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LETTER TO THE EDITOR

Dear Editor,

Three-handed gambler's ruin

Three gamblers play a fair game as follows: at each step one player is selected at random to give up a chip (i.e. one unit of wealth), and one of the other two players is selected at random to receive it. After some time one of the three players is the first to lose all of his chips. Here we use martingales to find an expression for the variance of this time very simply. This expression was first obtained by Bruss *et al.* [1], using a different method.

1. Preliminaries

The three gamblers initially possess integer fortunes of sizes a, b, and c, respectively. They play a sequence of fair games such that in each independent round one player is selected uniformly at random to give up a chip, and another is selected uniformly at random to receive it. Let the respective fortunes of the players after n rounds be X_n, Y_n, and Z_n (of course, $(X_0, Y_0, Z_0) = (a, b, c)$).

Denote by S the random number of rounds played until the first instance at which any one of the players has no chips. A natural question is then to ask what are the distribution, mean, and variance of S. The fact that

$$\mathrm{E}(S) = \frac{3abc}{a+b+c} \tag{1}$$

was first established by Engel [2], who conjectured the form of the solution from computer simulations and then verified that it satisfied the appropriate difference equation and boundary condition. Later Bruss *et al.* [1] found the distribution of S and used it to show that

$$\operatorname{var}(S) = \frac{3abc}{a+b+c}\left[\frac{ab+bc+ca-1}{2} - \frac{3abc}{a+b+c}\right]. \tag{2}$$

They did this by evaluating the sum $\mathrm{E}(S^2) = \sum_{r>0} \mathrm{P}(S=r)r^2$ using an ingenious application of Cauchy's theorem for functions of a complex variable. Here we shall use martingales to obtain (2) much more easily, by means of the optional stopping theorem.

2. The martingales

First recall the familiar fact that $M_n = X_nY_nZ_n + \frac{1}{3}n(a+b+c)$ is a martingale with respect to (X_n, Y_n, Z_n). To see this we use the fact that rounds are independent and simply evaluate

$$\begin{aligned}
&\mathrm{E}(M_{n+1} \mid X_n = x,\ Y_n = y,\ Z_n = z) \\
&\quad = \tfrac{1}{6}\{(x+1)(y-1)z + (x-1)(y+1)z + (x+1)y(z-1) + (x-1)y(z+1) \\
&\qquad\quad + x(y+1)(z-1) + x(y-1)(z+1)\} + \tfrac{1}{3}(n+1)(a+b+c) \\
&\quad = \tfrac{1}{6}\{6xyz - 2(x+y+z)\} + \tfrac{1}{3}n(a+b+c) + \tfrac{1}{3}(a+b+c) \\
&\quad = M_n,
\end{aligned}$$

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where the final equality follows from $x+y+z = X_n+Y_n+Z_n = a+b+c$. This martingale is uniformly bounded and S is a stopping time; therefore, by the optional stopping theorem,

$$abc = \mathrm{E}(M_0) = \mathrm{E}(M_S) = 0 + \tfrac{1}{3}\,\mathrm{E}(S)(a+b+c),$$

which is (1).

Next we note that

$$V_n = X_n^2Y_n^2Z_n + X_n^2Y_nZ_n^2 + X_nY_n^2Z_n^2 + 4nX_nY_nZ_n + \tfrac{2}{3}tn^2 + \tfrac{1}{3}tn,$$

where $t = a+b+c$, is also a martingale with respect to (X_n, Y_n, Z_n). This is verified in the usual way: use the fact that rounds are independent and evaluate

$$\begin{aligned}
&\mathrm{E}(V_{n+1} \mid X_n = x,\ Y_n = y,\ Z_n = z)\\
&\quad= \tfrac{1}{6}\{(x+1)^2(y-1)^2z + (x-1)^2(y+1)^2z + (x+1)^2y^2(z-1)\\
&\qquad\quad + (x-1)^2y^2(z+1) + x^2(y-1)^2(z+1) + x^2(y+1)^2(z-1)\\
&\qquad\quad + (x+1)^2(y-1)z^2 + (x-1)^2(y+1)z^2 + (x+1)^2y(z-1)^2\\
&\qquad\quad + (x-1)^2y(z+1)^2 + x^2(y+1)(z-1)^2 + x^2(y-1)(z+1)^2\\
&\qquad\quad + (x+1)(y-1)^2z^2 + (x-1)(y+1)^2z^2 + (x-1)y^2(z+1)^2\\
&\qquad\quad + (x+1)y^2(z-1)^2 + x(y+1)^2(z-1)^2 + x(y-1)^2(z+1)^2\}\\
&\qquad + 4(n+1)\tfrac{1}{6}\{(x+1)(y-1)z + (x-1)(y+1)z + x(y+1)(z-1)\\
&\qquad\qquad\qquad + x(y-1)(z+1) + (x+1)y(z-1) + (x-1)y(z+1)\}\\
&\qquad + \tfrac{2}{3}t(n+1)^2 + \tfrac{1}{3}t(n+1)\\
&\quad= \tfrac{1}{6}\{6x^2y^2z + 4zx^2 + 4zy^2 - 4yx^2 - 4xy^2 - 8xyz + 2z\\
&\qquad\quad + 6x^2yz^2 + 4yx^2 + 4z^2y - 4x^2z - 4xz^2 - 8xyz + 2y\\
&\qquad\quad + 6xy^2z^2 + 4xy^2 + 4z^2x - 4y^2z - 4yz^2 - 8xyz + 2x\}\\
&\qquad + \tfrac{2}{3}(n+1)\{6xyz - 2(x+y+z)\} + \tfrac{2}{3}t(n+1)^2 + \tfrac{1}{3}t(n+1)\\
&\quad= x^2y^2z + x^2yz^2 + xy^2z^2 + 4nxyz + \tfrac{1}{3}(a+b+c)\\
&\qquad - \tfrac{4}{3}n(a+b+c) - \tfrac{4}{3}(a+b+c) + \tfrac{2}{3}tn^2 + \tfrac{4}{3}tn + \tfrac{2}{3}t + \tfrac{1}{3}nt + \tfrac{1}{3}t\\
&\quad= V_n.
\end{aligned}$$

This martingale is not uniformly bounded, but it is easy to see that $\mathrm{E}(S^2) < \infty$. Therefore, applying the appropriate form of the optional stopping theorem yields

$$\begin{aligned}
abc(ab+bc+ca) = \mathrm{E}(V_0) &= \mathrm{E}(V_S)\\
&= \tfrac{2}{3}t\,\mathrm{E}(S^2) + \tfrac{1}{3}t\,\mathrm{E}(S)\\
&= \tfrac{2}{3}t\,\mathrm{E}(S^2) + abc, \quad \text{by (1).}
\end{aligned} \tag{3}$$

Hence,

$$\mathrm{E}(S^2) = \frac{3}{2t}(ab+bc+ca-1),$$

which immediately gives (2).

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As an alternative to using the optional stopping theorem, we may proceed directly by noting that $V_{n\wedge S}$, where $n \wedge S$ is the smaller of n and S, is a martingale. Then $\mathrm{E}(V_0) = \mathrm{E}(V_{n\wedge S})$. Now, as $n \to \infty$,

$$V_{n\wedge S} \to \tfrac{2}{3}tS^2 + \tfrac{1}{3}tS \quad \text{almost surely.}$$

It is easy to see that $\mathrm{E}(V_{n\wedge S})$ is uniformly bounded; hence, by dominated convergence we obtain (3), as before.

References

[1] Bruss, F. T., Louchard, G. and Turner, J. W. (2003). On the N-tower problem and related problems. *Adv. Appl. Prob.* **35,** 278–294.

[2] Engel, A. (1993). The computer solves the three tower problem. *Amer. Math. Monthly* **100,** 62–64.

Yours sincerely,

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