

A CHARACTERISATION OF RIEMANNIAN FOLIATIONS AND TOTALLY UMBILICAL SUBMANIFOLDS

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We discuss characterisations of Riemannian foliations, totally geodesic submanifolds, and totally umbilical submanifolds by sharp inequalities. These derive from the same linear algebraic set up, characterising a linear endomorphism which is a multiple of the identity.

1. GENERAL SET UP

In the first section we derive two general theorems which will lead to the applications mentioned in the abstract.

Let V be a finite-dimensional real vector space and g an inner product on it. Further, let $V = V_1 \oplus V_2$ be an orthogonal decomposition of V with $\dim V_1 = m$ and $\dim V_2 = n$. The induced product on V_1 will be denoted by g_1 and that on V_2 by g_2 .

Next, let γ be a *symmetric* bilinear form on V_1 with values in V_2 . Then

$$f : u \mapsto g_2(\gamma(u, u), \gamma(u, u)),$$

where u is a unit vector of V_1 , is a function on the unit sphere $S^{m-1}(1)$ in V_1 . Here we identify V_1 with \mathbb{R}^m via an orthonormal basis $\{e_a, a = 1, \dots, m\}$. Further, we put

$$x = ru = \sum_{a=1}^m x_a e_a.$$

Then we define $F : V_1 \rightarrow \mathbb{R}; x \mapsto F(x)$ by

$$(1) \quad F(x) = r^4 f(u) = \sum_{a,b,c,d=1}^m x_a x_b x_c x_d g_2(\gamma_{ab}, \gamma_{cd})$$

where $\gamma_{ab} = \gamma(e_a, e_b)$.

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Our aim is to compute $\widehat{\Delta}f$ on $S^{m-1}(1)$ where $\widehat{\Delta}$ denotes the Laplace operator on this unit sphere. Therefore, let Δ denote the Laplace operator on \mathbb{R}^m . We shall first compute ΔF and then use the well-known relation (see for example [1])

$$(2) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{m-1}{r} \frac{\partial}{\partial r} + \widehat{\Delta}_r$$

where $\widehat{\Delta}_r$ denotes the Laplacian on $S^{m-1}(r)$. Note that

$$\Delta F = \sum_{a=1}^m \frac{\partial^2 F}{\partial x_a^2}.$$

It follows easily that

$$\begin{aligned} \frac{\partial F}{\partial x_a} &= 4 \sum_{b,c,d} x_b x_c x_d g_2(\gamma_{ab}, \gamma_{cd}), \\ \frac{\partial^2 F}{\partial x_a^2} &= 4 \sum_{c,d} x_c x_d \{g_2(\gamma_{aa}, \gamma_{cd}) + 2g_2(\gamma_{ac}, \gamma_{ad})\}. \end{aligned}$$

Hence, we obtain

$$(3) \quad (\Delta F)(x) = 4r^2 \sum_{a=1}^m \{g_2(\gamma(e_a, e_a), \gamma(u, u)) + 2g_2(\gamma(e_a, u), \gamma(e_a, u))\}.$$

Next, put
$$\gamma(u, v) = \sum_{i=1}^n g_1(\Gamma_i u, v) e_i$$

for $u, v \in V_1$, in terms of an orthonormal basis $\{e_i, i = 1, \dots, n\}$ of V_2 and the symmetric endomorphisms $\Gamma_i : V_1 \rightarrow V_1, i = 1, \dots, n$. We also define a symmetric non-negative operator $\Gamma^2 : V_1 \rightarrow V_1$ by

$$\Gamma^2 u = \sum_{i=1}^n \Gamma_i^2 u$$

for $u \in V_1$. Then (3) can be written in the form

$$(4) \quad (\Delta F)(x) = 4r^2 \sum_{a=1}^m \{g_2(H_\gamma, \gamma(u, u)) + 2g_1(\Gamma^2 u, u)\}$$

where
$$H_\gamma = \sum_{a=1}^m \gamma(e_a, e_a).$$

Next, using (1) and (2) we get, by putting $r = 1$,

$$(5) \quad (\widehat{\Delta}F)(u) = 4\{g_2(H_\gamma, \gamma(u, u)) + 2g_1(\Gamma^2 u, u) - (m+2)g_2(\gamma(u, u), \gamma(u, u))\}$$

and finally, integration over $S^{m-1}(1)$ yields

$$(6) \quad \int_{S^{m-1}(1)} \{g_2(H_\gamma, \gamma(u, u)) + 2g_1(\Gamma^2 u, u) - (m + 2)g_2(\gamma(u, u), \gamma(u, u))\} d\mu = 0$$

where $d\mu$ is the canonical measure on $S^{m-1}(1)$.

The first general result follows now at once from (6).

THEOREM 1. $\gamma = 0$, that is, V_1 is totally geodesic in V , if and only if

$$(7) \quad g_2(H_\gamma, \gamma(u, u)) \geq (m + 2)g_2(\gamma(u, u), \gamma(u, u))$$

on $S^{m-1}(1)$.

To obtain the second result, we rewrite (6) in the following form :

$$(8) \quad \int_{S^{m-1}(1)} \{g_2(H_\gamma, \gamma(u, u)) - g_1(\Gamma^2 u, u) - (m - 1)g_2(\gamma(u, u), \gamma(u, u))\} d\mu \\ + 3 \int_{S^{m-1}(1)} \{g_1(\Gamma^2 u, u) - g_2(\gamma(u, u), \gamma(u, u))\} d\mu = 0.$$

Since
$$g_2(\gamma(u, u), \gamma(u, u)) = \sum_{i=1}^n g_1(\Gamma_i u, u)^2,$$

the Schwarz inequality implies

$$(9) \quad g_1(\Gamma^2 u, u) \geq g_2(\gamma(u, u), \gamma(u, u)).$$

Then we get

THEOREM 2. V_1 is totally umbilical in V , that is, $\Gamma_i = \lambda_i I$ for $i = 1, \dots, n$, if and only if

$$(10) \quad g_2(H_\gamma, \gamma(u, u)) - g_1(\Gamma^2 u, u) \geq (m - 1)g_2(\gamma(u, u), \gamma(u, u))$$

on $S^{m-1}(1)$.

PROOF: First, suppose (10) is satisfied. Then we have by (8) and (9)

$$\sum_{i=1}^n \{g_1(\Gamma_i u, \Gamma_i u) - g_1(\Gamma_i u, u)^2\} = 0$$

and hence, by the Schwarz inequality,

$$g_1(\Gamma_i u, \Gamma_i u) = g_1(\Gamma_i u, u)^2$$

for $i = 1, \dots, n$. Thus, $\Gamma_i u$ must be proportional to u for all u , which proves the result.

Conversely, let $\Gamma_i = \lambda_i I$, $i = 1, \dots, n$. Then we have

$$\gamma(u, u) = \sum_{i=1}^n \lambda_i e_i, \quad H_\gamma = m \sum_{i=1}^n \lambda_i e_i, \quad \Gamma^2 = \left(\sum_{i=1}^n \lambda_i^2 \right) I$$

and the equality holds in (10). □

REMARK. (6) may also be checked directly by using the integration formulas (see for example [3, 6]). Then we have

$$\int_{S^{m-1}(1)} g_2(H_\gamma, \gamma(u, u))d\mu = \frac{c_{m-1}}{m} g_2(H_\gamma, H_\gamma) = \frac{c_{m-1}}{m} |H_\gamma|^2,$$

$$\int_{S^{m-1}(1)} g_1(\Gamma^2 u, u)d\mu = \frac{c_{m-1}}{m} \text{trace } \Gamma^2,$$

$$\int_{S^{m-1}(1)} g_2(\gamma(u, u), \gamma(u, u))d\mu = \frac{c_{m-1}}{m(m+2)} \{|H|^2 + 2 \text{trace } \Gamma^2\},$$

where c_{m-1} denotes the volume of $S^{m-1}(1)$.

In the rest of this note we shall consider applications of these two theorems by specialising the symmetric bilinear form γ . The first one treats foliations and the second one submanifolds of Riemannian manifolds.

2. APPLICATIONS TO FOLIATIONS

Let \mathfrak{F} be a foliation on a Riemannian manifold (M, g) . We refer to [8, 9] for more detailed information. We denote vector fields tangent to \mathfrak{F} by U, V, \dots and vector fields orthogonal (transversal) to \mathfrak{F} by X, Y, \dots . The metric restricted to tangent vectors is denoted by g_L , and the metric restricted to normal vectors is denoted by g_Q . For the Lie derivative $\theta(U)g_Q$ we have then the following formula [9, (6.9)]:

LEMMA 3. $(\theta(U)g_Q)(X, Y) = -2g_L(\alpha^\perp(X, Y), U)$, where α^\perp is the second fundamental form of the normal bundle L^\perp of \mathfrak{F} .

In terms of the Levi Civita connection ∇^M of g and the tangential projection $\pi^L : TM \rightarrow L$, the symmetric bilinear form α^L is given by

$$\alpha^L(X, Y) = \frac{1}{2}\pi^L(\nabla_X^M Y + \nabla_Y^M X),$$

[8, 9, (6.8)].

Using α^\perp for γ in Theorem 1 we get the following characterisations of Riemannian foliations, given by $\alpha^\perp = 0$ according to [9, Corollary 6.10].

THEOREM 4. Let \mathfrak{F} be a foliation of codimension q on a Riemannian manifold (M, g) . Then \mathfrak{F} is Riemannian (that is, g is bundle-like) if and only if

$$g_L(\tau^\perp, \alpha^\perp(x, x)) \geq (q + 2)g_L(\alpha^\perp(x, x), \alpha^\perp(x, x))$$

for all unit normal vectors x of \mathfrak{F} .

Note that τ^\perp is the mean curvature vector field of the normal bundle given by

$$\tau^\perp = \sum_{a=1}^q \alpha^\perp(e_a, e_a)$$

for an orthonormal frame $\{e_a, a = 1, \dots, q\}$ of transversal vectors [9, (6.27)].

Next, we consider the class of foliations with a holonomy invariant transversal form ν . According to [9, (6.32)], these are characterised by $\tau^\perp = 0$. Assuming this condition, we rewrite (6) in the form

$$\int_{S^{q-1}(1)} \{(2 - \beta)g_Q((A^\perp)^2 x, x) - (q + 2 - \beta)g_L(\alpha^\perp(x, x), \alpha^\perp(x, x))\}d\mu + \beta \int_{S^{q-1}(1)} \{g_Q((A^\perp)^2 x, x) - g_L(\alpha^\perp(x, x), \alpha^\perp(x, x))\}d\mu = 0$$

at each point of M and for each unit normal vector x to \mathfrak{F} at this point. Here $(A^\perp)^2 : L^\perp \rightarrow L^\perp$ is the endomorphism Γ^2 corresponding to the bilinear form α^\perp . Then Theorem 2 yields the following result.

THEOREM 5. *Let \mathfrak{F} be a foliation with holonomy invariant transversal volume on the Riemannian manifold (M, g) . Then \mathfrak{F} is Riemannian if and only if there exists a number β such that*

$$(2 - \beta)g_Q((A^\perp)^2 x, x) \geq (q + 2 - \beta)g_L(\alpha^\perp(x, x), \alpha^\perp(x, x)) \quad \text{for } \beta > 0,$$

or

$$(2 - \beta)g_Q((A^\perp)^2 x, x) \leq (q + 2 - \beta)g_L(\alpha^\perp(x, x), \alpha^\perp(x, x)) \quad \text{for } \beta < 0$$

for each unit normal vector x to \mathfrak{F} .

3. APPLICATIONS TO SUBMANIFOLDS

For a second series of applications we consider submanifolds of a Riemannian manifold. For definitions and more detailed information we refer to [2], for example.

Let (M, g) be a Riemannian manifold of class C^∞ and suppose $\dim M = m$. Further, let $\tilde{\nabla}$ denote its Levi Civita connection and \tilde{R} its Riemann curvature tensor defined by

$$\tilde{R}_{XY} = \tilde{\nabla}_{[X, Y]} - [\tilde{\nabla}_X, \tilde{\nabla}_Y]$$

for tangent vector fields X, Y . Next, let (N, g) denote an n -dimensional submanifold of M where g also denotes the induced metric. Its Riemann curvature tensor will be denoted by R and the associated Ricci tensor by ρ .

Now, let $p \in N$ and let $\{e_A, A = 1, \dots, m\}$ be an orthonormal basis of $T_p M$ such that $e_i, i = 1, \dots, n$ are tangent to N and e_{n+1}, \dots, e_m normal to N at p . Further, let σ denote the second fundamental form of N and put, at p ,

$$g(\sigma(x, y), e_a) = -g(S_a x, y), \quad a = n + 1, \dots, m$$

for $x, y \in T_p N$. Here S_a denotes the shape operator corresponding to e_a . For $u \in T_p N$, the Gauss equation yields

$$\begin{aligned}
 (11) \quad \rho(u, u) &= \sum_{i=1}^n \tilde{R}_{ue_i ue_i} + g(H, \sigma(u, u)) - \sum_{a=n+1}^m g(S_a u, S_a u) \\
 &= \tilde{\rho}(u, u) - \sum_{a=n+1}^m \tilde{R}_{ue_a ue_a} + g(H, \sigma(u, u)) - \sum_{a=n+1}^m g(S_a u, S_a u)
 \end{aligned}$$

where $\tilde{\rho}$ is the Ricci tensor of (M, g) and H the mean curvature vector field of N in M .

For an arbitrary unit vector $u \in T_p N$, (6) becomes, using $\gamma = \sigma$ and (11),

$$(12) \quad \int_{S^{n-1}(1)} \left\{ \rho(u, u) - \sum_{i=1}^n \tilde{R}_{ue_i ue_i} - (n+2)g(\sigma(u, u), \sigma(u, u)) \right\} d\mu + 3 \int_{S^{n-1}(1)} g(S^2 u, u) d\mu = 0$$

and (8) may be written as

$$\begin{aligned}
 (13) \quad &\int_{S^{n-1}(1)} \left\{ \rho(u, u) - \sum_{i=1}^n \tilde{R}_{ue_i ue_i} - (n-1)g(\sigma(u, u), \sigma(u, u)) \right\} d\mu \\
 &+ 3 \int_{S^{n-1}(1)} \left\{ g(S^2 u, u) - g(\sigma(u, u), \sigma(u, u)) \right\} d\mu = 0
 \end{aligned}$$

where S^2 corresponds to Γ^2 . The relations (12), (13) yield at once

THEOREM 6. *N is a totally geodesic submanifold if and only if*

$$\rho(u, u) \geq (n+2)g(\sigma(u, u), \sigma(u, u)) + \sum_{i=1}^n \tilde{R}_{ue_i ue_i}$$

for all unit tangent vectors u .

THEOREM 7. *N is a totally umbilical submanifold if and only if*

$$\rho(u, u) \geq (n-1)g(\sigma(u, u), \sigma(u, u)) + \sum_{i=1}^n \tilde{R}_{ue_i ue_i}$$

for all unit tangent vectors u .

As a special case we then get

COROLLARY 8. *Let (M, g) be a space of constant curvature c . Then N is a totally geodesic submanifold if and only if*

$$\rho(u, u) \geq (n-1)c + (n+2)g(\sigma(u, u), \sigma(u, u))$$

for all unit tangent vectors u .

COROLLARY 9. *Let (M, g) be a space of constant curvature c . Then N is a totally umbilical submanifold if and only if*

$$\rho(u, u) \geq (n - 1)\{c + g(\sigma(u, u), \sigma(u, u))\}$$

for all unit tangent vectors u .

REMARK. For $c = 0$ and $n = m - 1$ we obtain the result provided in [4] and used to obtain a characterisation for a compact and connected hypersurface in \mathbb{R}^{n+1} to be a sphere in terms of lower bounds on the Ricci curvature.

Finally, we consider some special cases in almost Hermitian theory. For more details about submanifolds in this context we refer to [7]. Let (M, g, J) be a Kähler manifold and let N be a holomorphic submanifold of M . Then N is minimal and we get

THEOREM 10. *Let N be a holomorphic submanifold of a Kähler manifold (M, g, J) . Then N is totally geodesic if and only if*

$$\rho(u, u) \geq (n - 1)g(\sigma(u, u), \sigma(u, u)) + \sum_{i=1}^n \tilde{R}_{u e_i u e_i}$$

for all unit tangent vectors u .

Note that the same result holds when (M, g, J) is a quasi-Kähler manifold, that is, an almost Hermitian manifold satisfying

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

for all tangent vectors X, Y [5].

Next, let (M, g, J) be a complex space form, that is, a Kähler manifold of constant holomorphic sectional curvature, say c . Further, we note that the second fundamental form σ of a holomorphic submanifold N in a Kähler manifold satisfies

$$(14) \quad \sigma(Ju, u) = J\sigma(u, u), \quad \sigma(Ju, Ju) + \sigma(u, u) = 0$$

for any tangent vector u . This and the Gauss equation yield that the holomorphic sectional curvature $H(u)$ of N corresponding to the unit vector u is given by

$$H(u) = c - 2g(\sigma(u, u), \sigma(u, u)).$$

So since

$$\sum_{i=1}^n \tilde{R}_{u e_i u e_i} = \frac{n + 2}{4} c$$

in the complex space form M , we obtain

COROLLARY 11. *Let N be a holomorphic submanifold of a Kähler manifold of constant holomorphic sectional curvature c . Then N is totally geodesic if and only if for any unit vector u tangent to N we have*

$$\rho(u, u) \geq \frac{n+2}{4}c + (n-1)g(\sigma(u, u), \sigma(u, u))$$

or equivalently,

$$\rho(u, u) \geq \frac{3n}{4}c - \frac{n-1}{2}H(u).$$

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