with certain finite-order unitaries at the singular points. More precisely, B_θ is isomorphic to (see [7, Theorem 6.2.1])

$$(3.1)\qquad S_\theta := \left\{ F \in C(\mathbb{S}^2, M_q \otimes M_4) : \begin{array}{rcc} F(0,0) & \longleftrightarrow & W_0^{-1}\otimes D, \\ F\big(\frac{1}{2},\frac{1}{2}\big) & \longleftrightarrow & (W_0^{-1}{W_0'}^{-1}W_0)\otimes D, \\ F\big(\frac{1}{2},0\big) & \longleftrightarrow & \Gamma_0 W_2 \otimes D^2 \end{array} \right\}$$

where $(0,0)$, $\big(\frac{1}{2},\frac{1}{2}\big)$, $\big(\frac{1}{2},0\big)$ are the singular points and $X \leftrightarrow Y$ means that X and Y commute, and where $D = \mathrm{diag}(1,-1,i,-i)$ (which is $\hat{Z}$ in the notation of [7]). It is easy to see that the canonical trace on S_θ, which arises from that of A_θ given in Section 2.4, is given by

$$\tau(F) = \frac{1}{q}\iint_T \mathrm{Tr}_{4q}\big(F(x,y)\big)\,dx\,dy.$$

(Which is clearly normalized.)

To obtain the isomorphism $B_\theta \to S_\theta$, one considers (as in [7, p. 1190]) the intermediate algebra

$$\mathcal{T}_\theta := \left\{ g \in C(T, M_q \otimes M_4) : \begin{matrix} g(x,x) = (\alpha_1\alpha_0 \otimes \mathrm{Ad}\, D^{-1})\big(g(1-x,x)\big) \\ g(x,0) = (\alpha_2\gamma_0 \otimes \mathrm{Ad}\, D^2)\big(g(1-x,0)\big) \end{matrix} \right\}. \tag{3.2}$$

At this point we draw the reader's attention to the Appendix below for the corrections to [7] to be used freely henceforth (and already included in (3.1) and (3.2) above).

There are isomorphisms

$$B_\theta \xrightarrow{\ \gamma\ } \mathcal{T}_\theta \xrightarrow{\ \beta\ } S_\theta \tag{3.3}$$

given by, for $f \in A_\theta$ (viewed as a function on the unit square as in Section 2),

$$\gamma(f) = \frac{1}{4}\begin{bmatrix} f_0 & f_2 & f_1 & f_3 \\ f_2 & f_0 & f_3 & f_1 \\ f_3 & f_1 & f_0 & f_2 \\ f_1 & f_3 & f_2 & f_0 \end{bmatrix}, \quad \text{and} \quad \gamma(W) = I_q \otimes D = \begin{bmatrix} I_q & & & \\ & -I_q & & \\ & & iI_q & \\ & & & -iI_q \end{bmatrix},$$

where

$$f_k := \sum_{j=0}^{3} i^{jk} \sigma^j(f)$$

which is restricted to the triangle T and belongs to the vector space

$$A_\theta^\tau(i^k) := \left\{ g \in C(T, M_q) : \begin{matrix} g(x,x) = \overline{i^k}\alpha_1\alpha_0\big(g(1-x,x)\big) \\ g(x,0) = (i^k)^2\alpha_2\gamma_0\big(g(1-x,0)\big) \end{matrix} \right\}, \tag{3.4}$$

where τ here is our σ^{-1} (see (8.1) of the Appendix.) Conversely, if g_k are functions in $A_\theta^\tau(i^k)$, $k = 0,1,2,3$, then it is not hard to see that there is a unique function $f \in A_\theta$ such that $f_k = g_k$ for each k.

The map β can be described as follows (after a careful examination of the proofs in Sections 4.2 and 6.2 of [7]). For $g \in \mathcal{T}_\theta$ one defines $\beta(g)$ to be the continuous function on T (as a 2-sphere) such that

$$\beta(g)(s) := (R_s \otimes D_s)\cdot g(s)\cdot (R_s \otimes D_s)^{-1} \tag{3.5}$$

for $s \in T - \{s_j\}$, where $s \mapsto R_s$ and $s \mapsto D_s$ are unitary-valued functions on T (with respective values in $M_q(\mathbb{C})$ and $M_4(\mathbb{C})$) that are continuous on $T - \{s_j\}$ and have edge-limits as indicated in the figure shown below. The mapping D_s can be chosen to be diagonal-valued, a fact used below. Necessarily, these functions have jump discontinuities at the singular points, but they are carefully chosen so that $\beta(g)(s)$ is well-defined and continuous on $\mathbb{S}^2$—see [7, p. 1190].

The algebra S_θ has ten trace functionals that arise from the three singular points. Given $F \in S_\theta$ at each such point one can take the trace of any one of the block

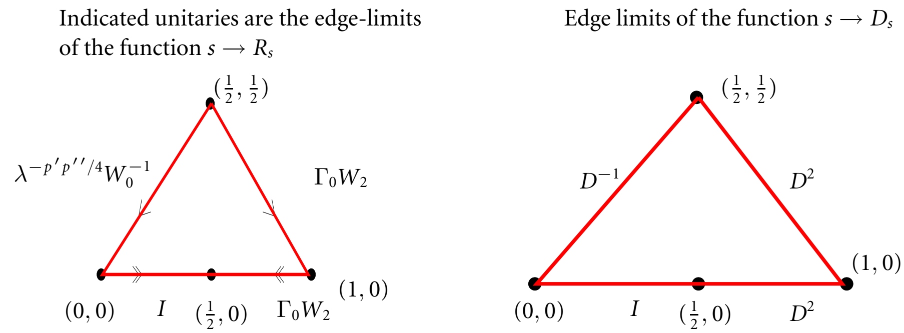

Figure 1: The triangle T and the 2-sphere S^2

decompositions of $F(s)$ relative to the corresponding unitary that it commutes with. But instead of doing this it will be more convenient to consider the following trace functionals

$$
\begin{aligned}
\tau_{1k}(F) &= \mathrm{Tr}\big(F(0,0)(W_0^{-1} \otimes D)^k \big) \quad k = 0,1,2,3 \\
\tau_{2k}(F) &= \mathrm{Tr}\Big(F\big(\tfrac{1}{2},\tfrac{1}{2}\big) \cdot \big((W_0^{-1} W_0'^{-1} W_0) \otimes D \big)^k \Big) \quad k = 0,1,2,3 \\
\tau_{0k}(F) &= \mathrm{Tr}\Big(F\big(\tfrac{1}{2},0\big) \, (\Gamma_0 W_2 \otimes D^2)^k \Big) \quad k = 0,1.
\end{aligned}
\tag{3.6}
$$

(These are in fact tracial maps on S_θ.) To simplify, denote the underlying unitaries in each case by $w_j \otimes Z_j$ and the respective singular points by $s_1 = (0,0)$, $s_2 = \big(\frac{1}{2},\frac{1}{2}\big)$, $s_0 = \big(\frac{1}{2},0\big)$, so that (3.6) can be written as

$$
\tau_{jk}(F) = \mathrm{Tr}\big(F(s_j)(w_j \otimes Z_j)^k \big).
$$

Let $Y := \{s_0, s_1, s_2\}$. Fixing $f \in A_\theta$ and expanding $\gamma(f)$ as

$$
\gamma(f) = \tfrac{1}{4}\Big(f_0 \otimes I_4 + \sum_{j=1}^{3} f_j \otimes (\text{matrices with zero diagonal}) \Big)
$$

one has, for s in $T - Y$,

$$
\begin{aligned}
\beta\big(\gamma(f)\big)(s) &= (R_s \otimes D_s) \cdot \gamma(f)(s) \cdot (R_s \otimes D_s)^{-1} \\
&= \tfrac{1}{4}\big(R_s f_0(s) R_s^* \big) \otimes I_4 + \tfrac{1}{4} \sum_{j=1}^{3} \big(R_s f_j(s) R_s^* \big) \\
&\qquad \otimes (\text{matrices with zero diagonal})
\end{aligned}
$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

and since $\beta\big(\gamma(W)\big) = \beta(I_q \otimes D) = I_q \otimes D$ (viewed as a constant function on T) and Z_j are all diagonal (being powers of D), one gets

$$\begin{aligned}
\tau_{jk}\Big(\beta\big(\gamma(fW^r)\big)\Big) &= \tau_{jk}\Big(\beta\big(\gamma(f)\big)\,(I_q \otimes D)^r\Big) \\
&= \operatorname{Tr}\Big(\beta\big(\gamma(f)\big)\,(s_j)(I_q \otimes D)^r(w_j \otimes Z_j)^k\Big) \\
&= \lim_{\substack{s\to s_j \\ s\in T-Y}} \operatorname{Tr}\Big(\beta\big(\gamma(f)\big)\,(s)(I_q \otimes D)^r(w_j \otimes Z_j)^k\Big) \\
&= \tfrac{1}{4} \lim_{\substack{s\to s_j \\ s\in T-Y}} \operatorname{Tr}\Big[\Big(\big(R_s f_0(s) R_s^*\big) \otimes I_4\Big)\,(I_q \otimes D)^r(w_j \otimes Z_j)^k\Big] \\
&= \tfrac{1}{4} \lim_{\substack{s\to s_j \\ s\in T-Y}} \operatorname{Tr}\Big(\big(R_s f_0(s) R_s^* w_j^k\big) \otimes (D^r Z_j^k)\Big) \\
&= \tfrac{1}{4} \lim_{\substack{s\to s_j \\ s\in T-Y}} \operatorname{Tr}\big(R_s f_0(s) R_s^* w_j^k\big) \cdot \operatorname{Tr}(D^r Z_j^k).
\end{aligned}$$

Now near each singular point s_j the unitary R_s can approach either of the edges joining at s_j (see the left figure above); and although it is not continuous at the edges, the limit of the trace $\operatorname{Tr}\big(R_s f_0(s) R_s^* w_j^k\big)$ will be independent of which edge it approaches (since $R_s f_0(s) R_s^*$ is continuous). Thus one can let $s \to s_j$ "toward" either edge. For example, for $s_1 = (0,0)$ one can let $R_s \to I_q$ so that (since $Z_1 = D$ and $w_1 = W_0^{-1}$ from (3.6)) one has

$$\lim_{\substack{s\to s_1 \\ s\in T-Y}} \operatorname{Tr}\big(R_s f_0(s) R_s^* w_1^k\big) = \lim_{\substack{s\to s_1 \\ s\in T-Y}} \operatorname{Tr}\big(f_0(s) \cdot R_s^* W_0^{-k} R_s\big) = \operatorname{Tr}\big(f_0(0,0) W_0^{-k}\big)$$

since f_0 is itself continuous on T. Hence, one gets the first set of traces

$$\tau_{1k}\Big(\beta\big(\gamma(f)\gamma(W^r)\big)\Big) = \tfrac{1}{4}\operatorname{Tr}\big(f_0(0,0) W_0^{-k}\big) \cdot \operatorname{Tr}(D^{r+k})$$

where

$$\operatorname{Tr}(D^n) = \begin{cases} 0 & \text{if } n \not\equiv 0 (\operatorname{mod} 4), \\ 4 & \text{if } n \equiv 0 (\operatorname{mod} 4), \end{cases}$$

Similarly, one gets

$$\tau_{2k}\Big(\beta\big(\gamma(f)\gamma(W^r)\big)\Big) = \tfrac{1}{4}\operatorname{Tr}\Big(f_0\big(\tfrac{1}{2},\tfrac{1}{2}\big)\,W_0'^{-k}\Big) \cdot \operatorname{Tr}(D^{r+k})$$

$$\tau_{0k}\Big(\beta\big(\gamma(f)\gamma(W^r)\big)\Big) = \tfrac{1}{4}\operatorname{Tr}\Big(f_0\big(\tfrac{1}{2},0\big)\,(\Gamma_0 W_2)^k\Big) \cdot \operatorname{Tr}(D^{r+2k}).$$

There is no danger of confusion to denote by U, V, W the unitaries in S_θ corresponding to the original unitaries U, V, W in B_θ under the isomorphism $\beta\gamma$. With

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

$f = U^m V^n$ these yield

$$\tau_{1k}(U^mV^nW^r) = \tfrac{1}{4}\,\mathrm{Tr}\big(f_0(0,0)W_0^{-k}\big)\cdot \mathrm{Tr}(D^{r+k})$$

$$\tau_{2k}(U^mV^nW^r) = \tfrac{1}{4}\,\mathrm{Tr}\Big(f_0\big(\tfrac{1}{2},\tfrac{1}{2}\big)\,W_0'^{-k}\Big)\cdot \mathrm{Tr}(D^{r+k}) \tag{3.7}$$

$$\tau_{0k}(U^mV^nW^r) = \tfrac{1}{4}\,\mathrm{Tr}\Big(f_0\big(\tfrac{1}{2},0\big)\,(\Gamma_0W_2)^k\Big)\cdot \mathrm{Tr}(D^{r+2k}).$$

We are now ready to relate the set of evaluation traces $\{\tau_{jk}\}$ with the original traces $\{T_{jk}\}$.

Proposition 3 *With $q = 4^d$ and p odd, where d is a positive integer, one has*

$$\tau_{11} = 4(1-i^p)T_{10}, \quad \tau_{21} = -4i(-1)^{\tilde p}(1-i^p)T_{11},$$
$$\tau_{12} = 8T_{20}, \quad \tau_{22} = -8T_{21}, \quad \tau_{01} = 4T_{22}.$$

In particular, for $p/q \in \mathbb{Q}'$, these yield

$$T_{10} = \tfrac{1+i}{8}\tau_{11}, \quad T_{11} = \tfrac{i-1}{8}\tau_{21}, \quad T_{20} = \tfrac{1}{8}\tau_{12}, \quad T_{21} = -\tfrac{1}{8}\tau_{22}, \quad T_{22} = \tfrac{1}{4}\tau_{01}.$$

Proof In this proof we will make free use of Lemma 2.5-B and equations (2.2.3), and in the computations to follow we shall take $f = U^mV^n$ so that

$$f_0 = \sum_{j=0}^{3} \sigma^j(U^mV^n) = U^mV^n + V^mU^{-n} + U^{-m}V^{-n} + V^{-m}U^n$$

or

$$f_0(x,y) = e\big((mx+ny)/q\big)\,U_0^mV_0^n + e\big((-nx+my)/q\big)\,V_0^mU_0^{-n}$$
$$+ e(-mx-ny)U_0^{-m}V_0^{-n} + e(nx-my)V_0^{-m}U_0^n.$$

This will be used below in evaluating the expressions in (3.7). For τ_{11} (and $r = 3$ and since $\mathrm{Tr}(D^4) = 4$) one has, using (2.2.3) and Lemma 2.5-B,

$$\tau_{11}(U^mV^nW^3) = \mathrm{Tr}\big(f_0(0,0)W_0^{-1}\big) = 4\,\mathrm{Tr}(U_0^mV_0^nW_0^3) = 4(1-i^p)\cdot T_{10}(U^mV^nW^3),$$

which holds for all m, n. For τ_{21} one takes $r = 3$ and gets (and recalling that p' is odd and that W_0' is given by (2.2.5))

$$\begin{aligned}
\tau_{21}(U^mV^nW^3) &= \mathrm{Tr}\Big(f_0\big(\tfrac{1}{2},\tfrac{1}{2}\big)\,W_0'^3\Big)\\
&= 4e\big((m+n)/2q\big)\ \mathrm{Tr}(U_0^mV_0^nW_0'^3)\\
&= 4e\big((m+n)/2q\big)\cdot(-i)(1-i^p)\lambda^{p'(m+n)/2}T_{11}(U^mV^nW^3)\\
&= 4e\big((m+n)/2q\big)\cdot(-i)(1-i^p)e\big(pp'(m+n)/2q\big)\,T_{11}(U^mV^nW^3)\\
&= e\big(q\tilde p(m+n)/2q\big)\cdot(-4i)(1-i^p)T_{11}(U^mV^nW^3)\\
&= (-1)^{\tilde p(m+n)}\cdot(-4i)(1-i^p)T_{11}(U^mV^nW^3)\\
&= -4i(-1)^{\tilde p}(1-i^p)T_{11}(U^mV^nW^3)
\end{aligned}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

where the last equality holds since $m+n$ is odd when $T_{11}(U^mV^nW^3)$ does not vanish. For τ_{12} one takes $r=2$ to obtain

$$\tau_{12}(U^mV^nW^2) = \mathrm{Tr}\big(f_0(0,0)W_0^2\big) = 4\,\mathrm{Tr}(U_0^mV_0^nW_0^2) = 8T_{20}(U^mV^nW^2).$$

For τ_{22}, one uses the identity $W_0'^2 = \lambda^{-p'p''/2}V_0^{p''}U_0^{p'}W_0^2$ and the fact that p, p', p'' are odd (see Section 2.3), and that $p=\tilde{q}+\tilde{p}$, to obtain

$$\begin{aligned}
\tau_{22}(U^mV^nW^2) &= \mathrm{Tr}\Big(f_0\big(\tfrac12,\tfrac12\big)\,W_0'^2\Big)\\
&= 4e\big((m+n)/2q\big)\ \mathrm{Tr}(U_0^mV_0^nW_0'^2)\\
&= 4e\big((m+n)/2q\big)\,\lambda^{-p'p''/2}\,\mathrm{Tr}(U_0^mV_0^n\cdot V_0^{p''}U_0^{p'}W_0^2)\\
&= 4e\big((m+n)/2q\big)\,\lambda^{-p'p''/2}\lambda^{p'(n+p'')}\,\mathrm{Tr}(U_0^{m+p'}V_0^{n+p''}W_0^2)
\end{aligned}$$

and using the first equation of Lemma 2.5-B this becomes

$$\begin{aligned}
&= 4e\big((m+n)/2q\big)\,\lambda^{p'p''/2}\lambda^{p'n}\lambda^{-(m+p')(n+p'')/2}\big(\delta_{\overline{n+p''},0}+(-1)^{p(m+p')}\delta_{\overline{n+p''},\overline{q}}\big)\\
&= 4e\big((m+n)/2q\big)\,\lambda^{p'p''/2}\lambda^{p'n}\lambda^{-(m+p')(n+p'')/2}\big(1-(-1)^m\big)\,\delta_{\overline{n},1}\\
&= 4\lambda^{-mn/2}(-1)^{\tilde{q}m+\tilde{p}n}\big(1-(-1)^m\big)\,\delta_{\overline{n},1}\\
&= 8(-1)^{\tilde{q}+\tilde{p}}\lambda^{-mn/2}\delta_{\overline{n},1}\delta_{\overline{m},1}\\
&= 8(-1)^pT_{21}(U^mV^nW^2) = -8T_{21}(U^mV^nW^2).
\end{aligned}$$

For τ_{01} one has (recalling that $W_2 = V_0^{-p''}$)

$$\begin{aligned}
\tau_{01}(U^mV^nW^2) &= \mathrm{Tr}\Big(f_0\big(\tfrac12,0\big)\,\Gamma_0W_2\Big)\\
&= 2e(m/2q)\,\mathrm{Tr}(U_0^mV_0^n\Gamma_0W_2) + 2e(-n/2q)\,\mathrm{Tr}(V_0^mU_0^{-n}\Gamma_0W_2)\\
&= 2e(m/2q)\,\mathrm{Tr}(U_0^mV_0^nV_0^{p''}W_0^2) + 2e(-n/2q)\,\mathrm{Tr}(V_0^mU_0^{-n}V_0^{p''}W_0^2)\\
&= 2e(m/2q)\,\mathrm{Tr}(U_0^mV_0^{n+p''}W_0^2)\\
&\qquad + 2e(-n/2q)\lambda^{-mn}\,\mathrm{Tr}(U_0^{-n}V_0^{m+p''}W_0^2)\\
&= 2e(m/2q)\lambda^{-m(n+p'')/2}\big(1+(-1)^{pm}\big)\,\delta_{\overline{n+p''},0}\\
&\qquad + 2e(-n/2q)\lambda^{-mn}\lambda^{n(m+p'')/2}\big(1+(-1)^{-pn}\big)\,\delta_{\overline{m+p''},0}\\
&= 2(-1)^{\tilde{q}m}\lambda^{-mn/2}\big(1+(-1)^m\big)\,\delta_{\overline{n},1}\\
&\qquad + 2(-1)^{\tilde{q}n}\lambda^{-mn/2}\big(1+(-1)^n\big)\,\delta_{\overline{m},1}\\
&= 4\lambda^{-mn/2}\delta_{\overline{m},0}\delta_{\overline{n},1} + 4\lambda^{-mn/2}\delta_{\overline{n},0}\delta_{\overline{m},1}\\
&= 4\lambda^{-mn/2}\delta_{\overline{m},\overline{n+1}}\\
&= 4T_{22}(U^mV^nW^2).
\end{aligned}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

When $p/q \in \mathbb{Q}'$, the second set of equations in the statement of the Proposition follow immediately from the first set using (2.3.2). ■

4 An Auxiliary Basis for $K_0(B_{p/q})$

As a step toward showing that the modules $P_j(\theta)$ generate $K_0(B_\theta)$ (for rational θ), we consider in this section an auxiliary basis for $K_0(B_\theta)$ that arises naturally from the realization of B_θ as a sphere with singularities and which enables one to show that the range of the reduced character $\mathbf{T}'$ on $K_0(B_\theta)$ (as defined in Section 2.1) is equal to its range on $\mathcal{R}_\theta$. To do this, we shall assume that θ is in $\mathbb{Q}'$, as defined in Section 2.3. (See Proposition 4-D.)

Assume that $\theta = p/q$ is any rational in $(0,1)$. Let F_1 be a rank one subprojection of the spectral projection of W_0^{-1} corresponding to the eigenvalue 1. Similarly, let F_2 be such a projection for $W_0^{-1}W_0'^{-1}W_0$, and F_0 for $\Gamma_0 W_2$. These are all projections in $M_q(\mathbb{C})$, and we think of them as being "located" at the singular points $(0,0)$, $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},0\right)$, respectively (*cf.* definition of S_θ in (3.1)). Thus, by definition, one has $W_0^{-1}F_1 = F_1$ (and similarly for F_2 and F_0). Now let

$$e_k^j := F_j \otimes E_k$$

for $j = 0,1,2$ and $k = 1,2,3,4$, where $E_k \in M_4(\mathbb{C})$ is the diagonal matrix that has 1 at the k-th diagonal entry and zeros elsewhere. These all have rank one in $M_q \otimes M_4$. It will be convenient to introduce the following notation. If e, f, g are projections of equal rank, denote by $[e,f,g]$ a smooth projection-valued function on $\mathbb{S}^2$ such that

$$[e,f,g]\left(\tfrac{1}{2},\tfrac{1}{2}\right) = e, \quad [e,f,g](0,0) = f, \quad [e,f,g]\left(\tfrac{1}{2},0\right) = g.$$

(Such a function clearly exists since the projections have equal rank.) So $[e,f,g]$ defines a projection in S_θ, and hence a unique positive class in $K_0(S_\theta)$. Now consider the following eight projections in S_θ:

$$\begin{array}{ll} [e_1^2, e_1^1, e_1^0], & [e_2^2+e_3^2, e_1^1+e_2^1, e_1^0+e_2^0], \\ [e_2^2, e_2^1, e_2^0], & [e_3^2+e_4^2, e_2^1+e_3^1, e_2^0+e_3^0], \\ [e_3^2, e_3^1, e_3^0], & [e_2^2, e_1^1, e_1^0], \\ [e_4^2, e_4^1, e_4^0], & [e_3^2, e_1^1, e_3^0]. \end{array} \tag{4.1}$$

We claim that these projections, together with one other class in the kernel of $\mathbf{T}'$, which will be $\kappa_{p,q}$ given by (2.1.7), form a basis for $K_0(S_\theta) \cong K_0(B_\theta)$.

Since $W_0^{-1} \otimes D$ has order four, let n_1, n_2, n_3, n_4 be its spectral dimensions corresponding to the eigenvalues $1, -1, i, -i$, respectively. (So, $\sum_j n_j = 4q$.) Similarly, let m_j be the spectral dimensions of $(W_0^{-1}W_0'^{-1}W_0) \otimes D$, and k, $4q-k$ those of $\Gamma_0 W_2 \otimes D^2$ (which has order two). The commutant of $W_0^{-1} \otimes D$ (respectively, $W_0^{-1}W_0'^{-1}W_0) \otimes D$) in $M_q \otimes M_4$ is isomorphic to $\bigoplus_j M_{n_j}$ (respectively, $\bigoplus_j M_{m_j}$). For $\Gamma_0 W_2 \otimes D^2$ the commutant algebra is isomorphic to $M_k \oplus M_{4q-k}$. (Although

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

these dimensions are known from [7] and [2], their exact values will not be needed here.) Identifying each commutant in this way with its corresponding matrix algebra direct sum, one has the surjective evaluation map

$$\mathcal{E}\colon S_\theta \longrightarrow \mathbb{F} := \Big(\bigoplus_j M_{m_j}\Big) \oplus \Big(\bigoplus_j M_{n_j}\Big) \oplus (M_k \oplus M_{4q-k})$$

$$\mathcal{E}(F) = \Big(F\big(\tfrac{1}{2},\tfrac{1}{2}\big); F(0,0); F\big(\tfrac{1}{2},0\big)\Big) \tag{4.2}$$

where $F\big(\frac{1}{2},0\big) \in M_k \oplus M_{4q-k}$. Under $\mathcal{E}$, the eight projections in (4.1) are mapped as follows:

$$\begin{aligned}
[e_1^2, e_1^1, e_1^0] &\mapsto (F_2,0,0,0); (F_1,0,0,0); (F_0,0), (0,0)\\
[e_2^2, e_2^1, e_2^0] &\mapsto (0,F_2,0,0); (0,F_1,0,0); (0,F_0), (0,0)\\
[e_3^2, e_3^1, e_3^0] &\mapsto (0,0,F_2,0); (0,0,F_1,0); (0,0), (F_0,0)\\
[e_4^2, e_4^1, e_4^0] &\mapsto (0,0,0,F_2); (0,0,0,F_1); (0,0), (0,F_0)\\
[e_2^2+e_3^2, e_1^1+e_2^1, e_1^0+e_2^0] &\mapsto (0,F_2,F_2,0); (F_1,F_1,0,0); (F_0,F_0), (0,0)\\
[e_3^2+e_4^2, e_2^1+e_3^1, e_2^0+e_3^0] &\mapsto (0,0,F_2,F_2); (0,F_1,F_1,0); (0,F_0), (F_0,0)\\
[e_2^2, e_1^1, e_1^0] &\mapsto (0,F_2,0,0); (F_1,0,0,0); (F_0,0), (0,0)\\
[e_3^2, e_1^1, e_3^0] &\mapsto (0,0,F_2,0); (F_1,0,0,0); (0,0), (F_0,0).
\end{aligned}$$

Letting J denote the kernel of $\mathcal{E}$, one has the short exact sequence

$$0 \longrightarrow J \xrightarrow{\ j\ } S_\theta \xrightarrow{\ \mathcal{E}\ } \mathbb{F} \longrightarrow 0 \tag{4.3}$$

where $j\colon J \hookrightarrow S_\theta$ is inclusion. Under the induced map

$$\mathcal{E}_*\colon K_0(S_\theta) \to K_0(\mathbb{F}) \cong \mathbb{Z}^4 \oplus \mathbb{Z}^4 \oplus (\mathbb{Z}\oplus\mathbb{Z}),$$

one gets (since F_j has rank one)

$$\begin{aligned}
[e_1^2, e_1^1, e_1^0] &\mapsto (1,0,0,0); (1,0,0,0); 1,0\\
[e_2^2, e_2^1, e_2^0] &\mapsto (0,1,0,0); (0,1,0,0); 1,0\\
[e_3^2, e_3^1, e_3^0] &\mapsto (0,0,1,0); (0,0,1,0); 0,1\\
[e_4^2, e_4^1, e_4^0] &\mapsto (0,0,0,1); (0,0,0,1); 0,1\\
[e_2^2+e_3^2, e_1^1+e_2^1, e_1^0+e_2^0] &\mapsto (0,1,1,0); (1,1,0,0); 2,0\\
[e_3^2+e_4^2, e_2^1+e_3^1, e_2^0+e_3^0] &\mapsto (0,0,1,1); (0,1,1,0); 1,1\\
[e_2^2, e_1^1, e_1^0] &\mapsto (0,1,0,0); (1,0,0,0); 1,0\\
[e_3^2, e_1^1, e_3^0] &\mapsto (0,0,1,0); (1,0,0,0); 0,1.
\end{aligned} \tag{4.4}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Remark 4-A Since J is the ideal of all functions $\mathbb{S}^2 \to M_{4q}$ vanishing at the three singular points s_j, it is isomorphic to $R_0 \otimes M_{4q}$ where

$$R_0 := \{f \in C(T, \mathbb{C}) : f(s_0) = f(s_1) = f(s_2) = 0\}. \tag{4.5}$$

Hence $K_0(J) \cong K_0(R_0) \cong \mathbb{Z}$, as can easily by checked. (See also the proof of Lemma 5-B below.) Similarly, one has $K_1(J) \cong K_1(R_0) \cong \mathbb{Z}^2$. ∎

Now consider the following part of the six-term exact K-theory sequence associated with (4.3)

$$\mathbb{Z} \cong K_0(J) \xrightarrow{j_*} K_0(S_\theta) \xrightarrow{\mathcal{E}_*} K_0(\mathbb{F}) \xrightarrow{\delta_0} K_1(J) \cong \mathbb{Z}^2 \longrightarrow 0 \tag{4.6}$$

where δ_0, the connecting homomorphism, is surjective (as $K_1(S_\theta) = 0$, by [7]). Since $K_0(S_\theta) \cong \mathbb{Z}^9$ and as the elements in $\mathbb{Z}^{10}$ given by

$$(1,0,0,0);(0,0,0,0);0,0 \quad \text{and} \quad (0,0,0,0);(0,0,0,0);0,1 \tag{4.7}$$

together with those in (4.4) constitute a 10×10 matrix whose determinant is 1, it follows that $\mathcal{E}_*\big(K_0(S_\theta)\big)$ is spanned by the images of the eight projections in (4.1). These, together with the image under j_* of a generator ξ of $K_0(J)$, constitute a basis for $K_0(S_\theta)$. The remaining basis element $j_*(\xi)$ will be shown to be $\pm\kappa_{p,q}$ (Corollary 5-D).

Remark 4-B By showing that the two K_0-elements corresponding to (4.7) are mapped onto generators of $K_1(J)$ via δ_0 one actually gets another proof, using (4.6), that, for the rational case, $K_0(S_\theta) \cong \mathbb{Z}^9$ and $K_1(S_\theta) = 0$.

Now let us calculate $\tau_{11}, \tau_{21}, \tau_{12}, \tau_{22}, \tau_{01}$ on these eight projections. We do this only for τ_{11} since for the others the computation is similar and shall only state the results for the other τ_{ij}. For $k = 1,2,3,4$ one gets

$$\begin{aligned}\tau_{11}[e_k^2, e_k^1, e_k^0] &= \operatorname{Tr}\big(e_k^1(W_0^{-1} \otimes D)\big) = \operatorname{Tr}(F_1 W_0^{-1} \otimes E_k D)\\ &= \operatorname{Tr}(F_1 W_0^{-1}) \operatorname{Tr}(E_k D) = \operatorname{Tr}(E_k D)\end{aligned}$$

since $\operatorname{Tr}(F_1 W_0^{-1}) = \operatorname{Tr}(F_1) = 1$ (by the choice of F_1). And as $\operatorname{Tr}(E_k D) = 1, -1, i, -i$, for $k = 1,2,3,4$, respectively, one gets the value for $\tau_{11}[e_k^2, e_k^1, e_k^0]$. In the same manner,

$$\begin{aligned}\tau_{11}[e_2^2 + e_3^2, e_1^1 + e_2^1, e_1^0 + e_2^0] &= \operatorname{Tr}((e_1^1 + e_2^1) \cdot ((W_0^{-1} \otimes D)) = 0\\ \tau_{11}[e_3^2 + e_4^2, e_2^1 + e_3^1, e_2^0 + e_3^0] &= \operatorname{Tr}((e_2^1 + e_3^1) \cdot ((W_0^{-1} \otimes D)) = -1 + i\\ \tau_{11}[e_2^2, e_1^1, e_1^0] &= 1\\ \tau_{11}[e_3^2, e_1^1, e_3^0] &= 1.\end{aligned}$$

Doing the same for the other traces one can summarize the data in Table 2:

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Projection	τ	τ_{11}	τ_{21}	τ_{12}	τ_{22}	τ_{01}
$[e_1^2, e_1^1, e_1^0]$	$\frac{1}{4q}$	1	1	1	1	1
$[e_2^2, e_2^1, e_2^0]$	$\frac{1}{4q}$	-1	-1	1	1	1
$[e_3^2, e_3^1, e_3^0]$	$\frac{1}{4q}$	i	i	-1	-1	-1
$[e_4^2, e_4^1, e_4^0]$	$\frac{1}{4q}$	$-i$	$-i$	-1	-1	-1
$[e_2^2 + e_3^2, e_1^1 + e_2^1, e_1^0 + e_2^0]$	$\frac{1}{2q}$	0	$i-1$	2	0	2
$[e_3^2 + e_4^2, e_2^1 + e_3^1, e_2^0 + e_3^0]$	$\frac{1}{2q}$	$i-1$	0	0	-2	0
$[e_2^2, e_1^1, e_1^0]$	$\frac{1}{4q}$	1	-1	1	1	1
$[e_3^2, e_1^1, e_3^0]$	$\frac{1}{4q}$	1	i	1	-1	-1

Table 2

(The canonical trace values are immediate from the expression for τ following equation (3.1).) We now assume that $\theta = p/q \in \mathbb{Q}'$ and use Proposition 3, together with Table 2, to obtain Table 3 for $\mathbf{T}'$.

One is now in a position to check that each of these $\mathbf{T}'$-images is in the $\mathbb{Z}$-span of $\mathbf{T}'(P_j)$, $j = 1, \ldots, 9$, as given in Table 1 (recall that in Table 1, ϕ_j, ϕ_j' are the real and imaginary components of T_{1j}). In so doing, however, for the projections of trace $1/4q$ (in Table 3) one encounters equations of the form

$$p(4n+1) - qb = 1$$

to which integral solutions n, b are required, where b is given *a priori* to be in $4\mathbb{Z} + \delta$ for some already prescribed $\delta = 0, 1, 2, 3$. This is guaranteed by the following simple fact.

Lemma 4-C *If $q = 4^d$, $p = 4k+1$, and $\delta \in \mathbb{Z}$ are given, then there exists $a \in 4\mathbb{Z}+1$ and $b \in 4\mathbb{Z}+\delta$ such that $pa - qb = 1$.*

Proof Pick integers a, b such that $pa - qb = 1$. Since q is even, a is odd, so write $a = 2a' + 1$. Substituting this into $(4k+1)a - 4^d b = 1$ implies that a' is even, so that $a \in 4\mathbb{Z}+1$. Now for any integer t one has $p(a + 4^d t) - q(b + tp) = 1$, and $b + pt = b + 4kt + t$ is in $4\mathbb{Z} + \delta$ if one chooses $t = \delta - b$, done. ■

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Projection	τ	T_{10}	T_{11}	T_{20}	T_{21}	T_{22}
$[e_1^2, e_1^1, e_1^0]$	$\frac{1}{4q}$	$\frac{1+i}{8}$	$\frac{-1+i}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{4}$
$[e_2^2, e_2^1, e_2^0]$	$\frac{1}{4q}$	$-\frac{1+i}{8}$	$\frac{1-i}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{4}$
$[e_3^2, e_3^1, e_3^0]$	$\frac{1}{4q}$	$\frac{-1+i}{8}$	$-\frac{1+i}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{4}$
$[e_4^2, e_4^1, e_4^0]$	$\frac{1}{4q}$	$\frac{1-i}{8}$	$\frac{1+i}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{4}$
$[e_2^2 + e_3^2, e_1^1 + e_2^1, e_1^0 + e_2^0]$	$\frac{1}{2q}$	0	$-\frac{i}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
$[e_3^2 + e_4^2, e_2^1 + e_3^1, e_2^0 + e_3^0]$	$\frac{1}{2q}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0
$[e_2^2, e_1^1, e_1^0]$	$\frac{1}{4q}$	$\frac{1+i}{8}$	$\frac{1-i}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{4}$
$[e_3^2, e_1^1, e_3^0]$	$\frac{1}{4q}$	$\frac{1+i}{8}$	$-\frac{1+i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{4}$

Table 3: Values of $\mathbf{T}'$ for $p/q \in \mathbb{Q}'$

For the other two projections of trace $1/2q$ one requires integer solutions n, b to

$$p(2n+1) - \tfrac{q}{2}b = 1$$

where b is prescribed to be in $4\mathbb{Z} + \delta$. This however can be solved in exactly the same fashion. For completion we do this. Since $q = 4^d$, pick integers a, b such that $pa - \frac{q}{2}b = 1$. Again a is odd, so write $a = 2a' + 1$ so that $p(2a'+1) - \frac{q}{2}b = 1$. Now for any integer t one has

$$p\big(2a' + 1 + \tfrac{q}{2}t\big) - \tfrac{q}{2}(b + pt) = 1$$

where $2a' + 1 + \frac{q}{2}t$ is clearly in $2\mathbb{Z} + 1$ and $b + pt = b + 4kt + t$ which can be chosen in $4\mathbb{Z} + \delta$ by taking $t = \delta - b$. We have therefore proved the following.

Proposition 4-D *For any $\theta \in \mathbb{Q}'$, one has $\mathbf{T}'\big(K_0(B_\theta)\big) = \mathbf{T}'(\mathcal{R}_\theta)$.*

5 The Connes Chern Character of a Bott Projection

The objective of this section is to identify the generator of $\mathrm{Ker}(\mathcal{E}_*) \subset K_0(S_\theta)$, compute Connes' canonical cyclic 2-cocycle, and therefore show that this generator is $\pm\kappa_{p,q}$ (as defined by (2.1.7)). This is done by proving the following.

Proposition 5-A *For any positive rational $\theta = p/q < 1$, the class $\kappa_{p,q} \in K_0(S_\theta)$ is the image of a generator of $K_0(J) \cong \mathbb{Z}$ under the canonical map $j_*\colon K_0(J) \to K_0(S_\theta)$.*

First we need the following lemma.

Lemma 5-B *With R_0 defined as in (4.5), the group $K_0(R_0) \cong \mathbb{Z}$ is generated by an element of the form $\xi_0 = [P_0] - [1]$ where*

$$P_0 = \begin{bmatrix} 1-f & g \\ \bar{g} & f \end{bmatrix} \tag{5.1}$$

and f, g are smooth and rapidly decreasing functions at the boundary of T.

Proof Consider the short exact sequence

$$0 \longrightarrow C_0(T) \xrightarrow{\ j\ } R_0 \xrightarrow{\ \alpha\ } C_0\big((0,1)\big) \oplus C_0\big((0,1)\big) \longrightarrow 0$$

where $C_0(T)$ is the subalgebra of R_0 of functions vanishing along the boundary of T, and where α, here, restricts a function to the two line segments connecting the singular points $(0,0)$, $\big(\frac{1}{2}, \frac{1}{2}\big)$ and $(0,0)$, $\big(\frac{1}{2}, 0\big)$. Looking at the corresponding exact sequence of K_0 groups one gets

$$K_0\big(C_0(T)\big) \xrightarrow{\ j_*\ } K_0(R_0) \xrightarrow{\ \alpha_*\ } K_0\Big(C_0\big((0,1)\big) \oplus C_0\big((0,1)\big)\Big) = 0$$

so j_* is onto, and since $K_0\big(C_0(T)\big) \cong \mathbb{Z} \cong K_0(R_0)$, one deduces that j_* is an isomorphism. Since it is known that a generator of $K_0\big(C_0(T)\big)$ has the form $[P_0] - [1]$, where P_0 has the form (5.1), one gets a generator of $K_0(R_0)$ that has exactly the same form. Finally, since the smooth rapidly decreasing functions at the boundary of T are dense in $C_0(T)$, and are closed under the holomorphic functional calculus, one can modify P_0 so that f and g are smooth and rapidly decreasing. ■

Lemma 5-C *One has $C_1\big(j_*(\xi)\big) = \pm 4q$.*

Let $\rho\colon R_0 \to J$ denote the homomorphism $\rho(f) = fe_{11}$ where $e_{11} = e_{11}^{(q)} \otimes e_{11}^{(4)} \in M_q \otimes M_4$. Then the induced map $\rho_*\colon K_0(R_0) \to K_0(J)$ is an isomorphism and so mapping ξ_0 to a generator ξ of $K_0(J)$. More precisely, letting $\tilde{\rho}\colon \tilde{R}_0 \to \tilde{J}$ denote the unitized map associated with ρ ($\tilde{J}$ being the unitization of J), so that $\tilde{\rho}(f + z1) = fe_{11} + zI_{4q}$ ($f \in R_0$), one has

$$\xi = \rho_*(\xi_0) = \rho_*([P_0]-[1]) = \tilde{\rho}_*([P_0]-[1]) = [\tilde{\rho}(P_0)]-[\tilde{\rho}(1)] = [P]-[I_{4q}] \tag{5.2}$$

where

$$P := \begin{bmatrix} I_{4q} - fe_{11} & ge_{11} \\ \bar{g}e_{11} & fe_{11} \end{bmatrix} \tag{5.3}$$

is a projection belonging to $M_2(\tilde{J}) \subset M_2(S_\theta)$. Thus, $C_1(\xi) = C_1(P)$. To calculate $C_1(P)$, one takes P back to B_θ via the isomorphism $\beta\gamma$ (of Section 3), then following

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

it by the injection $\Psi\colon B_\theta \to M_4(A_\theta)$ and so obtain $C_1(P) = c_1\Big(\Psi\big((\beta\gamma)^{-1}(P)\big)\Big)$ (as defined in Section 2.1). Now

$$(\beta\gamma)^{-1}(P) = \begin{bmatrix} 1_{B_\theta} - (\beta\gamma)^{-1}(fe_{11}) & (\beta\gamma)^{-1}(ge_{11}) \\ (\beta\gamma)^{-1}(\bar{g}e_{11}) & (\beta\gamma)^{-1}(fe_{11}) \end{bmatrix}$$

and since $fe_{11} = \frac{1}{4}f(e_{11}^{(q)} \otimes I_4)(I_{4q} + I_q \otimes D + I_q \otimes D^2 + I_q \otimes D^3)$ one gets

$$\begin{aligned}(\beta\gamma)^{-1}(fe_{11}) &= (\beta\gamma)^{-1}\big(\tfrac{1}{4}f(e_{11}^{(q)} \otimes I_4)\big) \cdot (1 + W + W^2 + W^3) \\ &= \gamma^{-1}\big(\tfrac{1}{4}f \cdot R^{-1}e_{11}^{(q)}R \otimes I_4\big) \cdot (1 + W + W^2 + W^3) \\ &= \tfrac{1}{4}F(1 + W + W^2 + W^3)\end{aligned}$$

where F is the unique function in A_θ such that $\gamma(F) = f \cdot R^{-1}e_{11}^{(q)}R \otimes I_4$. Similarly, let G be the function in A_θ such that $\gamma(G) = g \cdot R^{-1}e_{11}^{(q)}R \otimes I_4$. Let $\tilde{f} := f \cdot R^{-1}e_{11}^{(q)}R$ so that it is easy to check that it belongs to A_θ^σ. Letting T_1, T_2, T_3, T denote the closed triangles formed by the diagonal lines $y = x$, $y = 1 - x$ of the unit square (with T_1 being the left, T_2 the top, T_3 the right, and T the bottom triangle) one sets

$$F(x,y) = \begin{cases} \sigma_0\big(\tilde{f}(y, 1-x)\big) & \text{if } (x,y) \in T_3 \\ \sigma_0^2\big(\tilde{f}(1-x, 1-y)\big) & \text{if } (x,y) \in T_2 \\ \sigma_0^3\big(\tilde{f}(1-y, x)\big) & \text{if } (x,y) \in T_1 \\ \tilde{f}(x,y) & \text{if } (x,y) \in T \end{cases} \tag{5.4}$$

where σ_0 is the order-four automorphism of M_q defined in Section 2.2. (It is easily checked that F is well-defined and belongs to the fixed point subalgebra of A_θ under σ.) A similar formula holds for G in terms of $\tilde{g} := g \cdot R^{-1}e_{11}^{(q)}R$. By definition, one easily checks that $\gamma(F) = \tilde{f} \otimes I_4$ and $\gamma(G) = \tilde{g} \otimes I_4$. One thus has

$$\Psi\big((\beta\gamma)^{-1}(fe_{11})\big) = \tfrac{1}{4}\Psi\big(F(1 + W + W^2 + W^3)\big) = F \otimes E$$

where E is the rank one projection

$$E := \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence,

$$Q := \Psi\big((\beta\gamma)^{-1}(P)\big) = \begin{bmatrix} I_q \otimes I_4 - F \otimes E & G \otimes E \\ G^* \otimes E & F \otimes E \end{bmatrix}$$

and belongs to $M_2\big(M_4(A_\theta)\big)$. (Here, I_q is the identity element of A_θ.)

Now it is clear that we need to compute $(\varphi_q \# \mathrm{Tr}_4 \# \mathrm{Tr}_2)(Q,Q,Q) = \psi \# \mathrm{Tr}_2(Q,Q,Q)$, where $\psi := \varphi_q \# \mathrm{Tr}_4$ and φ_q is given in Section 2.4, and show that it is divisible by $4q$. Thus, using the cyclicity property, one obtains

$$\begin{aligned}
c_1[Q] &= (\varphi_q \# \mathrm{Tr}_4 \# \mathrm{Tr}_2)(Q,Q,Q) = \psi \# \mathrm{Tr}_2(Q,Q,Q) \\
&= \psi(-F\otimes E, -F\otimes E, -F\otimes E) + \psi(-F\otimes E, G\otimes E, G^*\otimes E) \\
&\quad + \psi(G\otimes E, G^*\otimes E, -F\otimes E) + \psi(G\otimes E, F\otimes E, G^*\otimes E) \\
&\quad + \psi(G^*\otimes E, -F\otimes E, G\otimes E) + \psi(G^*\otimes E, G\otimes E, F\otimes E) \\
&\quad + \psi(F\otimes E, F\otimes E, F\otimes E) + \psi(F\otimes E, G^*\otimes E, G\otimes E) \\
&= 3\psi(-F\otimes E, G\otimes E, G^*\otimes E) + 3\psi(F\otimes E, G^*\otimes E, G\otimes E) \\
&= 3(\varphi_q \# \mathrm{Tr}_4)(-F\otimes E, G\otimes E, G^*\otimes E) + 3(\varphi_q \# \mathrm{Tr}_4)(F\otimes E, G^*\otimes E, G\otimes E) \\
&= -3\varphi_q(F,G,G^*) + 3\varphi_q(F,G^*,G)
\end{aligned}$$

which by the expression for φ_q in Section 2.4 becomes

$$\begin{aligned}
&= -3\frac{q}{2\pi i}\int_0^1\int_0^1 \mathrm{Tr}\left(F\left[\frac{\partial G}{\partial x}\frac{\partial G^*}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial G^*}{\partial x}\right]\right)dx\,dy \\
&\quad + 3\frac{q}{2\pi i}\int_0^1\int_0^1 \mathrm{Tr}\left(F\left[\frac{\partial G^*}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial G^*}{\partial y}\frac{\partial G}{\partial x}\right]\right)dx\,dy \\
&= -12\cdot\frac{q}{2\pi i}\iint_T \mathrm{Tr}\left(\tilde f\left[\frac{\partial \tilde g}{\partial x}\frac{\partial \tilde g^*}{\partial y} - \frac{\partial \tilde g}{\partial y}\frac{\partial \tilde g^*}{\partial x}\right]\right)dx\,dy \\
&\quad + 12\cdot\frac{q}{2\pi i}\iint_T \mathrm{Tr}\left(\tilde f\left[\frac{\partial \tilde g^*}{\partial x}\frac{\partial \tilde g}{\partial y} - \frac{\partial \tilde g^*}{\partial y}\frac{\partial \tilde g}{\partial x}\right]\right)dx\,dy
\end{aligned}$$

(5.5)

$$= 4\cdot\big(-3\varphi_q(\tilde f,\tilde g,\tilde g^*) + 3\varphi_q(\tilde f,\tilde g^*,\tilde g)\big)$$

where, in the last equality, f and g (and hence $\tilde f$, $\tilde g$) have been extended to the unit square by defining them to be zero outside the triangle T. (Since f, g are smooth and rapidly decreasing at the boundary of T—Lemma 5-D—the resulting extensions are smooth on the unit square.) Letting P' denote the projection

$$P' = \begin{bmatrix} I_q - \tilde f & \tilde g \\ \tilde g^* & \tilde f \end{bmatrix}$$

whose entries belong to the subalgebra $\widetilde{\mathfrak{A}} = \mathfrak{A} + \mathbb{C}I_q$ of A_θ, where $\mathfrak{A} := M_q \otimes C_0\big((0,1)^2\big)$, one has

$$-3\varphi_q(\tilde f,\tilde g,\tilde g^*) + 3\varphi_q(\tilde f,\tilde g^*,\tilde g) = (\varphi_q \# \mathrm{Tr}_2)(P',P',P').$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Now $[P'] - [I_q]$ is in $K_0(\mathfrak{A}) \cong \mathbb{Z}$, which is generated by $[P_{\mathrm{Bott}}] - [I_q]$, where

$$P_{\mathrm{Bott}} := \begin{bmatrix} I_q - fe_{11}^{(q)} & ge_{11}^{(q)} \\ \overline{g}e_{11}^{(q)} & fe_{11}^{(q)} \end{bmatrix}$$

as is easily checked. So there is an integer n such that $[P'] - [I_q] = n([P_{\mathrm{Bott}}] - [I_q])$ in $K_0(\mathfrak{A})$. This still holds in $K_0(A_\theta)$. Continuing with our computation in (5.5) above we have

$$\begin{aligned} C_1\big(j_*(\xi)\big) &= c_1[Q] = 4(\varphi_q \,\#\, \mathrm{Tr}_2)(P', P', P') = 4\langle [P'], \varphi_q\rangle = 4\langle [P'] - [I_q], \varphi_q\rangle \\ &= 4n\langle [P_{\mathrm{Bott}}] - [I_q], \varphi_q\rangle \\ &= 4n(\varphi_q \,\#\, \mathrm{Tr}_2)(P_{\mathrm{Bott}}, P_{\mathrm{Bott}}, P_{\mathrm{Bott}}) \\ &= 4n[-3\varphi_q(fe_{11}^{(q)}, ge_{11}^{(q)}, \overline{g}e_{11}^{(q)}) + 3\varphi_q(fe_{11}^{(q)}, \overline{g}e_{11}^{(q)}, ge_{11}^{(q)})] \\ &= 4n(-6)\varphi_q(fe_{11}^{(q)}, ge_{11}^{(q)}, \overline{g}e_{11}^{(q)}) \\ &= -4qn \cdot \frac{6}{2\pi i}\int_0^1\int_0^1 f\left[\frac{\partial g}{\partial x}\frac{\partial \overline{g}}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial \overline{g}}{\partial x}\right] dx\,dy \\ &= -4qn \cdot \langle [P_0], \varphi_1\rangle \end{aligned}$$

where P_0 is given by (5.1) (with f, g extended as above), and $\langle [P_0], \varphi_1\rangle$ is an integer.[1]

Therefore, $C_1\big(j_*(\xi)\big) = c_1[Q]$ is divisible by $4q$, which proves Lemma 5-C. To complete the proof of Proposition 5-A, it is easy to see directly that $j_*(\xi)$, given by (5.2) and (5.3) and using (3.6), maps to zero by the components of $\mathbf{T}'$. Thus $\mathrm{Ker}(\mathcal{E}_*) \subseteq \mathrm{Ker}(\mathbf{T}')$ and from the above calculation,

$$\mathbf{T}\big(j_*(\xi)\big) = (0; 0, 0, 0, 0; 0, 0, 0; -4qN)$$

where $N = n\langle [P_0], \varphi_1\rangle$. But we already know (from Section 2.1) that $\mathbf{T}(\kappa_{p,q}) = (0; 0, 0, 0, 0; 0, 0, 0; 4q)$, thus $\mathbf{T}\big(j_*(\xi)\big) = -N\mathbf{T}(\kappa_{p,q})$. As $\mathbf{T}$ is injective on $K_0(B_\theta)$ (since θ is rational), one gets $j_*(\xi) = -N\kappa_{p,q}$ in $K_0(B_\theta)$.

Corollary 5-D *For $\theta \in \mathbb{Q}'$, one has* $\mathrm{Ker}(\mathbf{T}') = \mathbb{Z}j_*(\xi) = \mathbb{Z}\kappa_{p,q}$.

Proof It was already shown in Section 4 that the eight classes in Table 3, *i.e.*, of (4.1), together with $j_*(\xi)$, yield a basis for $K_0(S_\theta)$. Since the values of $\mathbf{T}'$ (given by Table 3) are independent and $\mathbf{T}'$ vanishes on $j_*(\xi)$, it follows that $j_*(\xi)$ generates $\mathrm{Ker}(\mathbf{T}')$. But since $j_*(\xi) = -N\kappa_{p,q}$ and $\mathbf{T}'(\kappa_{p,q}) = 0$ it follows that $N = \pm 1$, so that $j_*(\xi) = \pm\kappa_{p,q}$. Thus, $\mathrm{Ker}(\mathbf{T}') = \mathbb{Z}j_*(\xi) = \mathbb{Z}\kappa_{p,q}$. ∎

This completes the proof of Proposition 5-A. (Note that in the above computation $n = \pm 1$ and $\langle [P_0], \varphi_1\rangle = \pm 1$ hold automatically.)

[1] Although Connes [4] showed that c_1 is integer-valued for $0 < \theta < 1$ by computing it for the Powers-Rieffel projection, this can still be done in the case $\theta = 1$ for the Bott projection (5.1) to obtain the same result. In fact, one can show that $\langle [P_0], \varphi_1\rangle = 1$, but we shall not need this here.

6 Conclusions

Proposition 6-A *For $\theta \in \mathbb{Q}'$, the set $\{P_1(\theta), \dots, P_9(\theta)\}$ is a basis for the group $K_0(B_\theta)$.*

Proof Since the modules $P_j(\theta)$ are already independent (for each θ), it is enough to show that they generate. So pick any x in $K_0(S_\theta)$. From Proposition 4-D (since $\theta \in \mathbb{Q}'$) one can write $\mathbf{T}'(x) = \sum_{j=1}^{9} n_j \mathbf{T}'([P_j])$ for some integers n_j. Therefore, by Corollary 5-D, one gets

$$x = \sum_{j=1}^{9} n_j [P_j] + m\kappa_{p,q}$$

for some integer m (where $\theta = p/q$). The result follows since $\kappa_{p,q}$ is, by definition, in $\mathcal{R}_\theta$. ■

The conclusion of this proposition will in fact remain true for all rationals as will be seen from Corollary 6-C below.

Remark It is not hard to see that the unbounded traces T_{ij}^t on the smooth *-subalgebra B_t (as defined by (2.1.2) with respect to the canonical unitary generators U_t, V_t, W_t) are strongly continuous in the parameter t, in the sense that if ξ is a locally defined continuous section of the continuous field of smooth *-subalgebras $\{B_t\}$ (so its values are smooth elements), then the map $t \mapsto T_{ij}^t\big(\xi(t)\big)$ is continuous. Further, the same can be seen to hold for Connes' canonical cyclic 2-cocycle

$$\varphi^t(x^0, x^1, x^2) = \frac{1}{2\pi i}\tau_t\big(x^0[\delta_1^t(x^1)\delta_2^t(x^2) - \delta_2^t(x^1)\delta_1^t(x^2)]\big)$$

on the smooth rotation algebra A_t, where δ_1^t, δ_2^t are the canonical derivations associated with the unitary generators U_t, V_t, and τ_t is the canonical trace on A_t. That is, if ξ^j, $j = 0, 1, 2$ are locally defined continuous sections of the continuous field of smooth *-subalgebras $\{B_t\}$, then the map $t \mapsto \varphi^t\big(\xi^0(t), \xi^1(t), \xi^2(t)\big)$ is continuous.

Theorem 6-B **(Range of the Connes Chern Character)** *For any $0 < \theta < 1$ one has the range of the Connes Chern character:*

$$\mathbf{T}\big(K_0(B_\theta)\big) = \mathbf{T}(\mathcal{R}_\theta)$$

where $\mathcal{R}_\theta$ is the subgroup of $K_0(B_\theta)$ generated by $\{P_1(\theta), \dots, P_9(\theta)\}$. More specifically, the range is spanned by the rows in Table 1.

Proof Equality holds for $\theta \in \mathbb{Q}'$ in view of the preceding proposition which has $K_0(B_\theta) = \mathcal{R}_\theta$. So fix any $0 < \theta < 1$ and fix a positive class $[e] \in K_0(B_\theta)$, where e is a smooth projection in some matrix algebra over B_θ. Let $t \mapsto e_t$ be a continuous section of smooth projections of the continuous field of matrix algebras over B_θ (all of the same size), defined in a neighborhood of θ, such that $e_\theta = e$. The values

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

of canonical traces $t \mapsto \tau_t(e_t)$ defines a continuous function which takes values in $\frac{1}{4}(\mathbb{Z}+\mathbb{Z}t)$ for each t. There are integers m, n (independent of t) such that

$$\tau_t(e_t) = \tfrac{1}{4}(m+nt) \quad \text{and} \quad C_1^t(e_t) = -n$$

for t near θ (where we wrote C_1^t to specify the dependence of C_1, as in Section 2.1, upon t). To see this, note that $C_1^t(e_t)$ is a constant integer, by the continuity of Connes' cyclic 2-cocycle φ^t as easily seen from $C_1^t(e_t) = (\varphi^t \# \operatorname{Tr}) \cdot \big(\Psi_t(e_t), \Psi_t(e_t), \Psi_t(e_t)\big)$ (where $\Psi_t = \Psi$ is given as in (2.1.6)). As $\tau_t(e_t)$ is itself continuous, the result follows.

Now write

$$\mathbf{T}^t([e_t]) = \big(\tfrac{1}{4}(m+nt); \phi_0^t(e_t), \phi_0'^t(e_t), \phi_1^t(e_t), \phi_1'^t(e_t); T_{20}^t(e_t), T_{21}^t(e_t), T_{22}^t(e_t); -n\big)$$

where $\mathbf{T}^t$ denotes the Connes Chern character on $K_0(B_t)$. For $t \in \mathbb{Q}'$, Proposition 6-A and Table 1 show that $\phi_j^t(e_t)$, $\phi_j'^t(e_t)$, $T_{ij}^t(e_t)$ can only take values in $\frac{1}{8}\mathbb{Z}$, and since these are continuous in t, they must be constant for t in a neighborhood of θ. Now note that for each such t one has

$$\text{(6.1)} \qquad \mathbf{T}^t\big([e_t] - n[P_7(t)] - m[P_3(t)]\big) = (0; a_0, a_0', a_1, a_1'; b_0, b_1, b_2; 0)$$

for some constants $a_0, a_0', a_1, a_1', b_0, b_1, b_2 \in \frac{1}{8}\mathbb{Z}$. Let $x_t = [e_t] - n[P_7(t)] - m[P_3(t)]$. Evaluating (6.1) at a rational r in $\mathbb{Q}'$ near θ, one gets

$$(0; a_0, a_0', a_1, a_1'; b_0, b_1, b_2; 0) = \mathbf{T}^r(x_r) = \mathbf{T}^r(g_r)$$

for some $g_r \in \mathcal{R}_r$, by Proposition 6-A. Writing

$$g_r = \sum_{k=1}^{9} n_k[P_k(r)]$$

for some integers n_k, define $g_t := \sum_{k=1}^{9} n_k[P_k(t)]$ so that $g_t \in \mathcal{R}_t$ for each t. Now it follows that for each t

$$\mathbf{T}^t(g_t) = (0; a_0, a_0', a_1, a_1'; b_0, b_1, b_2; 0).$$

To see this, note that $C_1^r(g_r) = 0$ implies $n_7 + n_8 + n_9 = 0$, so that the canonical trace of g_t (for arbitrary t near θ) is

$$n_1\tfrac{1}{2} + n_2\tfrac{1}{2} + n_3\tfrac{1}{4} + n_4\tfrac{1}{2} + n_5\tfrac{1}{2} + n_6\tfrac{1}{4} + (n_7+n_8+n_9)\frac{t}{4} = 0.$$

Therefore, $\mathbf{T}^t\big([e_t] - n[P_7(t)] - m[P_3(t)]\big) = \mathbf{T}^t(g_t)$, and the result follows upon evaluating this at $t = \theta$. ∎

Corollary 6-C *For each rational* $0 < \theta < 1$, *the set* $\{P_1(\theta), \dots, P_9(\theta)\}$ *is a basis for the group* $K_0(B_\theta)$.

Proof This follows from Theorem 6-B since $\mathbf{T}$ is injective on $K_0(B_\theta)$ (Theorem 2.1) and since $K_0(B_{p/q}) \cong \mathbb{Z}^9$ (see [6]). ■

Remark Although $P_1(t), \dots, P_6(t)$, given by (2.1.1), are continuous sections of the field of C*-algebras $\{B_t : 0 \le t \le 1\}$, the same is not as immediate (though it is true) for the Fourier module $P_7(t) = \mathcal{F}_t$ as a function of t. This explains the reason for the argument to follow. The author has shown (in unpublished work [17]) that in fact there is a finitely generated projective module over the C*-algebra of sections of the field which, at each $t \in (0,1)$, gives the class of $\mathcal{F}_t$. However, it will not be necessary to use this result here.

For each rational $r \in (0,1)$ there is a closed interval N_r containing r (in its interior) such that $[P_j(r)] = [\xi_j^r(r)]$ in $K_0(B_r)$, $j = 1, \dots, 9$, for some projection ξ_j^r in some matrix algebra over $\Gamma|N_r$, the C*-algebra of all continuous sections of the field of C*-algebras $\{B_t : t \in N_r\}$. Thus, if $\varepsilon_t \colon \Gamma|N_r \to B_t$ is the evaluation map at $t \in N_r$, then $\varepsilon_{r*}[\xi_j^r] = [P_j(r)]$ so that the induced map

$$\varepsilon_{r*} \colon K_0(\Gamma|N_r) \to K_0(B_r)$$

is surjective.

Claim For each j and $t \in N_r$ one has $\mathbf{T}\big(P_j(t)\big) = \mathbf{T}\big(\xi_j^r(t)\big)$. Moreover, for each $t \in N_r \cap \mathbb{Q}$ one has $[\xi_j^r(t)] = [P_j(t)]$ in $K_0(B_t)$. In particular, if $t \in N_r \cap \mathbb{Q}$, then the induced map $\varepsilon_{t*} \colon K_0(\Gamma|N_r) \to K_0(B_t)$ is surjective.

Proof Clearly, from Table 1, for each fixed j there are integers m, n and constants $a_0, a_0', a_1, a_1', b_0, b_1, b_2 \in \frac{1}{8}\mathbb{Z}$ such that for each $t \in (0,1)$

$$\mathbf{T}\big(P_j(t)\big) = \big(\tfrac{1}{4}(m+nt); a_0, a_0', a_1, a_1'; b_0, b_1, b_2; -n\big).$$

As in the first paragraph of the proof of Theorem 6-B one can write, for each $t \in N_r$,

$$\mathbf{T}\big(\xi_j^r(t)\big) = \big(\tfrac{1}{4}(m'+n't); c_0, c_0', c_1, c_1'; d_0', d_1', d_2'; -n'\big)$$

for some integers m', n' and constants c_k, c_k', d_ℓ' independent of t. Since $[\xi_j^r(r)] = [P_j(r)]$, evaluation at $t = r$ yields $n' = n$, $m' = m$, $c_k = a_k$, $c_k' = a_k'$, $d_\ell' = b_\ell$. Thus, $\mathbf{T}\big(P_j(t)\big) = \mathbf{T}\big(\xi_j^r(t)\big)$ for each $t \in N_r$ and the result follows from the injectivity of $\mathbf{T}$ when t is rational. ■

We now appeal to the slighly more general results obtained in Section 7—the conditions for which were modelled on the current problem.

From the above claim it follows that by applying Corollary 7.3-E(a) to the field $\{B_t : t \in N_r\}$ and the classes $\{[\xi_1^r], \dots, [\xi_9^r]\}$ in $K_0(\Gamma|N_r)$, one obtains a dense G_δ subset G_r of N_r such that for each $t \in G_r$ the set $\{[\xi_1^r(t)], \dots, [\xi_9^r(t)]\}$ is a basis for $K_0(B_t)$. Since from the above claim we have $\mathbf{T}\big(P_j(t)\big) = \mathbf{T}\big(\xi_j^r(t)\big)$ for each $t \in N_r$, and since from Table 1 above $\mathbf{T}\big(P_j(t)\big)$ are independent over $\mathbb{Z}$, it follows that $\mathbf{T}$ is

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

injective on $K_0(B_t)$ and thus $[\xi_j^r(t)] = [P_j(t)]$ so that $\{[P_1(t), \dots, [P_9(t)]\}$ is a basis for $K_0(B_t)$ for each $t \in G_r$.

Since the countable union of G_δ sets is also a G_δ set, the union $G = \bigcup\{G_r : r \in \mathbb{Q} \cap (0,1)\}$ is a G_δ subset of $(0,1)$, which is clearly dense in $(0,1)$. Therefore, $\{[P_1(\theta)], \dots, [P_9(\theta)]\}$ is a basis for $K_0(B_\theta)$ for each θ in G. One thus obtains the following.

Theorem 6-E *There is a dense G_δ subset G of $(0,1)$ (containing the rationals) such that the set $\{P_1(\theta), \dots, P_9(\theta)\}$ is a basis for the group $K_0(B_\theta)$ for each $\theta \in G$. In particular, $K_0(B_\theta) \cong \mathbb{Z}^9$.*

The result for K_1 is much easier since one essentially invokes a Baire category argument and uses the fact that it holds in the rational case [6]. (More precisely, see Theorem 7.2-B below.)

Theorem 6-F *There is a dense G_δ set of parameters θ in $(0,1)$ (containing the rationals) for which $K_1(B_\theta) = 0$.*

Now we can say something about the K-groups of the fixed point subalgebra A_θ^σ of the rotation algebra under the Fourier automorphism. (But not about the generators of its K_0—save using the isomorphism $K_0(A_\theta^\sigma) \cong K_0(B_\theta)$ implemented by the strong Morita equivalence between A_θ^σ and B_θ.)

Corollary 6-G *For a dense G_δ set of parameters θ in $(0,1)$, containing the rationals except for $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, one has $K_0(A_\theta^\sigma) \cong \mathbb{Z}^9$ and $K_1(A_\theta^\sigma) = 0$.*

Proof For the rational case the result was shown in [7, Corollary 3.2.6]. The irrational case follows from Theorems 6-E and 6-F since in this case the fixed point subalgebra and the crossed product B_θ are strongly Morita equivalent [12]. ∎

It now appears, using Theorem 6-E and Table 1, that techniques similar to those of [13, Theorem 4.1], could be carried out to show that the positive cone of $K_0(B_\theta)$ (for θ in a dense G_δ) can be characterized as the set of elements of positive trace. This, together with the vanishing of K_1, would be further evidence that B_θ is an AF-algebra for irrational θ. From this it will follow that the ordered group $K_0(B_\theta)$ is unperforated and is a dimension group. But these considerations will be left for a future paper.

7 Continuous Fields of C*-Algebras

In this section we generalize the situation we have so far obtained above to two hypotheses on a continuous field of C*-algebras. Under these hypotheses certain K-theoretical data which are known to hold for a dense set of fibers, of a continuous field of C*-algebras over $[0,1]$, are shown to continue to hold on a dense G_δ subset of the parameter space. For example, such data can be the free-rank of the K_0-group or the vanishing of the K_1-group. The basic result is that under these hypotheses there is a surjection $K_0(\Gamma) \to K_0(B_t)$ for each t, induced by evaluation, where $\{B_t\}$ is the field and Γ the C*-algebra of the field.

7.1 A Slightly General Situation

Let $\{B_t\}$ be a separable continuous field of unital C*-algebras with parameter space $[0,1]$, so that Γ, the C*-algebra of continuous (global) sections of the field, is separable. (We will need the separability of Γ so that $K_0(\Gamma)$ and $K_1(\Gamma)$ are countable groups.) The two hypotheses are:

(H1) There are positive classes $[P_1],\ldots,[P_N]$ in $K_0(\Gamma)$ and a dense subset Q of $[0,1]$ such that: for each $t\in Q$ and each $x\in K_0(B_t)$ there is a positive integer $m_{t,x}$ such that

$$m_{t,x}\cdot x\in \mathbb{Z}[P_1(t)]+\cdots+\mathbb{Z}[P_N(t)].$$

(H2) There is a dense subset Q of $[0,1]$ such that for each $t\in Q$

$$K_1(B_t)=0.$$

Under each of these assumptions, separately, it will be shown in this section that they continue to hold on a dense G_δ set containing Q (Theorems 7.3-C and 7.2-B). The main result will be to show that under (H1), the canonical map $\varepsilon_{t*}\colon K_0(\Gamma)\to K_0(B_t)$, induced by the evaluation map $\varepsilon_t\colon \Gamma\to B_t$ at t, is *almost surjective* for all t (Theorem 7.3-B), in the sense that each element in $K_0(B_t)$ has a non-zero integral multiple in the span of $[P_1(t)],\ldots,[P_N(t)]$.

Clearly, of particular interest is the case $m_{t,x}\equiv 1$ (since in the Fourier case dealt with above the nine modules form a basis for $K_0(B_\theta)$ when θ is rational—Corollary 6-C). In this case the map ε_{t*} is surjective for each t. Furthermore, under both (H1) and (H2), one obtains the short exact sequence

$$0\longrightarrow K_i(J_t)\xrightarrow{\ j_*\ }K_i(\Gamma)\xrightarrow{\ \varepsilon_{t*}\ }K_i(B_t)\longrightarrow 0$$

for each t in $[0,1]$ and for $i=0,1$, where $J_t=\mathrm{Ker}(\varepsilon_t)$ and $j\colon J_t\hookrightarrow\Gamma$ is the canonical inclusion. (See Corollary 7.3-E.)

Presumably, these hypotheses can be tested and applied to similar situations such as the order three and order six automorphisms of the rotation algebra and the resulting crossed products. These examples are discussed briefly at the end of this section.

Notation For each t let $\mathcal{R}_t$ be the subgroup of $K_0(B_t)$ defined by

$$\mathcal{R}_t:=\mathbb{Z}[P_1(t)]+\cdots+\mathbb{Z}[P_N(t)].$$

All sections of the field $\{B_t\}$ are assumed to be continuous. By a 'global' section is meant one that is continuous and defined over $[0,1]$. We will say that a group homomorphism $K\to H$ is *almost surjective* if for each $h\in H$ there is a positive integer m such that mh is in its range.

7.2 The K_1-Group

The results of this section are simple and probably well-known, but are included here for completeness (and since the author was unable to find a reference in the literature from which to derive it).

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

Lemma 7.2-A *Let $\{B_t : t \in [0,1]\}$ be a continuous field of unital C^*-algebras such that $K_1(B_t) = 0$ for each t in a dense subset Q of $[0,1]$. Then for each θ the canonical map $K_1(\Gamma) \to K_1(B_\theta)$ is surjective. More precisely, for each θ in $[0,1]$, a positive integer n, and each invertible w in $M_n(B_\theta)$, there exists a positive integer m and a global section ξ of the field $\{M_{n+m}(B_t) : t \in [0,1]\}$ such that $\xi(t)$ is invertible for each t and $\xi(\theta) = w \oplus I_m$.*

Proof First, choose a section $t \mapsto w(t)$ of the field defined on a small enough open interval J containing θ and consisting of invertible elements such that $w(\theta) = w$. Fix $r, s \in Q \cap J$ such that $r < \theta < s$. Since the K_1-groups of B_r and B_s are zero, there exists an integer m such that

$$w(r) \oplus I_m \in \mathrm{GL}^0_{n+m}(B_r), \quad \text{and} \quad w(s) \oplus I_m \in \mathrm{GL}^0_{n+m}(B_s).$$

Each can be written as a product of exponentials $w(r) \oplus I_m = e^{T_1} \cdots e^{T_k}$, $w(s) \oplus I_m = e^{S_1} \cdots e^{S_\ell}$ for some $T_j \in M_{n+m}(B_r)$, $S_i \in M_{n+m}(B_s)$, and some integers k and ℓ. Now extend each T_j to a global section $T_j(t)$ (of the field $\{M_{n+m}(B_t) : t \in [0,1]\}$) so that $T_j(r) = T_j$, and similarly S_i to a global section $S_i(t)$ such that $S_i(s) = S_i$. Define a global section ξ of invertible elements by

$$\xi(t) = \begin{cases} e^{T_1(t)} \cdots e^{T_k(t)} & 0 \le t \le r, \\ w(t) \oplus I_m & r \le t \le s, \\ e^{S_1(t)} \cdots e^{S_\ell(t)} & s \le t \le 1. \end{cases}$$

By construction, ξ is well-defined at r and s, continuous, and so defines a global section of invertible elements with the required condition. ■

From the short exact sequence

$$0 \longrightarrow J_\theta \xrightarrow{j_\theta} \Gamma \xrightarrow{\varepsilon_\theta} B_\theta \longrightarrow 0$$

where $J_\theta = \{\xi \in \Gamma : \xi(\theta) = 0\}$, $\varepsilon_\theta(\xi) = \xi(\theta)$, and j_θ the canonical inclusion, one has its associated six-term exact sequence

$$\begin{array}{ccccc} K_0(J_\theta) & \xrightarrow{j_{\theta*}} & K_0(\Gamma) & \xrightarrow{\varepsilon_{\theta*}} & K_0(B_\theta) \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\ K_1(B_\theta) & \xleftarrow{\varepsilon_{\theta*}} & K_1(\Gamma) & \xleftarrow{j_{\theta*}} & K_1(J_\theta) \end{array}$$

Theorem 7.2-B *Let $\{B_t : t \in [0,1]\}$ be a separable continuous field of unital C^*-algebras such that $K_1(B_t) = 0$ for each t in a dense subset Q of $[0,1]$. Then there is a dense G_δ subset G of $[0,1]$ containing Q such that $K_1(B_\theta) = 0$ for each $\theta \in G$.*

Proof Let $[\xi^1], [\xi^2], \dots$ be an enumeration of the elements of $K_1(\Gamma)$. By the six term exact sequence above, and since $K_1(B_r) = 0$, for each $r \in Q$ there is a surjection

$$j_{r*} \colon K_1(J_r) \to K_1(\Gamma).$$

For each $r \in Q$ and each $n = 1, 2, \dots$ choose $[\eta_r^n] \in K_1(J_r)$ such that $j_{r*}([\eta_r^n]) = [\eta_r^n] = [\xi^n]$ in $K_1(\Gamma)$. Thus $\xi^n(\eta_r^n)^{-1} \in \mathrm{GL}_*^0(\Gamma)$ so that $\xi^n(t)\big(\eta_r^n(t)\big)^{-1} \in \mathrm{GL}_*^0(B_t)$ for each t. Now since $\eta_r^n(r)$ is a matrix with scalar entries and $t \mapsto \eta_r^n(t)^{-1}$ is continuous, it follows that there is an open interval $I_n(r)$ containing r such that $\xi^n(t) \in \mathrm{GL}_*^0\, B_t)$ for $t \in I_n(r)$. Now let

$$\mathcal{U}_n = \bigcup_{r \in Q} I_n(r),$$

a dense open set in $[0, 1]$, and consider the dense G_δ set

$$G = \bigcap_{n=1}^{\infty} \mathcal{U}_n.$$

Now for $\theta \in G$ one has $K_1(B_\theta) = 0$. To see this, fix $\theta \in G$ so that for each n, $\theta \in I_n(r)$ for some $r \in Q$. So, $\xi^n(\theta) \in \mathrm{GL}_*^0(B_\theta)$. Hence, $[\xi^n(\theta)] = 0$ in $K_1(B_\theta)$ for all n. Since, by Lemma 7.2-A, the map $(ev_\theta)_1 \colon K_1(\Gamma) \to K_1(B_\theta)$ is surjective for all θ, so that $[\xi^1(\theta)], [\xi^2(\theta)], \dots$ constitute all the elements of $K_1(B_\theta)$, it follows that $K_1(B_\theta) = 0$. ∎

7.3 The K_0-Group

Throughout this section we shall assume that $\{B_t : t \in [0, 1]\}$ is a given continuous field of unital C*-algebras.

Lemma 7.3-A *Assume the field $\{B_t : t \in [0, 1]\}$ satisfies the hypothesis (H1). Let $e \colon (a, b) \to \bigcup_t B_t$ be any local section of projections of the field. Then each $r \in Q \cap (a, b)$ has a neighborhood on which $m_{r,x}[e(t)] \in \mathcal{R}_t$, where $x = [e(r)]$.*

Proof Put $m = m_{r,x}$. Since $r \in Q$, $m[e(r)] \in \mathcal{R}_r$, so that one can write

$$m[e(r)] = \sum_j n_j [P_j(r)],$$

for some integers n_j. By continuity, this equation holds in a neighborhood of r, which gives the result. ∎

Theorem 7.3-B *Assume that the field $\{B_t : t \in [0, 1]\}$ satisfies the hypothesis (H1). Fix θ and a projection e in a matrix algebra over B_θ. Then there are global sections of projections $p_1(t)$ and $p_2(t)$ of the field $\{M_m(B_t) : t \in [0, 1]\}$ (for some m) and a positive integer n_θ such that*

$$[p_2(\theta)] - [p_1(\theta)] = n_\theta [e]$$

in $K_0(B_\theta)$. In particular, for each θ the canonical map $K_0(\Gamma) \to K_0(B_\theta)$ is almost surjective. The integer n_θ is the least common multiple of two integers of the form $m_{r,x}$ appearing in (H1).

Proof Choose a section $e(t)$ of the field defined on a sufficiently small interval I containing θ such that $e(t)$ is a projection for each $t \in I$ and $e(\theta) = e$. Pick $r, s \in Q \cap I$ such that $r < \theta < s$. Using the projections $P_1, \dots, P_N$ of (H1), there exist global sections R_1, R_2, S_1, S_2 of projections (in possibly different matrix-size algebras) whose classes at each t belong to $\mathcal{R}_t$ and such that

$$m[e(r)] = [R_2(r)] - [R_1(r)] \in \mathcal{R}_r, \quad m'[e(s)] = [S_2(s)] - [S_1(s)] \in \mathcal{R}_s,$$

where $m = m_{r,[e(r)]}$ and $m' = m_{s,[e(s)]}$. Let $n = \mathrm{lcm}(m, m')$, so that by suitably modifying R_j, S_j one can assume without loss of generality that

$$n[e(r)] = [R_2(r)] - [R_1(r)] \in \mathcal{R}_r, \quad n[e(s)] = [S_2(s)] - [S_1(s)] \in \mathcal{R}_s. \tag{7.3.1}$$

These equations in fact hold in a neighborhood of r and s (with n held fixed), respectively, in view of Lemma 7.3-A. Without loss of generality one can assume

$$\mathrm{size}(R_2) = \mathrm{size}(R_1) + n\,\mathrm{size}(e) \quad \text{and} \quad \mathrm{size}(S_2) = \mathrm{size}(S_1) + n\,\mathrm{size}(e).$$

From (7.3.1) there exists positive integers p, q, p', q' and invertibles w and u in some matrix algebra over B_r and B_s, respectively, such that

$$e(r)^{(n)} \oplus R_1(r) \oplus I_p \oplus O_q = w[R_2(r) \oplus I_p \oplus O_q]w^{-1} \tag{7.3.2}$$

$$e(s)^{(n)} \oplus S_1(s) \oplus I_{p'} \oplus O_{q'} = u[S_2(s) \oplus I_{p'} \oplus O_{q'}]u^{-1}. \tag{7.3.3}$$

One can assume that w and u have the same size k after suitable enlargements. Letting $R_j'(t) = R_j(t) \oplus I_p \oplus O_q$ and $S_j'(t) = S_j(t) \oplus I_{p'} \oplus O_{q'}$, for $j = 1, 2$, these become

$$e(r)^{(n)} \oplus R_1'(r) = wR_2'(r)w^{-1} \tag{7.3.2$'$}$$

$$e(s)^{(n)} \oplus S_1'(s) = uS_2'(s)u^{-1} \tag{7.3.3$'$}$$

and equations (7.3.1) are unchanged when R_j and S_j are replaced by R_j' and S_j', respectively. Choose global sections of invertibles $\xi(t)$ and $\eta(t)$ in some matrix algebra over the field such that $\xi(r) = w \oplus w^{-1}$ and $\eta(s) = u \oplus u^{-1}$. (This is possible since $w \oplus w^{-1}$ is in the connected component of the identity, so is a product of exponentials, and hence extends to a global invertible section.) Thus (7.3.2$'$) and (7.3.3$'$) can be written as

$$e(r)^{(n)} \oplus R_1'(r) \oplus O_k = (w \oplus w^{-1})\big(R_2'(r) \oplus O_k\big)(w \oplus w^{-1})^{-1} \tag{7.3.2$''$}$$

$$e(s)^{(n)} \oplus S_1'(s) \oplus O_k = (u \oplus u^{-1})\big(S_2'(s) \oplus O_k\big)(u \oplus u^{-1})^{-1}. \tag{7.3.3$''$}$$

Adding $S_1'(r)$ to both sides of (7.3.2$''$) and adding $R_1'(s)$ to both sides of (7.3.3$''$) one obtains

$$\begin{aligned} &e(r)^{(n)} \oplus R_1'(r) \oplus O_k \oplus S_1'(r) \\ &\quad = (w \oplus w^{-1} \oplus I_{m_1})[R_2'(r) \oplus O_k \oplus S_1'(r)](w \oplus w^{-1} \oplus I_{m_1})^{-1} \end{aligned} \tag{7.3.2$'''$}$$

$$\begin{aligned} &e(s)^{(n)} \oplus S_1'(s) \oplus O_k \oplus R_1'(s) \\ &\quad = (u \oplus u^{-1} \oplus I_{n_1})[S_2'(s) \oplus O_k \oplus R_1'(s)](u \oplus u^{-1} \oplus I_{n_1})^{-1} \end{aligned} \tag{7.3.3$'''$}$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

where m_1 is the size of S_1' and n_1 the size of R_1'. Now let

$$p_1(t) = R_1'(t) \oplus O_k \oplus S_1'(t).$$

Let $C(t)$, for $r \leq t \leq s$, denote a continuous path of invertible matrices over the complex numbers such that $C(r) =$ Identity and $C(s)$ the permutation matrix such that

$$C(s)[X \oplus Y \oplus O_k \oplus Z]C(s)^{-1} = X \oplus Z \oplus O_k \oplus Y.$$

Now put

$$p_2(t) = \begin{cases} \big(\xi(t) \oplus I_{m_1}\big)\,[R_2'(t) \oplus O_k \oplus S_1'(t)]\big(\xi(t) \oplus I_{m_1}\big)^{-1} & 0 \leq t \leq r \\ C(t)[e(t)^{(n)} \oplus R_1'(t) \oplus O_k \oplus S_1'(t)]C(t)^{-1} & r \leq t \leq s \\ \big(\eta(t) \oplus I_{n_1}\big)\,[S_2'(t) \oplus O_k \oplus R_1'(t)]\big(\eta(t) \oplus I_{n_1}\big)^{-1} & s \leq t \leq 1. \end{cases}$$

It easily follows that $p_1(t)$ and $p_2(t)$ are well-defined continuous sections, by (7.3.2′′′) and (7.3.3′′′), and that $[p_2(\theta)] - [p_1(\theta)] = n[e] \in K_0(B_\theta)$ is the canonical image of $[p_2] - [p_1]$ in $K_0(\Gamma)$. ∎

Remark Note that the class of $p_1(t)$ is in $\mathcal{R}_t$ for each t, whereas the class of $p_2(t)$ is in $\mathcal{R}_t$ for all t outside some neighborhood of θ.

Theorem 7.3-C *Assume that the continuous field $\{B_t\}$ is separable and satisfies the hypothesis (H1). There is a dense G_δ subset G of $[0, 1]$ containing Q such that for each $\theta \in G$ and each $x \in K_0(B_\theta)$ there is a positive integer $N_\theta(x)$ such that*

$$N_\theta(x) \cdot x \in \mathbb{Z}[P_1(\theta)] + \cdots + \mathbb{Z}[P_N(\theta)].$$

In addition, the integer $N_\theta(x)$ is a product of two integers, one of the form $m_{r,x}$ and the other a least common multiple of two integers of the form $m_{r,x}$ (which appear in (H1)) .

Proof Let $x_1, x_2, \ldots$ be an enumeration of the elements of $K_0(\Gamma)$. By Lemma 7.3-A, for each x_j and each $r \in Q$ there is an open interval $I_j(r)$ containing r such that $m_{r,x_j(r)} \cdot x_j(t) \in \mathcal{R}_t$ for $t \in I_j(r)$, where $x(t) := \varepsilon_{t*}(x)$ (here, ε_{t*} is as defined in Section 7.2). Let

$$\mathcal{U}_j = \bigcup_{r \in Q} I_j(r)$$

a dense open set, and consider the dense G_δ set

$$G = \bigcap_{j=1}^{\infty} \mathcal{U}_j.$$

Now pick any element $y \in K_0(B_\theta)$ where $\theta \in G$. Then by Theorem 7.3-B there is a positive integer n_θ with $n_\theta y = x_k(\theta)$ for some k. Now θ being in $\mathcal{U}_k$ is in $I_k(r)$ for some $r \in Q$ so that $m_{r,x_k(r)} \cdot x_k(\theta) \in \mathcal{R}_\theta$. Thus $m_{r,x_k(r)} \cdot n_\theta \cdot y = m_{r,x_k(r)} \cdot x_k(\theta) \in \mathcal{R}_\theta$ so that the result holds with $N_\theta(x) = m_{r,x_k(r)} \cdot n_\theta$ is of the exact form as in the statement. ■

Remark From the preceding proof we note that if $K_0(\Gamma)$ is finitely generated, then the conclusion of Theorem 7.3-C holds on a dense open subset of $[0, 1]$.

Corollary 7.3-D **(Conservation of Torsion-Free Rank)** *If, in addition to (H1) for a separable continuous field of C^*-algebras, the groups $K_0(B_r)$ have torsion-free rank n for each $r \in Q$, then there is a dense G_δ subset G of $[0, 1]$ containing Q such that $K_0(B_t)$ has torsion-free rank n for $t \in G$.*

We now arrive at and state the main result of this section as follows.

Corollary 7.3-E *Assume the hypotheses (H1) and (H2) hold for a separable continuous field of C^*-algebras, and that $m_{r,x} = 1$ for each $r \in Q$ and x, so that the classes $[P_1(t)], \dots, [P_N(t)]$ generate $K_0(B_t)$ for each $t \in Q$.*

(a) There is a dense G_δ subset G of $[0, 1]$ containing Q such that for each $\theta \in G$, the classes $[P_1(\theta)], \dots, [P_N(\theta)]$ generate $K_0(B_\theta)$.
(b) The canonical map $\varepsilon_{\theta *} \colon K_i(\Gamma) \to K_i(B_\theta)$, induced by evaluation, is surjective for each θ in $[0, 1]$ and for $i = 0, 1$.
(c) For each θ in $[0, 1]$ one has the short exact sequences of K-groups

$$0 \longrightarrow K_0(J_\theta) \xrightarrow{j_*} K_0(\Gamma) \xrightarrow{\varepsilon_{\theta *}} K_0(B_\theta) \longrightarrow 0$$
$$0 \longrightarrow K_1(J_\theta) \xrightarrow{j_*} K_1(\Gamma) \xrightarrow{\varepsilon_{\theta *}} K_1(B_\theta) \longrightarrow 0.$$

Proof Part (a) follows since $n_\theta = 1$ in Theorem 7.3-B and $N_\theta(x) = 1$ in Theorem 7.3-C. Part (b) follows from Theorem 7.3-B and (c) from the above six term exact sequence, (b), and Lemma 7.2-A. ■

Remarks Corollary 7.3-E holds with "generate" replaced by "form a basis for". Note that the conclusions in (b) and (c) hold for all θ and not just on the G_δ set. Also, for a dense G_δ of θ's, conclusion (c) implies that there is an isomorphism $j_* \colon K_1(J_\theta) \to K_1(\Gamma)$.

The Fourier Automorphism Case

Going back to the Fourier case with $B_\theta = A_\theta \rtimes_\sigma \mathbb{Z}_4$, Corollary 6-C shows that the nine canonical modules form a basis for $K_0(B_\theta)$ in the rational case, so that by Corollary 7.3-E(a) one has $K_0(B_\theta) \cong \mathbb{Z}^9$ for θ in a dense G_δ and that the nine modules (evaluated at θ) form a basis for it. In addition, Corollary 7.3-E(b) shows that for each θ there is a canonical surjection

$$\varepsilon_{\theta *} \colon K_i(\Gamma) \to K_i(B_\theta).$$

Also, Corollary 7.3-E(c) entails an isomorphism

$$K_0(\Gamma) \cong \mathbb{Z}^9 \oplus K_0(J_\theta)$$

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press

so that, in particular, all the groups $K_0(J_\theta)$, for θ in the G_δ, are isomorphic. It seems reasonable to expect that these should hold for any θ—under the hypotheses (H1) and (H2).

Since for rational θ, $K_1(B_\theta) = 0$, Theorem 7.2-B implies that the same holds on a dense G_δ (containing the rationals), and so Corollary 7.3-E(c) yields the isomorphism $j_*\colon K_1(J_\theta) \to K_1(\Gamma)$ (induced by the canonical inclusion) for each θ in the dense G_δ. (In particular, all such ideals J_θ also have the same K_1-group.)

Other Finite-Order Automorphisms

A few applications for other finite order automorphisms of the rotation algebra can be made as follows.

Example 1 Consider the flip automorphism of the rotation algebra, given by $\phi(U) = U^{-1}$, $\phi(V) = V^{-1}$. The associated crossed product, $C_\theta = A_\theta \rtimes_\phi \mathbb{Z}_2$ was studied by several authors. In [9], Kumjian was able to use Natsume's exact sequence for K-groups of amalgamated products, and the fact that $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$, to obtain the isomorphisms $K_0(C_\theta) \cong \mathbb{Z}^6$ and $K_1(C_\theta) = 0$ for all θ. One can, however, show that these isomorphisms hold on a dense G_δ without using Natsume's sequence (nor the fact that $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$) by applying Corollary 7.3-E and verifying the hypothesis (H1) for six canonical projections in C_θ in the rational case, which is easy to do. (For example, see Lemma 2.3 of [13] for an explicit form of these projections.) That hypothesis (H2) holds, *i.e.*, that K_1 vanishes in the rational case, follows from [2, Theorem 6.1].

Example 2 The order six automorphism β of the rotation algebra A_θ is defined by

$$\beta(U) = V, \quad \beta(V) = U^{-1}V.$$

Its square β^2 gives an order three automorphism. The associated crossed product algebras are $A_\theta \rtimes_\beta \mathbb{Z}_6$ and $A_\theta \rtimes_{\beta^2} \mathbb{Z}_3$. In [6, Corollary 2.0.10], it was shown that their K_1-groups vanish when θ is rational. Therefore from Theorem 7.2-B above there is a dense G_δ subset of $[0, 1]$ on which the K_1-groups vanish. In [6] it was also shown that for rational θ one has $K_0(A_\theta \rtimes_\beta \mathbb{Z}_6) \cong \mathbb{Z}^{10}$ and $K_0(A_\theta \rtimes_{\beta^2} \mathbb{Z}_3) \cong \mathbb{Z}^8$. In order for these isomorphisms to hold for at least a dense G_δ set of parameters θ, at least by our techniques, one needs to come up with ten (respectively, eight) canonical modules and show that they are independent in K_0.

Closing Comments Some questions come to mind when looking at continuous fields satisfying either (H1) or (H2), or both. Does Theorem 7.3-C hold for all θ? If the groups $K_0(B_t)$ are torsion-free for t in a dense set, are they always torsion-free? Is the canonical image of $K_0(J_\theta)$ in $K_0(\Gamma)$ always the same subgroup? (Which is true for each θ in the G_δ set.) Are all the groups $K_0(J_\theta)$ isomorphic? It seems that in some important examples, like the flip and the Fourier automorphisms, one would expect $K_0(J_\theta)$ to be all isomorphic. It is known that continuous fields of C*-algebras can behave quite strangely in such a way that no fiber, not even a dense set of them, can practically predict any facts about other fibers. However, what if some additional and more stringent assumptions are made on the field? For example, define two C*-algebras to be K_0-*similar* if they are isomorphic to two fibers in some continuous field $\{B_t\}$ over $[0,1]$ for which there are classes $\xi_1,\dots,\xi_n$ in $K_0(\Gamma)$ such that at each t the classes $\xi_1(t),\dots,\xi_n(t)$ form a basis for $K_0(B_t)$. (Assume the groups are torsion-free.) For example, the continuous field of rotation algebras yields a K_0-similarity between its fibers. What connection does this equivalence relation have, if any, with other known relations, such as KK-equivalence, strong Morita equivalence, or even shape equivalence?

8 Appendix. Corrections to: Quartic Algebras Paper

In this appendix we point out some corrections to the paper [7] which are crucial for the proofs of the present paper (particularly, in Section 3 above). (These do not affect the conclusions obtained in [7].)

In [7] the authors use τ to denote the automorphism inverse to our Fourier automorphism σ. Thus, $\tau(U) = V^{-1}$, $\tau(V) = U$, and with the realization of $A_{p/q}$ given as functions on the square by (2.2.1) one has

$$\tau(f)(x,y) = \tau_0\big(f(1-y,x)\big),$$

where $\tau_0 = \alpha_1\alpha_0$ (as on page 1172 of [7]). (The latter automorphisms are defined as in Section 2.2 above.) For $E = -1, \pm i$, the authors define the subspace (see [7, p. 1189])

$$A_\theta^\tau(E) := \{x \in A_\theta : \tau(x) = Ex\}.$$

In their proof of Theorem 6.2.1 (page 1190), the authors state the identification

$$A_\theta^\tau(E) = \left\{ f \in C(T, M_q) : \begin{array}{c} f(x,x) = E\alpha_1\alpha_0\big(f(1-x,x)\big) \\ f(x,0) = E\alpha_1\alpha_2\gamma_0\big(f(1-x,0)\big) \end{array} \right\}$$

where $\alpha_0, \alpha_1, \alpha_2, \gamma_0$ are as defined in Section 2.2, and where T is the triangle shown in the figures of Section 3. Three corrections are to be noted here. The identification should read (after examination of the proof)

$$A_\theta^\tau(E) = \left\{ f \in C(T, M_q) : \begin{array}{c} f(x,x) = \overline{E}\alpha_1\alpha_0\big(f(1-x,x)\big) \\ f(x,0) = E^2\alpha_2\gamma_0\big(f(1-x,0)\big) \end{array} \right\}. \tag{8.1}$$

These appear to have stemmed from the last two equations on page 1173, which read, for a given $f \in A_\theta^\tau$ (the fixed point subalgebra),

$$f(x,x) = (\tau f)(x,x) = \tau_0\big(f(1-x,x)\big) = \alpha_1\alpha_0\big(f(1-x,x)\big), \tag{8.2}$$

$$f(x,0) = (\sigma f)(x,0) = \alpha_1\alpha_2\gamma_0\big(f(1-x,0)\big), \tag{8.3}$$

where their "σ" here denotes the flip automorphism, *i.e.*, our σ^2. Recall that their flip "σ" is given by

$$\text{“}\sigma\text{”}(f)(x,y) = \sigma_0\big(f(1-x,1-y)\big) = \alpha_1\alpha_2\gamma_0\big(f(1-x,1-y)\big).$$

Equation (8.2) is correct, but in (8.3) there should not be an α_1. (In fact, (8.3) does not hold for $f = U + V + U^{-1} + V^{-1}$ which is in A_θ^τ.) When considering these equations more generally for typical $f \in A_\theta^\tau(E)$, so that $\tau(f) = Ef$, these equations become

$$Ef(x,x) = (\tau f)(x,x) = \tau_0\big(f(1-x,x)\big) = \alpha_1\alpha_0\big(f(1-x,x)\big),$$
$$E^2 f(x,0) = (\sigma f)(x,0) = \alpha_2\gamma_0\big(f(1-x,0)\big)$$

and these yield (8.1) as the corrected identification.

Finally, on page 1190, the authors have obtained the isomorphism

$$A_\theta \rtimes_\tau \mathbb{Z}_4 \cong \left\{ f \in C(T, M_{4q}) : \begin{array}{c} f(x,x) = (\alpha_1\alpha_0 \otimes \mathrm{Ad}\, D)\big(f(1-x,x)\big), \\ f(x,0) = (\alpha_1\alpha_2\gamma_0 \otimes \mathrm{Ad}\, D^2)\big(f(1-x,0)\big) \end{array} \right\}$$

which, in view of (8.1), should now be

$$(8.4)\quad A_\theta \rtimes_\tau \mathbb{Z}_4 \cong \left\{ f \in C(T, M_{4q}) : \begin{array}{c} f(x,x) = (\alpha_1\alpha_0 \otimes \mathrm{Ad}\, D^{-1})\big(f(1-x,x)\big) \\ f(x,0) = (\alpha_2\gamma_0 \otimes \mathrm{Ad}\, D^2)\big(f(1-x,0)\big) \end{array} \right\}.$$

This is the algebra that we called $\mathcal{T}_\theta$ in Section 3 above.

Acknowledgements The author wishes to thank George Elliott for many helpful and stimulating conversations in the course of this paper and for raising the problem related to the Fourier automorphism. And also for his insightful comments on the introduction of this paper. This research was partly supported by a grant from NSERC (Natural Science and Engineering Council of Canada).

Addendum After submission of this paper, in [18] the author constructed an order four automorphism of the irrational rotation algebra A_θ that mimics the Fourier automorphism on K_1 (*i.e.*, sends the classes $[U]$, $[V]$ to $[V]$, $[U^{-1}]$, respectively) and such that the fixed point subalgebra is an AF-algebra.

References

[1] O. Bratteli, G. A. Elliott, D. E. Evans and A. Kishimoto, *Non-commutative spheres I.* Internat. J. Math. (2) **2**(1990), 139–166.

[2] ———, *Non-commutative spheres II: rational rotations.* J. Operator Theory **27**(1992), 53–85.

[3] O. Bratteli and A. Kishimoto, *Non-commutative spheres III. Irrational Rotations.* Comm. Math. Phys. **147**(1992), 605–624.

[4] A. Connes, *C^* algèbre et géométrie différentielle.* C. R. Acad. Sci. Paris Ser. A–B **290**(1980), 599–604.

[5] ———, *Noncommutative Geometry.* Academic Press, 1994.

[6] C. Farsi and N. Watling, *Fixed point subalgebras of the rotation algebra.* C.R. Math. Rep. Acad. Sci. Canada (2) **13**(1991), 75–80.

[7] ———, *Quartic algebras.* Canad. J. Math. (6) **44**(1992), 1167–1191.

[8] E. Grosswald, *Representations of integers as sums of squares.* Springer-Verlag, New York, 1985.

[9] A. Kumjian, *On the K-theory of the symmetrized non-commutative torus.* C. R. Math. Rep. Acad. Sci. Canada (3) **12**(1990), 87–89.

[10] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed product C^*-algebras.* J. Operator Theory **4**(1980), 93–118.

[11] ______, *K-groups of reduced crossed products by free groups.* J. Operator Theory **8**(1982), 131–156.
[12] J. Rosenberg, *Appendix to 'Crossed products of UHF algebras by product type actions'.* Duke Math. J. (1) **46**(1979), 25–26.
[13] S. G. Walters, *Projective modules over the non-commutative sphere.* J. London Math. Soc. (2) **51**(1995), 589–602.
[14] ______, *Inductive limit automorphisms of the irrational rotation algebra.* Comm. Math. Phys. **171**(1995), 365–381.
[15] ______, *Chern characters of Fourier modules.* Canad. J. Math. (3) **52**(2000), 633–672.
[16] ______, *On the irrational quartic algebra.* C.R. Math. Rep. Acad. Sci. Canada (3) **21**(1999), 91–96.
[17] ______, *Gluing Hilbert modules in a continuous field of C^*-algebras.* Unpublished note, 1998.
[18] ______, *On the inductive limit structure of order four automorphisms of the irrational rotation algebra.* Preprint, 2000, 8 pages.

Department of Mathematics and Computer Science
The University of Northern British Columbia
Prince George, BC
V2N 4Z9
email: walters@hilbert.unbc.ca
website: http://hilbert.unbc.ca/walters

https://doi.org/10.4153/CJM-2001-026-x Published online by Cambridge University Press