BULL. AUSTRAL. MATH. SOC. 06B15, 06D05
VOL. 22 (1980), 125-132. (06D20, 54H10)

AN ORDERED SHEAF REPRESENTATION OF SUBRESIDUATED LATTICES

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Kennison's concept of an ordered sheaf is used to show that any member of the variety of subresiduated lattices is canonically isomorphic to the algebra of all ordered sections in a certain ordered sheaf, whose base is the Priestley space of the residuating sublattice.

0. Introduction

Recently in [5], Kennison introduced the notion of an ordered sheaf of finitary algebras and its associated algebra of ordered sections. In such a sheaf, the total space is Hausdorff, the base space is an ordered topological space, and a representation of an algebra as the ordered sections supplies a good generalization of the representation of an algebra as the algebra of all sections in a Hausdorff sheaf. Nevertheless, it would not seem to be an easy task to give a non-trivial ordered sheaf representation for each algebra in some variety. Here we show that such a representation is possible for the variety of subresiduated lattices; this variety was introduced by Epstein and Horn in their recent work [4] on modal logics.

1. Subresiduated lattices

The more intuitive definition of a *subresiduated lattice* is a pair (A, Q) , where Q is a bounded distributive lattice (with largest element

Received 20 December 1979.

https://doi.org/10.1017/S0004972700006390 Published online by Cambridge University Press

1 and smallest 0) and Q is a sublattice of A containing 0, 1 such that for each $x, y \in A$ there is an element $p \in Q$ with the property that for all $q \in Q$, $x \wedge q \leq y$ if and only if $q \leq p$. This element p is denoted by $x \to y$. For $x \in A$, $x! = 1 \to x$ is the largest element of the *subresiduating sublattice* Q which is beneath x and Q is the sublattice $\{x \in A : x = x!\} = \{y! : y \in A\}$. Moreover, the following equations are identically satisfied:

(R1) $(x \wedge y) \to y = 1$,

(R2) $x \to y \leq z \to (x \to y)$,

(R3) $x \wedge (x \to y) \leq y$,

(R4) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$.

The concept of a subresiduated lattice was recently introduced by Epstein and Horn [4] to describe the Lindenbaum algebras of certain modal logics. In their paper [4, Theorem 1], they showed that an algebra $(A; \wedge, \vee, \to, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is a subresiduated lattice with $Q = \{x \in A; x = 1 \to x\}$ as the subresiduating sublattice if and only if $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and (R1)-(R4) hold identically. Thus, the class of subresiduated lattices will be considered as a variety R of algebras of type $(2, 2, 2, 0, 0)$. The subresiduating sublattice $\{x \in A; x = x!\}$ of an R-algebra A is denoted by $Q(A)$. The most familiar subclass of R is the class H of all *Heyting* (or relatively pseudocomplemented or, in the terminology of [6, Chapter 4], pseudo-Boolean) algebras, that is, bounded distributive lattices $(A; \wedge, \vee, 0, 1)$ such that for each $x, y \in A$, there is an element $x \to y \in A$ with the property that $x \wedge z \leq y$ $(z \in A)$ if and only if $z \leq x \to y$. When $\to$ is considered as a fundamental binary operation, H is nothing more than the subvariety of R which consists of all R-algebras A satisfying the identity $x = x!$ (that is $A = Q(A)$). Perhaps, it is also worth noting that a *finite* sublattice Q of a bounded distributive lattice $(A; \wedge, \vee, 0, 1)$ subresiduates A, provided $0, 1 \in Q$; for $x, y \in A$, $x \to y$ is the supremum of all $q \in Q$ for which $x \wedge q \leq y$.

Let A be an R-algebra and F be a filter (dual ideal) of the sublattice $Q(A)$. Then, F induces an R-congruence on A which is given by $x \equiv y \pmod F$ $(x, y \in A)$ if and only if $x \wedge f = y \wedge f$ for

https://doi.org/10.1017/S0004972700006390 Published online by Cambridge University Press

some $f \in F$ if and only if $x \to y$, $y \to x \in F$ if and only if $(x \to y) \wedge (y \to x) \in F$.

The quotient R-algebra is denoted by A/F ; the congruence class of $x \in A$ is denoted $x/F \in A/F$. The above congruence is the unique R-congruence Θ on A such that $F = \{q \in Q(A) : q \equiv 1 \ (\Theta)\}$. In fact, the map $F \to \text{mod } F$ is a lattice-isomorphism of the lattice of filters of $Q(A)$ onto the lattice of R-congruences of A ; the details are given in [4, Theorem 2].

2. The ordered sheaf

A *sheaf of R-algebras* is a triple (E, π, X) such that the following properties are fulfilled:

(1) E and X are topological spaces and $\pi : E \to X$ is a local homeomorphism from E onto X ;

(2) for each $x \in X$, the stalk $E_x = \pi^{-1}(\{x\})$ is an R-algebra;

(3) the functions $(a, b) \mapsto a \wedge b$, $(a, b) \mapsto a \vee b$, and $(a, b) \mapsto (a \to b)$ from the subspace

$$\{(a, b) \in E \times E : \pi(a) = \pi(b)\}$$

of $E \times E$ into E are continuous;

(4) the functions 0 and 1 which assign to each $x \in X$, the zero 0_x and the unit 1_x of E_x respectively, are continuous.

A *section* σ of such a sheaf (E, π, X) is a continuous map $\sigma : X \to E$ such that $\pi\sigma(x) = x$ for each $x \in X$. Under pointwise defined operations, the set $\Gamma(X)$ of all sections forms an R-algebra with smallest element 0 and largest element 1 ; this is ensured by conditions (3) and (4). For a background on sheaves of finitary algebras of a given type, we refer to Davey [3]; see also, Cignoli [1] and Kennison [5].

The space E of a sheaf (E, π, X) is called the *total space* while the space X is called the *base space*. A sheaf is said to be *Hausdorff* if the total space E is a Hausdorff topological space. An important fact about sheaves is that the set $\{x \in X : \sigma(x) = \tau(x)\}$, where two sections agree, is open. Hence, in a Hausdorff sheaf $\{x \in X : \sigma(x) = \tau(x)\}$ is *clopen* (closed and open).

By an ordered (topological) space, we mean a set X which is both a topological space and a partially ordered set. Such a space is said to be *order-disconnected* if whenever $x \not\leq y$, there is a clopen increasing subset U of X such that $x \in U$ and $y \in X \backslash U$; a subset U of a partially ordered set is *increasing* if whenever $x \leq y$ and $x \in U$ then $y \in U$.

Specializing Kennison's definition [5, Section 1] of an ordered sheaf of finitary algebras to R-algebras, we have: *an ordered sheaf of R-algebras* is a quadruple (E, π, X, e) such that

(1) (E, π, X) is a *Hausdorff* sheaf of R-algebras and the base space X is an ordered topological space, and

(2) for each pair $x, y \in X$ with $x \leq y$, there is a so-called *order map* $e(x, y) : E_x \to E_y$ which is an *onto* R-homomorphism and $e(x, x)$ equals the identity of E_x, $e(y, z)e(x, y) = e(x, z)$, when defined.

An *ordered section* σ of an ordered sheaf (E, π, X, e) is a section of the sheaf (E, π, X) such that $e(x, y)(\sigma(x)) = \sigma(y)$ whenever $x \leq y$. The set $\Gamma_0(X)$ of all ordered sections forms an R-subalgebra of the R-algebra $\Gamma(X)$. It should be noted that a Hausdorff sheaf (E, π, X) gives rise to an ordered sheaf (E, π, X, e) by defining the order on X to be the equality relation and the order maps $e(x, y)$ to be the corresponding identity functions, wherein $\Gamma_0(X) = \Gamma(X)$. Thus, the concept of an ordered sheaf extends the notion of a Hausdorff sheaf.

We are going to represent an R-algebra A as the algebra $\Gamma_0(X)$ in a certain ordered sheaf (E, π, X); we proceed to describe X and then E.

Let L be a bounded distributive lattice. Then $S(L)$ denotes the *Stone space* of L. As a set, $S(L)$ is the set of all prime filters of L and a base for its topology is $\{s(a) : a \in L\}$, where $s(a) = \{F \in S(L) : a \in F\}$; $S(L)$ is a so-called spectral space and the map $a \mapsto s(a)$ is a lattice-isomorphism of L onto the compact-open subsets of $S(L)$. We use $P(L)$ to denote the *Priestley space* of L. As a set, $P(L) = S(L)$ and a base for its topology is

$$\{s(a) : a \in L\} \cup \{P(L)\backslash s(a) : a \in L\} ;$$

with $P(L)$ ordered by set-inclusion between its member prime filters, $P(L)$ is a compact order-disconnected ordered space and the map $a \mapsto s(a)$ is a lattice-isomorphism of L onto the clopen increasing subsets of $P(L)$. For more details and the connection with the dual of the category of bounded distributive lattices, the reader is urged to consult [2].

For a non-trivial (that is, $0 \neq 1$) R-algebra A, let $X(A)$ be the Priestley space $P(Q(A))$ and $E(A)$ be the disjoint union of the R-algebras $E(A)_F = A/F$, $F \in X(A)$. For any $x \in A$, let $\hat{x} : X(A) \to E(A)$ be the Gelfand transform of x, that is, $\hat{x}$ is defined by $\hat{x}(F) = x/F \in E(A)_F = A/F$ for each prime filter $F \in X(A)$. Also define $\pi(A) : E(A) \to X(A)$ by $\pi(A)(t) = F$, if $t = a/F$ for some $a \in A$ and (unique) $F \in X(A)$.

For any $x, y \in A$, $\{F \in X(A) : \hat{x}(F) = \hat{y}(F)\}$ is an open subset of $X(A)$. Indeed, suppose $\hat{x}(F) = \hat{y}(F)$. Then $x \equiv y \pmod F$ and so $(x \to y) \wedge (y \to x) \in F$. Then $s((x \to y) \wedge (y \to x))$ is a clopen (increasing) neighbourhood of F contained within the given set. It follows from general considerations ([3, Lemma 2.1]) that if $E(A)$ is endowed with the finest topology making the Gelfand transforms continuous (that is $E(A)$ is given the topology whose base for the opens is $\{\hat{x}(s(a)) : x \in A, a \in Q(A)\}$) then $(E(A), \pi(A), X(A))$ is a sheaf of R-algebras.

For $F \leq G$ (that is $F \subseteq G$) in $X(A)$, define

$$e(A)(F, G) : E(A)_F \to E(A)_G$$

to be the well-defined R-epimorphism $e(A)(F, G)(a/F) = a/G$ for each $a/F \in A/F$ with $a \in A$.

We now come to our theorem.

THEOREM. *Let A be a non-trivial R-algebra. Then $(E(A), \pi(A), X(A), e(A))$ is an ordered sheaf of R-algebras and the map $a \mapsto \hat{a}$ is an R-isomorphism of A onto the algebra $\Gamma_0(X(A))$ of all ordered sections.*

Proof. To show that we have an ordered sheaf, it suffices to prove the Hausdorffness of $E(A)$. Suppose $m, n \in E(A)$ with $m \neq n$. Then $m = \hat{a}(F)$ and $n = \hat{b}(G)$ for suitable $a, b \in A$ and $F, G \in X(A)$. Now $X(A)$ is order-disconnected, so if $F \neq G$, say $F \nsubseteq G$, there is a clopen increasing neighbourhood U $\big(= s(q)$ for some $q \in Q(A)\big)$ such that $F \in U$ and $G \in X\backslash U$. Thus $\hat{a}(U)$ and $\hat{b}(X\backslash U)$ are open in $E(A)$, $m \in \hat{a}(U)$, $n \in \hat{b}(X\backslash U)$ and $\hat{a}(U) \cap \hat{b}(X\backslash U) = \emptyset$; that is, if $F \neq G$ then m and n are separated in $E(A)$ by open neighbourhoods. The complication arises when m and n are in the same stalk, that is, $F = G$. In this case, $m \neq n$ means $\hat{a}(F) = \hat{b}(F)$, that is,

$$F \in \{G \in X(A) : (a \to b) \wedge (b \to a) \notin G\} = X(A)\backslash s\big((a \to b) \wedge (b \to a)\big) = T$$

(say), which is open in $X(A)$. Then $\hat{a}(T)$ and $\hat{b}(T)$ are disjoint open neighbourhoods of m and n, as required.

The definitions ensure that each $\hat{a}$ $(a \in A)$ is in $\Gamma_0\big(X(A)\big)$ and that the map $a \mapsto \hat{a}$ is an R-homomorphism. If $a, b \in A$ and $a \neq b$, say $a \nleq b$, then $a \to b \neq 1$ and so there is a prime filter H of $Q(A)$ such that $a \to b \notin H$. Then $\hat{a}(H) \neq \hat{b}(H)$ and so the homomorphism is one-to-one.

It remains to prove that the homomorphism is onto. Let $\sigma \in \Gamma_0\big(X(A)\big)$ be fixed. For $F \in X(A)$, $\sigma(F) \in E(A)_F = A/F$ and so there exists $x_F \in A$ such that $\sigma(F) = \hat{x}_F(F)$. Now $\{G \in X(A) : \sigma(G) = \hat{x}_F(G)\}$ is clopen as $E(A)$ is Hausdorff. Moreover, it is increasing. Indeed, it is easy to establish that the set where two ordered sections agree in an ordered sheaf is both clopen and increasing! Hence, there exists $b_F \in Q(A)$ such that $\{G \in X(A) : \sigma(G) = \hat{x}_F(G)\} = s\big(b_F\big)$. Now $\{s\big(b_F\big) : F \in X(A)\}$ is an open cover of compact $X(A) = P\big(Q(A)\big)$ (it is even an open cover of compact $S\big(Q(A)\big)$) so there exist integers $i = 1, \ldots, n$ for which $s\big(b_1\big) \cup \ldots \cup s\big(b_n\big) = X(A)$ and $\sigma(G) = \hat{x}_i(G)$ for all $G \in s\big(b_i\big)$, where b_i and x_i are simplified notations for b_{F_i} and x_{F_i}, respectively.

With the given notation, $\hat{x}_i(G) = \hat{x}_j(G)$ for all $i, j = 1, \ldots, n$ and $G \in s\big(b_i\big) \cap s\big(b_j\big) = s\big(b_i \wedge b_j\big)$, that is, $x_i \equiv x_j \pmod{G}$ for all prime filters G of $Q(A)$ such that $b_i \wedge b_j \in G$. Thus, for all $i, j = 1, \ldots, n$, $x_i \wedge b_i \wedge b_j = x_j \wedge b_i \wedge b_j$. Otherwise, there are subscripts r and s such that $x_r \wedge b_r \wedge b_s \neq x_s \wedge b_r \wedge b_s$, say $x_r \wedge b_r \wedge b_s \nleq x_s \wedge b_r \wedge b_s$ so that $b_r \wedge b_s \nleq x_r \rightarrow x_s$ in $Q(A)$. But then we must have a prime filter H of distributive $Q(A)$ such that $b_r \wedge b_s \in H$ and $x_r \rightarrow x_s \notin H$. As $x_r \equiv x_s \pmod{H}$, $x_r \rightarrow x_s \in H$ and we have the desired contradiction.

Let $x = (x_1 \wedge b_1) \vee \ldots \vee (x_n \wedge b_n)$. Because of the preceding paragraph, $x \wedge b_i = x_i \wedge b_i$ for each $i = 1, \ldots, n$. In other words, $\hat{x}(G) = \hat{x}_i(G)$ for all $G \in X(A)$ such that $b_i \in G$, that is, $G \in s(b_i)$.

It now follows that $\sigma(G) = \hat{x}(G)$ for all $G \in X(A)$, that is, $\sigma = \hat{x}$ and the Gelfand transform does map A onto $\Gamma_0(X(A))$. //

The assignment of the ordered sheaf of the above theorem to each non-trivial R-algebra can be expanded to yield a functor from the category R into the appropriate category of ordered sheaves and their morphisms. We will not pursue the details here; ordered sheaf morphisms are defined in [5, p. 39].

An important subclass of R consists of those R-algebras A in which each element q of the subresiduating sublattice $Q(A)$ has a complement p within $Q(A)$, or equivalently $q \vee q^* = 1$. Here x^* denotes the element $x \rightarrow 0$ of $Q(A)$, for any $x \in A$. This class is the class of R_5-algebras of [4]; R_5 is the subvariety of R which consists of all algebras satisfying (R1)-(R4) and the additional identity:

(R5) $x! \vee x^* \vee (x! \vee x^*)^* = 1$.

For details, see [4, Lemma 10]. For any R_5-algebra A, $Q(A)$ is a Boolean lattice and so the base space $X(A)$ is totally unordered and the order maps $e(A)$ reduce to identity functions. Moreover, for each $F \in X(A)$, $Q(A/F)$ is the two-element chain. In other words, the stalks of the total space $E(A)$ consist of those subdirectly irreducible R_5-algebras (special R_5-algebras) which are R-homomorphic images of A (*cf.* [4, Definition 14, Lemma 16, Corollary 17]). In this way the ordered sheaf representation can be used to characterize R_5-algebras. It can also be used to characterize the so-called B-algebras and P-algebras of [4]. However, in these cases it is not necessary to proceed from the ordered sheaf representation. In [1, Theorem 3.6], Cignoli gave a characterization and sheaf representation of P-algebras, as a special class of distributive lattices.

In conclusion, it should be mentioned that the representation of R_5-algebras, and hence B-algebras and P-algebras, can be obtained from a general representation theorem of Davey [3, Theorem 4-5], wherein the base space is the Stone space of a Boolean lattice of factor congruences of a finitary algebra.

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