SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WHICH GENERALIZE THE HEAT EQUATION

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1. Introduction. The distribution of heat in an infinite rod is closely bound up with the theory of the Weierstrass transform. This connection is exhibited most clearly in the theory of Widder [**3**]. Consider the heat equation

$$\frac{\partial u}{\partial h} = \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

Widder has shown that if $u(x, h)$ satisfies 1.1 and if $u(x, h) \geqslant 0$ for $-\infty < x < \infty$, $a < h < b$ then we have

$$u(x, h'') = \int_{-\infty}^{\infty} k(x - y, h'' - h')u(y, h')\, dy \tag{1.2}$$

if $a < h', h'' < b$, $h'' > h'$. Here

$$k(x, t) = (4\pi t)^{-\frac{1}{2}} \exp\,(-x^2/4t) \tag{1.3}$$

$$k(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp\,[-ty^2 - ixy]\, dy. \tag{1.4}$$

Let $\mathfrak{E}_m$ and $\mathfrak{H}_n$ be real m- and n-dimensional vector spaces respectively. We shall write the elements X, Y, Z of $\mathfrak{E}_m$ and H, H', H'' of $\mathfrak{H}_n$ as column matrices,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \qquad Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix},$$

$$H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \qquad H' = \begin{pmatrix} h'_1 \\ h'_2 \\ \vdots \\ h'_n \end{pmatrix}, \qquad H'' = \begin{pmatrix} h''_1 \\ h''_2 \\ \vdots \\ h''_n \end{pmatrix}.$$

In place of $u(x, h)$, we consider $u(X, H)$ depending upon the $m + n$ real variables $x_1, \ldots, x_m, h_1, \ldots, h_n$, and instead of the heat equation 1.1, we consider the system of partial differential equations

$$\frac{\partial u}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i\, \partial x_j} a^l_{ij} \qquad l = 1, \ldots, n. \tag{1.5}$$

Here the a's are real constants and

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$$a_{ij}^{l} = a_{ji}^{l}.$$

In the present paper we shall develop for such systems a theory analogous to that of Widder.

Let $\mathfrak{T}_{\frac{1}{2}(m+1)}$ be the space of real symmetric $m \times m$ matrices,

$$T = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1m} \\ t_{21} & t_{22} & \ldots & t_{2m} \\ & \ldots & & \ldots \\ t_{m1} & t_{m2} & \ldots & t_{mm} \end{pmatrix}, \qquad T' = \begin{pmatrix} t'_{11} & t'_{12} & \ldots & t'_{1m} \\ t'_{21} & t'_{22} & \ldots & t'_{2m} \\ & \ldots & & \ldots \\ t'_{m1} & t'_{m2} & \ldots & t'_{mm} \end{pmatrix}.$$

We may define a partial ordering in $\mathfrak{T}_{\frac{1}{2}m(m+1)}$ by writing $T' > T$ if $T' - T$ is positive definite. We introduce corresponding to the system 1.5 a mapping $H \to H_a$ of $\mathfrak{H}_n$ into $\mathfrak{T}_{\frac{1}{2}m(m+1)}$, the (i, j) entry of H_a being defined as

$$\sum_{l=1}^{n} a_{ij}^{l} h_l \qquad (i, j = 1, 2, \ldots, m).$$

We shall show that if $u(X, H)$ satisfies 1.5 and if $u(X, H) \geqslant 0$ for $X \in \mathfrak{C}_m$, $H \in \mathfrak{R}$ where $\mathfrak{R}$ is an open convex subset of $\mathfrak{H}_n$, then

$$u(X, H'') = \int_{\mathfrak{C}_m} k(X - Y, H''_a - H'_a) u(Y, H')\, dY \tag{1.6}$$

if $H', H'' \in \mathfrak{R}$, $H''_a > H'_a$. The function $k(X, T)$ is defined by the formula

$$k(X, T) = (2\pi)^{-m} \int_{\mathfrak{C}_m} \exp\left[- Y^*TY - iY^*X\right] dY; \tag{1.7}$$

here T must be positive definite. An explicit formula for $k(X, T)$ (due to Czüber) is given in §3.

If A is any matrix, A^* is its transpose; thus

$$Y^*X = \sum_{i=1}^{m} y_i x_i,$$
$$Y^*TY = \sum_{i,j=1}^{m} y_i t_{ij} y_j.$$

2. A special case. The equation of heat transfer in m-dimensional space is

$$\frac{\partial u}{\partial h} = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} \quad \text{or} \quad \frac{\partial u}{\partial h} = \Delta u. \tag{2.1}$$

In this section we shall establish our theory for this special case. If $m = 1$ in 2.1 we have the equation considered by Widder. It is to be noted that only relatively minor adjustments are needed to make the extension from 1 to m dimensions. We set

$$k(X, t) = (4\pi t)^{-\frac{1}{2}m} \exp\left[-\frac{1}{4t} X^*X\right], \tag{2.2}$$
$$k(X, t) = (2\pi)^{-m} \int_{\mathfrak{C}_m} \exp\left[- tY^*Y - iY^*X\right] dY.$$

Let us write $|X|$ for

$$(X^*X)^{\frac{1}{2}} = (\sum_1^m x_i^2)^{\frac{1}{2}}.$$

We begin with a discussion of the convolution transform

2.3 $$\psi(X, h) = \int_{\mathfrak{E}_m} k(X - Y, h)\phi(Y)\, dY$$

under radial symmetry. A function ϕ defined on $\mathfrak{E}_m$ is said to be radically symmetric if $\phi(X') = \phi(X'')$ whenever $|X'| = |X''|$. If in formula 2.3 $\phi(Y)$ is radially symmetric, then for each h, $\psi(X, h)$ is also radially symmetric and the m-fold integration in our formula may be collapsed to a single integration with respect to a new kernel, which we need later and which we now compute. Since $\phi(Y)$ depends only upon $|Y| = \rho$ we write $\phi(\rho)$ instead of $\phi(Y)$. Similarly since $\psi(X, h)$ depends only upon $|X| = r$ and h, we write $\psi(r, h)$ instead of $\psi(X, h)$.

Introducing spherical coordinates,

$$\begin{aligned} y_1 &= \rho \cos \phi_1 \\ y_2 &= \rho \sin \phi_1 \cos \phi_2 \\ y_3 &= \rho \sin \phi_2 \sin \phi_2 \cos \phi_3 \\ &\cdots \\ y_{m-1} &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \cos \phi_{m-1} \\ y_m &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \sin \phi_{m-1}, \end{aligned}$$

and setting $x_1 = r$, $x_2 = x_3 = \ldots = x_m = 0$, we have

$$\psi(r, h) = (4\pi t)^{-\frac{1}{2}m} \int_0^\infty \int_{-\pi}^{\pi} \int_0^\pi \cdots \int_0^\pi \phi(\rho) e^{-(r^2 - 2r\rho \cos\phi_1 + \rho^2)/4t}$$
$$\rho^{m-1} \sin^{m-2}\phi_1 \ldots \sin\phi_{m-2}\, d\phi_1 \ldots d\phi_{m-1}\, d\rho.$$

From this we obtain

$$\psi(r, t) = (4\pi t)^{-\frac{1}{2}m} \frac{2\pi(\pi)^{\frac{1}{2}(m-3)}}{\Gamma(\frac{1}{2}m - \frac{1}{2})} \int_0^\infty \phi(\rho) e^{-(r^2+\rho^2)/4t} \rho^{m-1} d\rho \int_0^\pi e^{r\rho \cos\phi_1/2t} \sin^{m-2}\phi_1\, d\phi_1.$$

By Watson [2, p. 79] we have

$$\int_0^\pi e^{r\rho \cos\phi_1/2t} \sin^{m-2} \phi_1\, d\phi_1 = \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})\sqrt{\pi}}{(r\rho/4t)^{\frac{1}{2}m-1}} I_{\frac{1}{2}m-1}\left(\frac{r\rho}{2t}\right).$$

We finally get

2.4 $$\psi(r, t) = (2t)^{-1} \int_0^\infty e^{-(r^2+\rho^2)/4t} \rho^{\frac{1}{2}m} r^{-\frac{1}{2}m+1} I_{\frac{1}{2}m-1}\left(\frac{r\rho}{2t}\right) \phi(\rho)\, d\rho.$$

The function $u(X, h)$ is said to belong to S, $u(X, h) \in S$, at (X_0, h_0) if there exists an open set in $E_m \times H$ containing (X_0, h_0), throughout which the partial derivatives

$$\frac{\partial u}{\partial h}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (i, j = 1, \ldots, m)$$

are continuous and if

$$\frac{\partial u}{\partial h} = \Delta u \qquad \text{for } X = X_0, h = h_0.$$

We denote the set of points

$$|X| < R, \qquad 0 < h \leqslant c$$

by D_{Rc} and the set of points

$$|X| \leqslant R, \qquad t = 0,$$

$$|X| = R, \qquad 0 < t \leqslant c,$$

by B_{Rc}.

LEMMA 2a. *If*

1. $u(X, h) \in S$ *for* $(X, h) \in D_{Rc}$;
2. $\varliminf_{X \to X_0, h \to h_0} u(X, h) \geqslant 0$ *for* $(X_0, h_0) \in B_{Rc}$

then $u(X, h) \geqslant 0$ *in* D_{Rc}.

Suppose that there exists a point $(X_1, h_1) \in D_{Rc}$ at which we have $u(X_1, h_1) = -l < 0$. Form the function

$$v(X, h) = u(X, h) + k(h - h_1)$$

where $k > 0$ is chosen so small that

$$\varliminf_{X \to X_0, h \to h_0} v(X, h) \geqslant -\tfrac{1}{2}l \qquad (X_0, h_0) \in B_{Rc}.$$

For $\delta > 0$ suitably chosen $v(X, h) > -l$ within a distance δ of B_{Rc}. Consequently the minimum of $v(X, h)$ in D_{Rc} is attained at some point (X_2, h_2) of D_{Rc}. The function $v(X, h)$ satisfies the partial differential equation

2.5 $$\Delta v = \frac{\partial v}{\partial h} - k.$$

At (X_2, h_2) we must have

2.6 $$\Delta v \geqslant 0,$$

2.61 $$\frac{\partial v}{\partial h} \leqslant 0.$$

(If $h_2 \neq c$ the equality holds in 2.61.) Equations 2.5 and 2.6, 2.61 are in contradiction.

LEMMA 2b. *If*

1. $u(X, h) \in S$, $X \in \mathfrak{E}_m, 0 < h \leqslant c$;
2. $\lim_{X \to X_0,\ h \to 0+} u(X, h) = 0$, $X_0 \in \mathfrak{E}_m$;
3. $M(r) = \operatorname{l.u.b.}_{|X| = r,\ 0 < h \leqslant c} |u(X, h)|$;
4. $M(r) = O(e^{ar^2})$, $r \to +\infty$

for some $a > 0$; *then* $u(X, H) \equiv 0$ *for* $X \in \mathfrak{E}_m$, $0 < h \leqslant c$.

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Take $\lambda > 1$ and form the auxiliary function

$$U_R(X, h) = \lambda M(R)(\pi c)^{\frac{1}{2}} h^{-1} e^{-(|X|^2+R^2)/4h} R^{\frac{1}{2}m} |X|^{-\frac{1}{2}m+1} I_{\frac{1}{2}m-1}\left(\frac{R|X|}{2h}\right).$$

Using the asymptotic formula

2.7 $$I_\nu \sim e^x/\sqrt{2\pi x} \qquad (x \to +\infty),$$

we find that [2]

$$U_R(X, h)\big|_{|X|=R} \backsim \lambda(c/h)^{\frac{1}{2}} M(R), \qquad R \to \infty,$$

uniformly for $0 < h \leqslant c$. Thus when R is large

$$U_R(X, h) \geqslant M(R), \qquad |X| = R, 0 < h \leqslant c.$$

The function $U_R(X, h) - u(X, h)$ belongs to S for $X \in \mathfrak{E}_m$, $0 < h \leqslant c$. When R is large

$$\varliminf_{X \to X_0,\ h \to h_0} U_R(X, h) - u(X, h) \geqslant 0, \qquad (X_0, h_0) \in B_{Rc}.$$

By Lemma 2a we have

$$u(X, h) \leqslant U_R(X, h), \qquad (X, h) \in D_{Rc}.$$

Fix X and h and let $R \to +\infty$. Making use of 2.7 we find that $u(X, h) = 0$ for $0 < h < 1/4a$, $X \in \mathfrak{E}_m$. If $4a \leqslant 1/c$, our proof is complete. Otherwise repeat the above argument with $u(X, h)$ replaced by $u(X, h + 1/4a)$, and so forth.

$L(\mathfrak{E}_m)$ is the class of functions $\phi(X)$ defined for $X \in \mathfrak{E}_m$ and such that

$$\int_{\mathfrak{E}_m} |\phi(X)|\, dX$$

exists.

Exactly as in [3] we may establish

LEMMA 2c. *If*

1. $\phi(X)e^{-a|X|^2} \in L(\mathfrak{E}_m)$ *for some* $a > 0$,

2. $F(X, h) = \displaystyle\int_{\mathfrak{E}_m} k(X - Y, h)\phi(Y)\, dY,$

then $F(X, h)$ *is defined and belongs to* S *in the strip* $0 < h < 1/4a$.

LEMMA 2d. *Under the assumptions of Lemma* 2c, *we have*

$$\varliminf_{X \to X_0} \phi(X) \leqslant \varliminf_{X \to X_0, h \to 0+} F(X, h); \quad \varlimsup_{X \to X_0, h \to 0+} F(X, h) \leqslant \varlimsup_{X \to X_0} \phi(X).$$

LEMMA 2e. *If* $\phi(Y)$ *is integrable for* $|Y| \leqslant A$ *then for any* $c > 0$, *we have*

$$\lim_{|X| \to \infty} \left(\operatorname*{l.u.b.}_{0<h<c} \left| \int_{|Y| \leqslant A} k(X - Y, h)\phi(Y)\, dY \right| \right) = 0.$$

LEMMA 2f. *If*

1. $u(X, h) \in S, \qquad 0 < h < c, X \in \mathfrak{E}_m;$
2. $u(X, h) \geqslant 0, \qquad 0 < h < c, X \in \mathfrak{E}_m;$

then we have

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$$\int_{\mathfrak{E}_m} k(X - Y, \delta')u(Y, \delta)\, dY \leqslant u(X, \delta' + \delta),$$
$$0 < \delta, 0 < \delta', \delta + \delta' < c, X \in \mathfrak{E}_m.$$

Consider the function

$$v_A(X, h) = u(X, h + \delta) - \int_{|Y| \leqslant A} k(X - Y, h)u(Y, \delta)\, dY$$

where A is a positive constant. If $\delta' < c' < c - \delta$ then $v_A(X, h)$ belongs to S for $(X, h) \in D_{Rc'}$ for any R. Let $l > 0$ be given. Using Lemmas 2d and 2e we see that, if R is sufficiently large,

$$\varliminf_{X \to X_0, h \to h_0} v_A(X, h) \geqslant - l, \qquad (X_0, h_0) \in B_{R,c'}.$$

Lemma 2a implies that $v_A(X, h) \geqslant - l$ for $(X, h) \in D_{R,c'}$. Letting $l \to 0 +$ and $R \to + \infty$, we find that $v_A(X, h) \geqslant 0$ for $X \in \mathfrak{E}_m$, $0 < h < c'$. Setting $h = \delta'$ we have

$$\int_{|Y| \leqslant A} k(X - Y, \delta')u(Y, \delta)\, dY \leqslant u(X, \delta' + \delta).$$

Letting A increase without limit, we obtain our desired result.

Lemma 2g. *If*

1. $u(X, h) \in S, \qquad 0 \leqslant h < c, X \in \mathfrak{E}_m;$
2. $u(X, h) \geqslant 0, \qquad 0 \leqslant h < c, X \in \mathfrak{E}_m;$
3. $u(X, 0) = 0, \qquad X \in \mathfrak{E}_m;$

then $u(X, h) = 0 \qquad 0 \leqslant h < c, X \in \mathfrak{E}_m.$

We set

$$w(X, h) = \int_0^h u(X, t)\, dt.$$

It is easily verified that $w(X, h) \in S$ for $0 \leqslant h < c$, $X \in \mathfrak{E}_m$. We have $w(X, h_1) \geqslant w(X, h_2)$ if $c > h_1 \geqslant h_2 \geqslant 0$. It follows that

$$\frac{\partial}{\partial h} w(X, h) \geqslant 0$$

and hence that $\Delta w(X, h) \geqslant 0$ for $0 \leqslant h < c$; equivalently $w(X, h)$ is a subharmonic function of X for each value of h, $0 \leqslant h < c$.

Let δ be an arbitrary number $0 < \delta < c$ and let δ' be such that $0 < \delta' < c - \delta$. By Lemma 2f we have

$$w(0, \delta' + \delta) \geqslant (4\pi\delta)^{-\frac{1}{2}m} \int_{\mathfrak{E}_m} e^{-|Y|^2/4\delta}\, w(Y, \delta')\, dY.$$

Again we have

$$\int_{\mathfrak{E}_m} e^{-|Y|^2/4\delta}\, w(Y, \delta')\, dY \geqslant M(X) \int_{|Y-X| \leqslant 1} w(Y, \delta')\, dY$$

where

$$M(X) = \underset{|Y-X|\leqslant 1}{\text{g.l.b.}} [e^{-|Y|^2/4\delta}].$$

Since $w(Y, \delta')$ is subharmonic

$$\int_{|Y-X|\leqslant 1} w(Y, \delta')\, dY \geqslant \pi w(X, \delta').$$

It follows that

$$w(X, \delta') = O(\exp a|X|^2)$$

for $a > 1/4\delta$. Since

$$\frac{\partial}{\partial h} w\,(X, h) \geqslant 0$$

we find that

$$\underset{0\leqslant h\leqslant \delta', |X|=R}{\text{l.u.b.}} |w(X, \delta)| = O(e^{aR^2}).$$

Applying Lemma 2b we see that $w(X, h) = 0$ for $0 \leqslant h \leqslant \delta'$, $X \in \mathfrak{E}_m$. Since $\delta' < c$ is arbitrary we have $w(X, h) = 0$ for $0 \leqslant h < c$, $X \in \mathfrak{E}_m$. This in turn implies that $u(X, h) = 0$ for $X \in \mathfrak{E}_m$, $0 \leqslant h < c$.

THEOREM 2h. *If*

1. $u(X, h) \in S$ *for* $X \in \mathfrak{E}_m$, $a < h < b$;
2. $u(X, h) \geqslant 0$ *for* $X \in \mathfrak{E}_m$, $a < h < b$;
3. $a < h'$, $h'' < b$, $h'' > h'$;

then

$$u(X, h'') = \int_{\mathfrak{E}_m} k(X - Y, h'' - h')u(Y, h')\, dY.$$

By Lemma 2f we have

$$\int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h') \leqslant u(X, h)$$

for $h' < h < b$. This together with Lemma 2c implies that

$$\int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h')\, dY \in S$$

for $h' < h < b$, $X \in \mathfrak{E}_m$. Thus

$$v(X, h) = u(X, h) - \int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h')\, dY$$

belongs to S and is non-negative for $h' < h < b$, $X \in \mathfrak{E}_m$. Moreover, by Lemma 2d,

$$\lim_{h\to h',\ Y\to X_0} v(X, h) = 0.$$

Lemma 2g implies that $v(X, h) \equiv 0$. Setting $h = h''$, we obtain the desired result.

3. The main theorem. Let $T \in \mathfrak{T}_{\frac{1}{2}m(m+1)}$ be positive definite and let $X \in \mathfrak{E}_m$. We set

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3.1 $$k(X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp\,[-\,Y^*TY - iY^*X]\,dY.$$

Since T is positive definite, there exists a constant $\epsilon > 0$ such that $Y^*TY \geqslant \epsilon|Y|^2$. This insures the validity of our definition. Let $\Phi = [\phi_{ij}]_{i,j=1,\ldots,m}$ be a real unimodular matrix. We assert that

3.2 $$k(\Phi^{-1}X, T) = k(X, \Phi T\Phi^*).$$

We have

$$k(\Phi^{-1}X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp\,[-\,Y^*TY - iY^*\Phi^{-1}X]\,dY.$$

Making the change of variable $Y = \Phi^*Z$ we obtain

$$k(\Phi^{-1}X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp\,[-\,Z^*\Phi T\Phi^*Z - iZX]\,dZ = k(X, \Phi T\Phi^*).$$

This formula may be used to compute $k(X, T)$ explicitly, see [**1**, p. 185]. By this method, Czüber has shown that

3.3 $$k(X, T) = \frac{\exp\left(\begin{vmatrix} 0 & X^* \\ X & T \end{vmatrix} \Big/ 4|T|\right)}{(4\pi)^{\frac{1}{2}m}|T|^{\frac{1}{2}}}.$$

In particular if T is a diagonal matrix with equal entries,

$$T = \begin{pmatrix} \tau & 0 & \ldots & 0 \\ 0 & \tau & \ldots & 0 \\ \cdot & \cdot & \cdot\;\cdot & \cdot \\ 0 & 0 & \ldots & \tau \end{pmatrix},$$

then $k(X, T)$ is equal to $k(X, \tau)$.

Consider the system of partial differential equations

3.4 $$\frac{\partial u}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i\,\partial x_j} a_{ij}^l \qquad (a_{ij}^l = a_{ji}^l;\ l = 1, \ldots, n),$$

where the a's are real constants. The function $u(X, H)$ is said to belong to $S(a)$ at (X_0, H_0) if there exists an open set in $E_m \times H_n$ containing (X_0, H_0) throughout which the partial derivatives

$$\frac{\partial u}{\partial h_l}\,(l = 1, \ldots, n), \qquad \frac{\partial^2 u}{\partial x_i\,\partial x_j}\,(i, j = 1, \ldots, m)$$

are continuous and if equation 3.4 holds for $X = X_0$, $H = H_0$.

LEMMA 3a. *If:*

1. $u(X, H) \in S(a)$, *where* $\mathfrak{N}$ *is an open subset of* $\mathfrak{H}_n$, $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$;
2. $\Phi = [\phi_{ij}]_{i,j=1,\ldots,m}$ *is a real unimodular matrix;*
3. $w(X, h) = u(\Phi^{-1}X, h)$;

then

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3.5 $$\frac{\partial w}{\partial h_l} = \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 w}{\partial x_\alpha \, \partial x_\beta} b^l_{\alpha,\beta}$$

for $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$, *where*

$$b^l_{\alpha,\beta} = \sum_{i,j=1}^{m} a^l_{ij} \phi_{\alpha i} \phi_{\beta j}.$$

(Briefly we have $w(X, H) \in S(b)$ for $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$.)

We have $w(X, H) = u(Y, H)$ where $X = \Phi Y$. Now

$$\frac{\partial u}{\partial h_l} = \frac{\partial w}{\partial h_l}, \quad \frac{\partial^2 u}{\partial y_i \, \partial y_j} = \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 w}{\partial x_\alpha \, \partial x_\beta} \phi_{\alpha i} \phi_{\beta j}.$$

Substituting in 3.4 we obtain 3.5.

If $H \in \mathfrak{H}_n$, then $H \to H_b$ is a mapping of $\mathfrak{H}_n$ into $\mathfrak{T}_{\frac{1}{2}m(m+1)}$, the (i, j) entry of H_b being

$$\sum_{l=1}^{n} h_l \, b^l_{i,j}.$$

It is easily verified that

3.6 $$\Phi H_a \Phi^* = H_b.$$

Lemma 3b. *If*:

1. $u(X, H)$ *is continuous for* $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$ *where* $\mathfrak{N}$ *is an open subset of* $\mathfrak{H}_n$,
2. $H', H'' \in \mathfrak{N}$, $H''_a > H'_a$,
3. $u(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_a - H'_a) u(Y, H') \, dY$,
4. $w(X, H) = u(\Phi^{-1}X, H)$,

then $w(X, H)$ *is defined and continuous for* $X \in E_m$, $H \in N$, $H''_b - H'_b > 0$, *and*

$$w(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_b - H'_b) w(Y, H_,) \, dY.$$

In 3, replace X by $\Phi^{-1}X$ and make the change of variable $Y = \Phi^{-1}Z$ to obtain

$$u(\Phi^{-1}X, H'') = \int_{\mathfrak{E}_m} k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) u(\Phi^{-1}Z, H') \, dZ.$$

$$w(X, H'') = \int_{\mathfrak{E}_m} k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) w(Z, H') \, dZ.$$

By 3.2 and 3.6 we have

$$\begin{aligned} k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) &= k(Z - X, \Phi H''_a \Phi^* - \Phi H'_a \Phi^*), \\ &= k(Z - X, H''_b - H'_b). \end{aligned}$$

Our lemma follows.

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THEOREM 3c. *If*:

1. $u(X, H) \in S(a)$, $X \in E_m$, $H \in \mathfrak{R}$, *where* $\mathfrak{R}$ *is an open convex subset of* $\mathfrak{H}_n$,
2. $u(X, H) \geqslant 0$, $\qquad X \in \mathfrak{E}_m, H \in \mathfrak{R}$
3. $H', H'' \in \mathfrak{R}$, $\qquad H''_a > H'_a$,

then

$$u(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_a - H'_a)u(Y, H')\, dY.$$

Let Φ be a real unimodular $m \times m$ matrix. Because of Lemmas 3a and 3b, it is sufficient to establish the corresponding relation for $w(X, H) = u(\Phi^{-1}X, H)$,

$$w(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_b - H'_b)w(Y, H')\, dY.$$

By 3.6 we have

$$H''_b - H'_b = \Phi(H''_a - H_a)\Phi^*.$$

It is possible to choose Φ so that $H''_b - H'_b$ is a diagonal matrix with equal entries, i.e.,

$$H''_b - H'_b = \begin{pmatrix} \tau & 0 & \dots & 0 \\ 0 & \tau & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \tau \end{pmatrix}.$$

Having chosen Φ in this manner, we consider

$$w\left[X, \left(1 - \frac{t}{\tau}\right)H' + \frac{t}{\tau}H''\right] = v\ (X, t).$$

Since $\mathfrak{R}$ is convex and open, $v(X, t)$ is defined for $a < t < b$ where $a < 0$, $b > \tau$. We have

$$\frac{\partial v}{\partial t} = \sum_{l=1}^{n} \frac{\partial w}{\partial h_l}\left[\frac{1}{\tau}(h''_l - h'_l)\right].$$

Since

$$\frac{\partial w}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 w}{\partial x_i\, \partial x_j} b^l_{ij} = \sum_{i,j=1}^{m} \frac{\partial^2 v}{\partial x_i\, \partial x_j} b^l_{ij}$$

we find that

$$\frac{\partial v}{\partial t} = \frac{1}{\tau} \sum_{i,j=1}^{m} \left[\sum_{l=1}^{n} b^l_{ij}\left(h''_l - h'_l\right)\right] \frac{\partial^2 v}{\partial x_i\, \partial x_j},$$

$$\frac{\partial v}{\partial t} = \Delta v.$$

Thus $v(X, t) \in S$ for $X \in \mathfrak{E}_m$, $a < t < b$ where $a < 0$, $b > \tau$. Applying Theorem 2h we have

$$v(X, \tau) = \int_{\mathfrak{E}_m} k(X - Y, \tau)v(Y, 0)\, dY.$$

Now

$$v(X, \tau) = w(X, H''), \quad v(X, O) = w(X, H').$$

Thus

$$w(X, H'') = \int_{\mathfrak{E}_m} k(X - Y, \tau) w(Y, H')\, dY,$$

$$w(X, H'') = \int_{\mathfrak{E}_m} k(X - Y, H''_b - H'_b) w(Y, H')\, dY,$$

a relation which we have seen to be quivalent to our theorem.

There are some systems of equations 3.4 for which Theorem 3c gives no information. This is because there do not exist real vectors H' and H'' such that $H''_a > H'_a$. The system consisting of the single equation

$$\frac{\partial u}{\partial h} = \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}$$

is of this type.

Making use of Theorem 3c and the concept of weak compactness, we may demonstrate the following result.

THEOREM 3d. *Let $\mathfrak{N}$ be an open subset of $\mathfrak{H}_n$ such that $H', H'' \in \mathfrak{N}$ imply* $\lambda''H'' + \lambda'H' \in \mathfrak{N}$ for $0 < \lambda'$, $0 < \lambda''$, $\lambda' + \lambda'' \leqslant 1$, *and such that $H \in \mathfrak{N}$ implies that H_a is positive definite. If:*

1. $u(X, H) \in S\,(a)$, $\qquad X \in \mathfrak{E}_m, H \in \mathfrak{N}$
2. $u(X, H) \geqslant 0$, $\qquad X \in \mathfrak{E}_m, H \in \mathfrak{N}$

then

$$u(X, H) = \int_{\mathfrak{E}_m} k(X - Y, H_a)\, dm(Y) \qquad X \in \mathfrak{E}_m, H \in \mathfrak{N},$$

where $m(Y)$ is a non-negative measure defined in the σ-field of Borel sets of E_m.

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