Canad. J. Math. Vol. **61** (5), 2009 pp. 1151–1181

Covering Maps and Periodic Functions on Higher Dimensional Sierpinski Gaskets

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Abstract. We construct covering maps from infinite blowups of the n-dimensional Sierpinski gasket SG_n to certain compact fractafolds based on SG_n. These maps are fractal analogs of the usual covering maps from the line to the circle. The construction extends work of the second author in the case $n = 2$, but a different method of proof is needed, which amounts to solving a Sudoku-type puzzle. We can use the covering maps to define the notion of periodic function on the blowups. We give a characterization of these periodic functions and describe the analog of Fourier series expansions. We study covering maps onto quotient fractalfolds. Finally, we show that such covering maps fail to exist for many other highly symmetric fractals.

1 Introduction

The purpose of this paper is to extend to the n-dimensional Sierpinski gasket SG_n the results of [S4] for SG_2. Let $\{q_i\}_{i=1}^{n+1}$ be $n+1$ points in $\mathbb{R}^n$ that span a nondegenerate simplex, and denote by $K = SG_n$ the attractor in $\mathbb{R}^n$ of the iterated function system (IFS) $\{F_i\}_{i=1}^{n+1}$, where $F_i x = \frac{1}{2}(x + q_i)$. We regard the points $\{q_i\}$ as *boundary* points of K, not because they are topologically boundary points, but because they play the role of boundary points in the analytic theory of Kigami (see [Ki, S3]). A *fractafold without boundary* $\mathcal{F}$ modeled on K is a finite or countable union of copies of K where each boundary point of each copy is identified with exactly one other such boundary point. Note that a neighborhood of an identified boundary point is isometric to a neighborhood of a nonboundary point in K (for example $F_1K \cap F_2K$). Thus every point in $\mathcal{F}$ has a neighborhood isometric to an interior neighborhood of K. In this way the fractafold is the analog of a flat manifold. The fractafold is compact if and only if the number of copies of K is finite. The theory of fractafolds based on SG_2 was introduced in [S2].

A special class of noncompact fractafolds are the infinite blowups K_ω defined by an infinite word ω (see Section 2 for the construction), where the large scale structure mirrors the small scale structure [S1, T]. When $n = 1$ we obtain the real line in this manner. The main question we address is the following: do covering maps $\pi\colon K_\omega \to \mathcal{F}$ exist for some compact fractafolds, and can we describe the covering maps explicitly? Here we want covering maps in a geometric sense, local isometry, rather than just a topological sense. One reason for this is that a covering map allows us to define *periodic functions* on K_ω to be lifts of functions on $\mathcal{F}$ via the covering

Received by the editors February 6, 2008.

The research of Ruan is supported in part by the National Science Foundation of China, grant 10601049, and by the Future Academic Star project of Zhejiang University. The research of Strichartz is supported in part by the National Science Foundation, grant DMS-0652440.

AMS subject classification: Primary: 28A80.

map. Note that K_ω does not admit nontrivial isometries, so we cannot define periodic functions via analogs of translations.

In the case $n = 2$, such covering maps are constructed in [S4] for a fracatafold $\mathcal{F}_0$ consisting of 4 copies of SG_2. Then, by a simple process of subdivision, this leads to an infinite family $\pi_k\colon (SG_2)_\omega \to \mathcal{F}_k$ of covering maps, where $\mathcal{F}_k$ consists of $4 \cdot 3^k$ copies of SG_2. Another set of covering maps is also obtained by taking the quotient of $\mathcal{F}_0$ under a finite group of fixed point free isometries, and then subdividing. It is conjectured that these are all the possible covering maps.

We cannot simply mimic the construct in [S4] for general n for the following reason: the construction used a slightly different IFS $\{\widetilde{F}_i\}$ whose attractor is still SG_2, but where all $\widetilde{F}_i$ are local inverses of a single expanding map. Such an IFS "with twists" exists for SG_n only when n is even. We use a different method here that works for all n. The fractafold $\mathcal{F}_0$ consists of $n+2$ copies $\{Y_i\}_{i=0}^{n+1}$ of SG_n, with each boundary point of Y_i identified with a boundary point of Y_j for $j \neq i$. To describe the covering map $\pi_0\colon K_\omega \to \mathcal{F}_0$ we need to place a label $L(C)$ in each 0-cell (isometric to SG_n) in K_ω that tells which copy Y_i it gets mapped to (the exact mapping is then determined by the condition that it must be an isometry on the cell and by the labels of all neighboring 0-cells). The key condition that we require is that the labels of $L(C)$ and its $n+1$ neighbors must contain all values in $\{0, 1, \dots, n+1\}$ exactly once. This reduces the construction of the covering map to the solution of a Sudoku-type puzzle! It turns out that the description of the labeling is easier if we simultaneously label cells of larger size. The details are given in Section 2. Once we have the single covering map $\pi_0\colon K_\omega \to \mathcal{F}_0$, we easily obtain an infinite family of covering maps $\pi_k\colon K_\omega \to \mathcal{F}_k$ where the fractafolds $\mathcal{F}_k$ are obtained inductively from $\mathcal{F}_0$ by subdivision (smaller cells in $\mathcal{F}_0$ become larger cells in $\mathcal{F}_k$).

In Section 3 we give two characterizations of periodic functions, extending the results of [S4] for $n = 2$. The first is relatively straightforward and requires that the function be essentially the same on all cells with the same label. The second requires that the function be invariant under certain transformations of central cycles in all large cells. In the case $n = 2$ these transformations are in fact isometries of certain subsets of K_ω, but this is not the case in general.

The fractafold $\mathcal{F}_0$ has an isometry group which is isomorphic to the permutation group on $n+2$ letters. Any subgroup $\widetilde{G}$ that acts without fixed points on $\mathcal{F}_0$ yields a quotient fractafold $\mathcal{F}_0/\widetilde{G}$ and a covering map from K_ω to $\mathcal{F}_0/\widetilde{G}$. The periodic functions associated to this covering map are just the periodic functions associated to $\mathcal{F}_0$ that are invariant under the action of $\widetilde{G}$. In [S4] we identified one such a subgroup when $n = 2$. In Section 4 we characterize such subgroup in terms of the associated permutation subgroup: every nonidentity permutation must have either zero or one fixed point, and the remaining cycles must all have the same odd length. We give a number of examples. There is at least one in every dimension, but the simplest type of example depends on the parity of n. The problem of enumerating (up to conjugacy) all examples appears to be very challenging.

In Section 5 we discuss the analog of Fourier series for our periodic functions. These functions may be expanded in periodic eigenfunctions of the Kigami Laplacian on SG_n, using the method of spectral decimation of Fukushima and Shima [FS].

This is a rather straightforward extension of the $n = 2$ case in [S4], so we keep the discussion brief.

It is natural to ask if similar covering maps can be constructed for other highly symmetric self-similar fractals. In Section 6 we show that in most cases the answer is no. We explain in detail why there are no covering maps for the pentagasket, but it is clear that the same reasoning applies to other fractals. In fact we know of no other fractals where it is reasonable to expect covering maps to exist.

There is another approach to our problem that is based on the theory of self-similar groups [GS, N]. This will be treated in a forthcoming paper by Grigorchuk, Nekrashevych, and Šunić.

2 Definition of Covering Map

Let $n \geq 2$ be a positive integer. Let $K = SG_n \subset \mathbb{R}^n$ generated by $n + 1$ maps $F_i x = \frac{1}{2}(x + q_i)$ where $\{q_i\}_{i=1}^{n+1}$ span a nondegenerate simplex.

Denote $\Omega = \{1, 2, \ldots, n+1\}$ and $\Omega' = \Omega \cup \{0\}$. For $m \geq 1$, define

$$\Omega^m = \{i_1 i_2 \cdots i_m \mid i_j \in \Omega \text{ for all } 1 \leq j \leq m\}$$

to be the set of all words with length m. Define $\Omega^* = \bigcup_{m=1}^{\infty} \Omega^m$ to be the set of all words with finite length. Define

$$\Omega^\infty = \{i_1 i_2 \cdots i_j \cdots \mid i_j \in \Omega \text{ for all } j\}$$

to be the set of all words with infinite length. For any $\mathcal{I} \in \Omega^m$, we define $|\mathcal{I}| = m$.

For any $i_1 i_2 \cdots i_m \in \Omega^*$, we write $F_{i_1 i_2 \cdots i_m}$ for $F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_m}$ and define $K_{i_1 i_2 \cdots i_m} = F_{i_1 i_2 \cdots i_m}(K)$, $q_{i_1 i_2 \cdots i_m} = F_{i_1 i_2 \cdots i_{m-1}}(q_m)$.

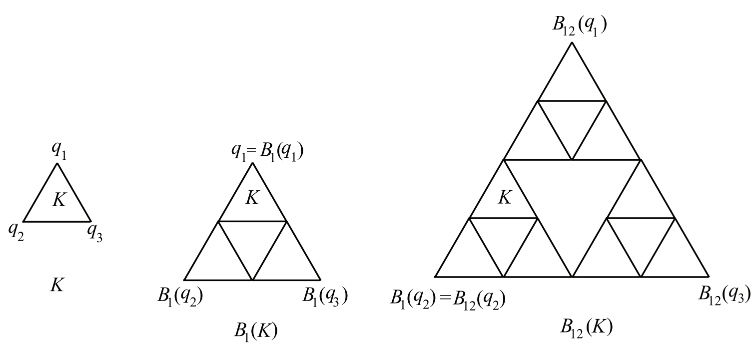

Figure 2.1: The figures of K, $B_1(K)$ and $B_{12}(K)$.

For any $i_1 i_2 \cdots i_k \in \Omega^*$, we define

$$B_{i_1 i_2 \cdots i_k}(x) = F^{-1}_{i_k i_{k-1} \cdots i_1}(x) = F^{-1}_{i_1} \circ F^{-1}_{i_2} \circ \cdots \circ F^{-1}_{i_k}(x), \quad x \in \mathbb{R}^n.$$

https://doi.org/10.4153/CJM-2009-054-5 Published online by Cambridge University Press

Then $B_{i_1 i_2 \cdots i_k}(x)$ is a linear map with derivative 2^k. It is easy to check that for $k \geq 2$,

$$B_{i_1 i_2 \cdots i_k}(x) = B_{i_1 i_2 \cdots i_{k-1}}(B_{i_k}(x)) = 2B_{i_1 i_2 \cdots i_{k-1}}(x) - B_{i_1 i_2 \cdots i_{k-1}}(q_{i_k}), \quad x \in \mathbb{R}^n.$$

See Figure 2.1 for an example, $n = 2$. For any word $\omega = \omega_1\omega_2\omega_3 \cdots \in \Omega^\infty$, we define

$$K_\omega = \bigcup_{m=1}^{\infty} B_{\omega_1\omega_2\cdots\omega_m}(K).$$

K_ω is called an infinite blow-up of K.

Let $\omega \in \Omega^\infty$. If for at least two i's in Ω, there are infinite many j such that $\omega_j = i$, then ω is called *nondegenerate*. Throughout the paper, we always suppose that ω is nondegenerate.

Let $[\omega]_m = \omega_1\omega_2 \cdots \omega_m$ denote the truncation of ω of length m. For any $\omega' \in \Omega^*$, we call $B_{[\omega]_m}(F_{\omega'}K)$ a cell of K_ω, and the set of all $n+1$ vertices of the cell are called the boundary of the cell, denoted by $\partial B_{[\omega]_m}(F_{\omega'}K)$. For $k = |\omega'| - m$ we define k to be the *order* of $B_{[\omega]_m}(F_{\omega'}K)$, denoted by $ord(B_{[\omega]_m}(F_{\omega'}K))$. We also call $B_{[\omega]_m}(F_{\omega'}K)$ a *k-cell* of K_ω, or a k-cell for short.

Let C and $\widetilde{C}$ be two cells of K_ω. $\widetilde{C}$ is called a *k-subcell* of C if $\widetilde{C} \subset C$ and $ord(\widetilde{C}) = k$. If $\widetilde{C} \subset C$ and $ord(\widetilde{C}) = ord(C) + 1$, we call $\widetilde{C}$ a *child cell* of C and call C the *parent cell* of $\widetilde{C}$. We also use $U(C)$ to denote the parent cell of C and define $U^{m+1}(C) = U(U^m(C))$ for positive integer m.

Let C be an m-cell for some integer m. By the definition of K_ω, there exists a point $p_C \in \mathbb{R}^n$ such that $C = 2^{-m}K + p_C$. Define the translation T_C by

$$T_C(x) = 2^{-m}x + p_C, \quad x \in \mathbb{R}^n.$$

For any $\omega' \in \Omega^*$, we define $C_{\omega'} = T_C(K_{\omega'})$. See Figure 2.2 for an example with $n = 2$.

The following fact is straightforward.

Lemma 2.1 *Let C be a cell of K_ω. Then, for any $\omega', \omega'' \in \Omega^*$, we have $C_{\omega'\omega''} = (C_{\omega'})_{\omega''}$.*

Proof Suppose $ord(C) = m$ and $|\omega'| = k$. By definition, we have

$$C_{\omega'\omega''} = 2^{-m}K_{\omega'\omega''} + p_C, \quad (C_{\omega'})_{\omega''} = 2^{-m-k}K_{\omega''} + p_{C_{\omega'}}$$

and

$$C_{\omega'} = 2^{-m}K_{\omega'} + p_C = 2^{-m-k}K + p_{C_{\omega'}}. \tag{2.1}$$

Note that there exists a point $o_{\omega'} \in \mathbb{R}^n$ such that $F_{\omega'}(x) = 2^{-k}x + o_{\omega'}$. We then have $K_{\omega'} = F_{\omega'}(K) = 2^{-k}K + o_{\omega'}$. Thus, by (2.1), we have

$$2^{-m}\left(2^{-k}K + o_{\omega'}\right) = 2^{-m-k}K + p_{C_{\omega'}},$$

so that $p_{C_{\omega'}} = p_C + 2^{-m}o_{\omega'}$. It follows that

$$\begin{aligned} C_{\omega'\omega''} &= 2^{-m}K_{\omega'\omega''} + p_C = 2^{-m}F_{\omega'}(K_{\omega''}) + p_C \\ &= 2^{-m}(2^{-k}K_{\omega''} + o_{\omega'}) + p_C = 2^{-m-k}K_{\omega''} + p_{C_{\omega'}} = (C_{\omega'})_{\omega''}. \end{aligned}$$ ■

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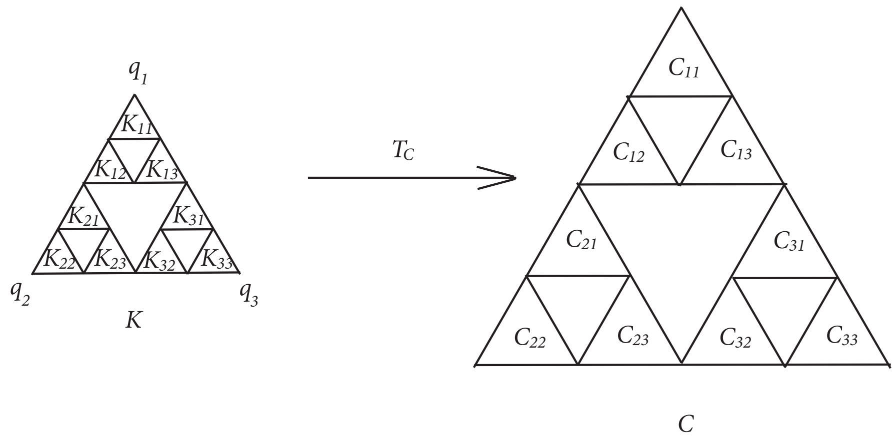

Figure 2.2: T_C, K_ω and C_ω.

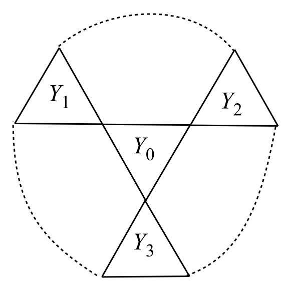

Figure 2.3: A fractafold, $n = 2$.

Definition 2.2 Let $\mathcal{F}$ be any compact fractafold without boundary. Then the map $\pi\colon K_\omega \to \mathcal{F}$ is called a *large locally isometric covering map* if every 0-cell C in $\mathcal{F}$ has a connected neighborhood $\mathcal{U}$ such that π is an isometry from each connected component of $\pi^{-1}(\mathcal{U})$ onto $\mathcal{U}$. In what follows, we shall simply write *covering map* for such maps.

Now suppose $\mathcal{F}$ is the specific fractafold consisting of $n + 2$ copies of K, each denoted by Y_i for $i \in \Omega'$, and each glued to all the others at boundary points such that

(i) $Y_i \cap Y_j$ is a singleton for any $i \neq j$,
(ii) and $Y_i \cap Y_j \cap Y_k = \varnothing$ for any distinct i, j, k.

See Figure 2.3 for an example with $n = 2$. Note that $\{Y_i\}$ are the only subsets of $\mathcal{F}$ isometric to K, so any covering map must map each 0-cell C in K_ω to one of the Y_i. Define the *label* $L(C)$ to be that i. So each covering map defines a labeling, and each labeling may be used to define a covering map provided the conditions of the following lemma hold.

Lemma 2.3 *There is a one-to-one correspondence between covering maps and labelings satisfying the following.*

(2.2) *For every* 0*-cell* C*, all values in* Ω' *occur exactly once among the labels of* C *and the* $n+1$ *neighboring* 0*-cells in* K_ω*.*

Proof Suppose we have a covering map and any 0-cell C with $L(C) = i$. Then each boundary point v of C is mapped to a boundary point $\pi(v)$ of Y_i, and each boundary point of Y_i also belongs to a distinct Y_j for $j \neq i$. If $\pi(v) \in Y_j$ then $\pi(C^{(j)}) = Y_j$, where $C^{(j)}$ is the neighboring 0-cell of C glued at v. Thus $L(C^{(j)}) = j$, and we obtain all labels in Ω' among C and $\{C^{(j)}\}$. Moreover, the isometry $\pi\colon C \to Y_i$ is determined by the images $\pi(v)$ of the boundary points of C, and these images are determined by the labeling, $\pi(C \cap C^{(j)}) = Y_i \cap Y_j$.

Conversely, given a labeling with the property (2.2), we can define $\pi|_C$ to be the isometry $C \to Y_{L(C)}$ such that $\pi(C \cap C') = Y_{L(C)} \cap Y_{L(C')}$ for all $n+1$ neighboring 0-cells C'. It is easy to verify that π is a covering map. ■

We will show that covering maps exist by constructing labeling satisfying (2.2). It will simplify matters if we extend the labels to all cells of nonpositive level.

Suppose there exists a labeling satisfying (2.2). Then for any (-1)-cell C, $L(C_i)$ takes different values in Ω' for different $i \in \Omega$. Thus, we can define the label $L(C)$ to be the unique value in $\Omega' \setminus \{L(C_i)\}_{i\in\Omega}$. We call this the *complementary method* to label (-1)-cells.

Now we suppose that C and C' are two distinct (-1)-cells and $C_k \cap C'_\ell \neq \varnothing$ for some $k, \ell \in \Omega$. Note that C'_ℓ and $C_i, i \in \Omega \setminus \{k\}$ are the $n+1$ neighboring 0-cells of C_k in K_ω, so we know from (2.2) that $L(C'_\ell)$ is the unique value in $\Omega' \setminus \{L(C_i)\}$. Thus $L(C'_\ell) = L(C)$ by the definition of $L(C)$. Similarly $L(C') = L(C_k)$. We also have a direct result $L(C) \neq L(C')$ since $L(C) \neq L(C_k)$. We call $L(C'_\ell) = L(C)$, $L(C') = L(C_k)$ and $L(C) \neq L(C')$ the *intersection property* of labeling.

Denote by $(2.2)_m$ the properties analogous to (2.2), if it holds for any $(-m)$-cell C. We also require that the $n+1$ neighboring cells of C have the same order as C in $(2.2)_m$.

By the above, we know that $(2.2)_0$ holds.

Using intersection property of labeling, for every (-1)-cell C' in K_ω, the labels of its $n+1$ neighboring (-1)-cells are $L(C'_1), \dots, L(C'_{n+1})$. This means that $(2.2)_1$ holds. Then for any (-2)-cell C, $L(C_i)$ takes different values in Ω' for different $i \in \Omega$. Thus, we can define the label $L(C)$ by the complementary method, *i.e.,* to be the unique value in $\Omega' \setminus \{L(C_i)\}$.

Now we suppose that C and C' are two distinct (-2)-cells and $C_k \cap C'_\ell \neq \varnothing$ for some $k, \ell \in \Omega$. Using the same method as above, by $(2.2)_1$, we can show that the intersection property of labeling still holds.

Continuing this procedure, we can define the label for any cell C with nonpositive order by the complementary method with $(2.2)_m$ holding for any $m \geq 0$ and with the intersection property of labeling. That is, we have following lemma.

Lemma 2.4 *Let* L *be a labeling satisfying* (2.2)*.Then we can label any cell with nonpositive order by the complementary method with* $(2.2)_m$ *holding for any* $m \geq 0$*.*

Furthermore, suppose C and C' are two distinct cells with the same nonpositive order and $C_k \cap C'_\ell \neq \varnothing$ for some $k, \ell \in \Omega$. Then $L(C) = L(C'_\ell)$, $L(C') = L(C_k)$ and $L(C) \neq L(C')$. ■

We can represent each (-1)-cell of K_ω by an $(n+1)$-gon in the plane. Each vertex represents one 0-cell, marked with the cell's label. These cells are all connected to each other, and to one other 0-cell outside the (-1)-cell. We write the label of the (-1)-cell in parenthesis in the center of the cell. For example, $n = 3$, we may have a (-1)-cell as in Figure 2.4.

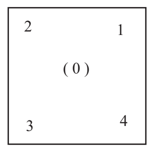

Figure 2.4: Labels in a (-1)-cell.

Suppose this (-1)-cell lies in a (-2)-cell as in Figure 2.5. Then we can use the intersection property to label all subcells of the (-2)-cell with nonpositive order as in Figure 2.6.

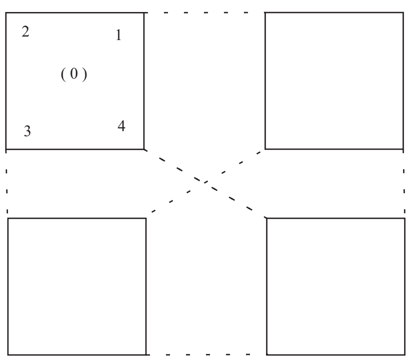

Figure 2.5: A (-2)-cell.

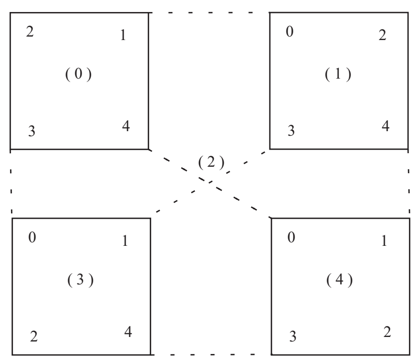

Figure 2.6: A totally labeled (-2)-cell.

Note that if there exists a labeling satisfying (2.2), then we can define labels from smaller to larger cells by the complementary method. The following interesting lemma shows that we can determine labels from the opposite direction.

Lemma 2.5 *Let L be a labeling satisfying* (2.2)*, with any cell with nonpositive order labeled by the complementary method. Then L satisfies the* recurrent property *of labeling: $L(C_{ij}) = L(C_j)$ if $i \neq j$ and $L(C_{ii}) = L(C)$ for any cell C with $ord(C) \leq -2$.*

Proof If $i \neq j$, since $q_{ij} = q_{ji} = K_i \cap K_j = K_{ij} \cap K_{ji}$, we have $C_{ij} \cap C_{ji} = T_C(q_{ij}) \neq \varnothing$. Thus $L(C_{ij}) = L(C_j)$ by the intersection property of labeling. It follows that

$$\{L(C_{ij}) |\, j \in \Omega \setminus \{i\}\} = \{0, 1, \dots, n+1\} \setminus \{L(C_i), L(C)\},$$

which implies that $L(C_{ii}) = L(C)$. ■

We call C' an *equalsize neighboring cell* of C if C' is a neighboring cell of C with $ord(C') = ord(C)$. The following lemma can be viewed as the inverse lemma of Lemma 2.5.

Lemma 2.6 *Let L be a labeling which assigns any cell with nonpositive order a value in Ω'. Suppose L satisfies the recurrent property and*

$$L(C_i) \text{ takes different values in } \Omega' \text{ for any } (-1)\text{-cell } C. \tag{2.3}$$

Then L satisfies (2.2).

Proof Suppose that C' and C'' are any two distinct (-1)-cells with $C'_k \cap C''_\ell \neq \varnothing$ for some $k, \ell \in \Omega$. It suffices to prove that we have the intersection property of labeling, *i.e.*, $L(C''_\ell) = L(C')$, $L(C'_k) = L(C'')$ and $L(C') \neq L(C'')$.

Suppose C' and C'' have same parent cell, say C. Then there exist $\alpha, \beta \in \Omega$ such that $C' = C_\alpha$ and $C'' = C_\beta$. Since $C_{\alpha\beta} \cap C_{\beta\alpha} = C_\alpha \cap C_\beta = T_C(q_{\alpha\beta})$ as in the proof of Lemma 2.5, we know from $\varnothing \neq C'_k \cap C''_\ell = C_{\alpha k} \cap C_{\beta\ell} = T_C(K_{\alpha k} \cap K_{\beta\ell})$ that $q_{\alpha\beta} = K_{\alpha k} \cap K_{\beta\ell}$, which implies that $k = \alpha$, $\ell = \beta$. Thus $k \neq \ell$, $C' = C_\ell$ and $C'' = C_k$. It follows that $L(C''_\ell) = L(C_{k\ell}) = L(C_\ell) = L(C')$. Similarly $L(C'_k) = L(C'')$. Since C' and C'' have the same parent cell, they have different labels by (2.3).

Now, suppose $m \geq 2$ is the minimal positive integer such that $U^m(C') = U^m(C'')$. Using the same discussion as above, we have

$$\begin{gathered} L(U^{m-1}(C')) = L(U^{m-2}(C'')), \quad L(U^{m-1}(C'')) = L(U^{m-2}(C')), \\ \text{and } \ L(U^{m-1}(C')) \neq L(U^{m-1}(C'')). \end{gathered} \tag{2.4}$$

In order to prove the intersection property in this case, we introduce following claim.

Claim A Let C be a cell of K_ω. If one equalsize neighboring cell of C is not contained in $U^2(C)$, then the labels of C and $U^2(C)$ are equal.

Proof of Claim A Denote $\widetilde{C} = U^2(C)$. Note that for any $i, j \in \Omega$ with $i \neq j$, the $n+1$ equalsize neighboring cells of $\widetilde{C}_{ij}$ are $\widetilde{C}_{it}$, $t \in \Omega \setminus \{j\}$ and $\widetilde{C}_{ji}$ which are all subcells of $\widetilde{C}$. Hence there exists $i \in \Omega$ such that $C = \widetilde{C}_{ii}$ so that $L(C) = L(\widetilde{C}_{ii}) = L(\widetilde{C})$ by the recurrent property. ■

Note that for any $0 \leq i < m$, $U^i(C')$ has an equalsize neighboring cell $U^i(C'')$ and $U^i(C'')$ is not contained in $U^{i+2}(C')$ if $i + 2 < m$. Then we know from Claim A that

$$L(C') = L(U^2(C')) = \cdots = L(U^{2r}(C)), \quad \text{if } 2r < m.$$

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Similarly

$$L(C''_\ell) = L(U(C'')) = \cdots = L(U^{2t+1}(C'')), \quad \text{if } 2t+1 < m.$$

It follows that $L(C') = L(U^{m-1}(C'))$ if m is odd, and $L(C') = L(U^{m-2}(C'))$ if m is even. Similarly, $L(C''_\ell) = L(U^{m-2}(C''))$ if m is odd, and $L(C''_\ell) = L(U^{m-1}(C''))$ if m is even. Thus $L(C') = L(C''_\ell)$ always holds by (2.4). Similarly $L(C'') = L(C'_k)$. We also have $L(C') \neq L(C'')$. ∎

Now we will prove the existence and uniqueness of covering maps.

Theorem 2.7 *Let $(\alpha_0, \alpha_1, \dots, \alpha_{n+1})$ be a permutation of $(0, 1, \dots, n+1)$ and C be a (-1)-cell. Then, there exists a unique covering map $\pi\colon K_\omega \to \mathcal{F}$ such that $\pi(C_i) = Y_{\alpha_i}$, $1 \le i \le n+1$.*

Proof First, we will prove the uniqueness. By Lemma 2.3 and Lemma 2.5, it suffices to show that there exists at most one labeling L defined on any cells of nonpositive order with (2.2) holding, with the recurrent property, with $L(C_i) = \alpha_i$ for $1 \le i \le n+1$ and such that $L(\widetilde{C})$ is the unique value in $\Omega' \setminus \{L(\widetilde{C}_i)\}_{i\in\Omega}$ for any cell $\widetilde{C}$ with negative order.

Define $C^m = U^{m-1}(C)$ for $m \in \mathbb{Z}^+$. We remark that $C^1 = C$. Define $t_m \in \Omega$ by $C^{m-1} = C^m_{t_m}$ for any m. Then by Lemma 2.1, for any $\omega' \in \Omega^*$ we have $C^{m-1}_{\omega'} = (C^m_{t_m})_{\omega'} = C^m_{t_m\omega'}$.

Clearly, $C^1 = C$ must be labeled by α_0. For any $m \ge 2$, from the recurrent property, we have

$$\begin{aligned} &L(C^m) = L(C^m_{t_m t_m}) = L(C^{m-1}_{t_m}), \\ &L(C^m_i) = L(C^m_{t_m i}) = L(C^{m-1}_i), \quad i \neq t_m. \end{aligned} \tag{2.5}$$

If $m = 2$, we can obtain $L(C^2)$ and $\{L(C^2_i)\}_{i\in\Omega\setminus\{t_2\}}$ by (2.5). On the other hand, $C^2_{t_2} = C^1$. Thus, all $\{L(C^2_{ij})\}_{i\in\Omega\setminus\{t_2\}, j\in\Omega}$ are determined by the recurrent property. Note that $C^2_{ij} = C^1_j$ if $i = t_2$, so we have already obtained labels for all subcells of C^2.

Generally, suppose for some $k \ge 2$, we have already defined labels for all subcells of C^k with nonpositive order. We can obtain $L(C^{k+1})$ and $\{L(C^{k+1}_i)\}_{i\in\Omega\setminus\{t_{k+1}\}}$ by (2.5). On the other hand, $C^{k+1}_{t_{k+1}} = C^k$. Thus, by the recurrent property, we can obtain $L(C^{k+1}_{i\omega'})$ for any $i \in \Omega \setminus \{t_{k+1}\}$, $\omega' \in \Omega^*$ with $1 \le |\omega'| \le k$. Note that $C^{k+1}_{i\omega'} = C^k_{\omega'}$ for $i = t_{k+1}$, so we have already obtained labels for all subcells of C^{k+1}.

By induction, the labels of all cells of K_ω are determined.

Now we will prove that the labeling L defined as above is suitable to define a covering map π satisfying $\pi(C_i) = Y_{\alpha_i}$, $1 \le i \le n+1$. By Lemma 2.3 and Lemma 2.6, and noting that L satisfies the recurrent property by the definition of labels above, it suffices to prove that $V(\widetilde{C})$ is a permutation of V_n for any (-1)-cell $\widetilde{C}$, where $V_n = (0, 1, \dots, n+1)$ and $V(\widetilde{C}) = (L(\widetilde{C}), L(\widetilde{C}_1), \dots, L(\widetilde{C}_{n+1}))$.

From (2.5) and $C^m_{t_m} = C^{m-1}$, it is easy to see by induction that for any $m \ge 1$, $V(C^m) = V(C^1)$, so that $V(C^m)$ is a permutation of V_n.

Let $m \geq 2$. For any $i \in \Omega$, if $i = \omega_m$, then $C_i^m = C^{m-1}$ so that $V(C_i^m) = V(C^{m-1})$ is a permutation of V_n. If $i \neq \omega_m$, then $L(C_{ii}^m) = L(C^m)$ and $L(C_{ij}^m) = L(C_j^m)$ for $j \neq i$ so that $V(C_i^m)$ is also a permutation of V_n. Continuing this procedure, we know that for any $\omega' \in \Omega^{m-1}$, $V(C_{\omega'}^m)$ is a permutation of V_n. Since m and ω' can be arbitrarily chosen, we know that $V(\widetilde{C})$ is a permutation of V_n for any (-1)-cell $\widetilde{C}$ of K_ω. ■

Let $\mathcal{F}_0 = \mathcal{F}$ be our original fractafold of $n+2$ copies of K. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be the sequence of fractafolds obtained by subdivision: the 1-cells of $\mathcal{F}_{k-1}$ become the 0-cells of $\mathcal{F}_k$, or the k-cells of $\mathcal{F}$ become the 0-cells of $\mathcal{F}_k$. Our goal is to construct a commutative diagram of covering maps as Figure 2.7.

$$
\begin{tikzcd}
K_\omega \arrow[d, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[drr, "\pi_2"] & & & \cdots \\
\mathcal{F}_0 & \mathcal{F}_1 \arrow[l, "\pi_1'"'] & \mathcal{F}_2 \arrow[l, "\pi_2'"'] & \cdots \arrow[l]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & K_\omega \arrow[dl, "\pi_{k-1}"'] \arrow[dr, "\pi_k"] & \\
\mathcal{F}_{k-1} & & \mathcal{F}_k \arrow[ll, "\pi_k'"']
\end{tikzcd}
$$

$$\pi_k' \circ \pi_k = \pi_{k-1}$$

Figure 2.7: The commutative diagram of covering maps.

To define the covering maps $\pi_k' \colon \mathcal{F}_k \to \mathcal{F}_{k-1}$ we follow the same procedure as in the $n = 2$ case (see [S4]). The 0-cells in $\mathcal{F}_k$ (k-cells in $\mathcal{F}_0$) will be labeled $Y_{j_1, j_2, \ldots, j_{k+1}}$ with consecutive values distinct ($j_1 \neq j_2$, $j_2 \neq j_3, \ldots$). For example, the 0-cell Y_0 in $\mathcal{F}_0$ splits into 1-cells $Y_{01}, Y_{02}, \ldots, Y_{0,n+1}$ in $\mathcal{F}_0$, which are 0-cells in $\mathcal{F}_1$. Similarly, Y_1 splits into $Y_{10}, Y_{12}, Y_{13}, \ldots, Y_{1,n+1}$, and $Y_{01} \cap Y_{10}$ intersect at a point. In general, if $Y_{j_1, j_2, \ldots, j_k}$ is a 0-cell in $\mathcal{F}_{k-1}$, then we replace it by $n+1$ 0-cells $Y_{j_1, j_2, \ldots, j_k, i}$ (for $i \neq j_k$) in $\mathcal{F}_k$. All these $n+1$ cells intersect each other at a point. In addition, each of these intersects one additional 0-cell, determined as follows: Suppose $Y_{j_1, \ldots, j_k}$ and $Y_{j_1', \ldots, j_k'}$ intersect in $\mathcal{F}_{k-1}$, then $Y_{j_1, \ldots, j_k, j_k'}$ intersects $Y_{j_1', \ldots, j_k', j_k}$ in $\mathcal{F}_k$. See Figure 2.8 and Figure 2.9. For this to make sense one needs the following lemma.

Lemma 2.8 *If $Y_{j_1, \ldots, j_k}$ intersects $Y_{j_1', \ldots, j_k'}$ in $\mathcal{F}_{k-1}$, then $j_k \neq j_k'$.*

Proof By induction on k. If $k = 2$, we know by definition that Y_{j_1, j_2} and $Y_{j_1', j_2'}$ intersect $\Leftrightarrow$ either (i) $j_1 = j_1'$ and $j_2 \neq j_2'$, or (ii) $j_1 = j_2'$ and $j_1' = j_2$. Note that in either case $j_2 \neq j_2'$.

Assume that the lemma holds when $k = \ell$ for some $\ell \geq 2$. We will prove the lemma holds for $k = \ell + 1$. Suppose $Y_{j_1, \ldots, j_{\ell+1}}$ and $Y_{j_1', \ldots, j_{\ell+1}'}$ intersect in $\mathcal{F}_\ell$. Then by

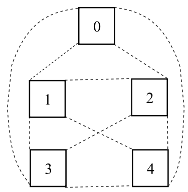

Figure 2.8: $\mathcal{F}_0$, $n = 3$, label i means the cell is Y_i.

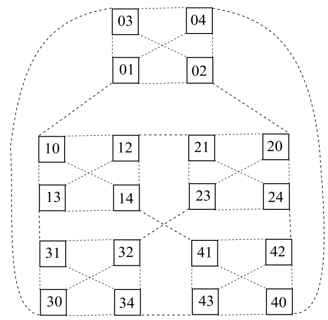

Figure 2.9: $\mathcal{F}_1$, $n = 3$, label ij means the cell is Y_{ij}.

definition there are two possibilities:

(i) $Y_{j_1,\dots,j_{\ell+1}}$ and $Y_{j'_1,\dots,j'_{\ell+1}}$ belong to same 0-cell in $\mathcal{F}_{\ell-1}$, *i.e.*, $(j_1,\dots,j_\ell) = (j'_1,\dots,j'_\ell)$ and $j_{\ell+1} \neq j'_{\ell+1}$.

(i) $Y_{j_1,\dots,j_\ell}$ and $Y_{j'_1,\dots,j'_\ell}$ intersect in $\mathcal{F}_{\ell-1}$ and $j_{\ell+1} = j'_\ell$, $j'_{\ell+1} = j_\ell$.

By inductive assumption on $k = \ell$, we know that $j_\ell \neq j'_\ell$ so that $j_{\ell+1} = j'_\ell \neq j_\ell = j'_{\ell+1}$. Thus in either case $j_{\ell+1} \neq j'_{\ell+1}$. ■

We then define π'_k by

$$\pi'_k(Y_{j_1,j_2,\dots,j_{k+1}}) = Y_{j_2,\dots,j_{k+1}} \tag{2.6}$$

and $\pi'_k|_{Y_{j_1,j_2,\dots,j_{k+1}}}$ is a similarity for any 0-cell $Y_{j_1,\dots,j_{k+1}}$ in $\mathcal{F}_k$.

Lemma 2.9 *If $Y_{j_1,\dots,j_{k+1}}$ intersects $Y_{j'_1,\dots,j'_{k+1}}$ in $\mathcal{F}_k$, then $Y_{j_2,\dots,j_{k+1}}$ intersects $Y_{j'_2,\dots,j'_{k+1}}$ in $\mathcal{F}_{k-1}$.*

Proof If $k = 1$. Let Y_{j_1,j_2} and $Y_{j'_1,j'_2}$ intersect in $\mathcal{F}_1$. Then by Lemma 2.8, we know that $j_2 \neq j'_2$ so that Y_{j_2} intersect Y'_{j_2} in $\mathcal{F}_0$.

Assume that the lemma holds when $k = \ell$ for some $\ell \geq 1$. We will prove the lemma holds for $k = \ell + 1$. Suppose $Y_{j_1,\dots,j_{\ell+2}}$ intersects $Y_{j'_1,\dots,j'_{\ell+2}}$ in $\mathcal{F}_{\ell+1}$. Then by definition there are two possibilities:

(i) $(j_1,\dots,j_{\ell+1}) = (j'_1,\dots,j'_{\ell+1})$ and $j_{\ell+2} \neq j'_{\ell+2}$.

(ii) $Y_{j_1,\dots,j_{\ell+1}}$ intersects $Y_{j'_1,\dots,j'_{\ell+1}}$ in $\mathcal{F}_\ell$ and $j_{\ell+2} = j'_{\ell+1}$, $j'_{\ell+2} = j_{\ell+1}$.

In case (i), we have $(j_2,\dots,j_{\ell+1}) = (j'_2,\dots,j'_{\ell+1})$ and $j_{\ell+2} \neq j'_{\ell+2}$ so that $Y_{j_2,\dots,j_{\ell+2}}$ intersects $Y_{j'_2,\dots,j'_{\ell+2}}$. In case (ii), by inductive assumption, we have $Y_{j_2,\dots,j_{\ell+1}}$ intersects $Y_{j'_2,\dots,j'_{\ell+1}}$. Combining this with $j_{\ell+2} = j'_{\ell+1}$ and $j'_{\ell+2} = j_{\ell+1}$, we obtain by definition that $Y_{j_2,\dots,j_{\ell+2}}$ intersects $Y_{j'_2,\dots,j'_{\ell+2}}$. Thus, in either case $Y_{j_2,\dots,j_{\ell+2}}$ intersects $Y_{j'_2,\dots,j'_{\ell+2}}$ in $\mathcal{F}_\ell$. ■

This shows π'_k is a covering map.

Now we will define covering maps $\pi_k\colon K_\omega \to \mathcal{F}_k$. Note that $\pi_0 = \pi$ is a covering map defined as above. Thus for each cell C of K_ω with nonpositive order, we have

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already defined a corresponding label $L(C)$. Fix a positive integer k. Let C be a 0-cell of K_ω. We define

$$\pi_k(C) = Y_{j_1,j_2,\dots,j_{k+1}}, \tag{2.7}$$

where $j_i = L(U^{k+1-i}(C))$ for any $1 \le i \le k+1$. In order to make π_k well defined, we need to show $j_i \ne j_{i+1}$ for any $1 \le i \le k$. But this holds because $L(\widetilde{C}) \ne L(\widetilde{C}_i)$ for any cell $\widetilde{C}$ with negative order and any $i \in \Omega$.

By definition, we know that for any 0-cell C of K_ω,

$$\pi_{k-1}(C) = Y_{\ell_1,\ell_2,\dots,\ell_k}, \tag{2.8}$$

where $\ell_i = L(U^{(k-1)+1-i}(C)) = L(U^{k+1-(i+1)}(C)) = j_{i+1}$. Using this equality, from (2.6), (2.7), and (2.8), it is easy to check that $\pi'_k \circ \pi_k = \pi_{k-1}$. That is, the diagram in Figure 2.7 commutes.

The following lemma shows that π_k is a covering map for any positive integer k.

Lemma 2.10 *If C and C' are two distinct 0-cells of K_ω and C intersects C', then $\pi_k(C)$ intersects $\pi_k(C')$.*

Proof Let $j_i = L(U^{k+1-i}(C))$, $j'_i = L(U^{k+1-i}(C'))$ for any $1 \le i \le k+1$. Then $\pi_k(C) = Y_{j_1,j_2,\dots,j_{k+1}}$ and $\pi_k(C') = Y_{j'_1,j'_2,\dots,j'_{k+1}}$. Let t be the minimal integer i in $[1, k+1]$ satisfying $U^{k+1-i}(C) \ne U^{k+1-i}(C')$. Then t is well defined because $i = k+1$ satisfies $U^{k+1-i}(C) \ne U^{k+1-i}(C')$.

If $t > 1$, then $j_i = j'_i$ for any $1 \le i < t$ and $j_t \ne j'_t$. Thus $Y_{j_1,\dots,j_t}$ intersects $Y_{j'_1,\dots,j'_t}$. If $t = k+1$, then this means $\pi_k(C)$ intersects $\pi_k(C')$. Otherwise, we have $t < k+1$. Noting that $U^{k+1-(t+1)}(C)$ and $U^{k+1-(t+1)}(C')$ have different parent cells, by the intersection property, we have $L(U^{k+1-t}(C)) = L(U^{k+1-(t+1)}(C'))$ and $L(U^{k+1-t}(C')) = L(U^{k+1-(t+1)}(C))$, *i.e.*, $j_t = j'_{t+1}$ and $j'_t = j_{t+1}$. Thus by definition, $Y_{j_1,\dots,j_{t+1}}$ intersects $Y_{j'_1,\dots,j'_{t+1}}$. Continuing this procedure, eventually, we can obtain that $Y_{j_1,\dots,j_{k+1}}$ intersects $Y_{j'_1,\dots,j'_{k+1}}$ so that $\pi_k(C)$ intersects $\pi_k(C')$.

If $t = 1$, then for any integer $i \in [2, k+1]$, $U^{k+1-i}(C)$ and $U^{k+1-i}(C')$ are distinct. Noting that $U^{k-1}(C)$ and $U^{k-1}(C')$ have different parent cells, by the intersection property, we have $L(U^k(C)) = L(U^{k-1}(C'))$, $L(U^k(C')) = L(U^{k-1}(C))$ and $L(U^k(C)) \ne L(U^k(C'))$, *i.e.*, $j_1 = j'_2$, $j'_1 = j_2$ and $j_1 \ne j'_1$. By definition, Y_{j_1,j_2} intersects $Y_{j'_1,j'_2}$. Then using the same method as in the case $t > 1$, we can obtain that $\pi_k(C)$ intersects $\pi_k(C')$. ∎

3 Characterization of Periodic Functions

Let $\pi\colon K_\omega \to \mathcal{F}$ be a covering map. We call $f_\omega\colon K_\omega \to \mathbb{R}$ a periodic function on K_ω if and only if f_ω is continuous and there exists a continuous function f on $\mathcal{F}$ such that $f_\omega = f \circ \pi$. The set of all periodic functions is denoted by $\mathrm{Per}(K_\omega)$. It is clear that $\mathrm{Per}(K_\omega)$ is independent of the choice of covering map.

For any two 0-cells C and C' with the same label, we define $h_{C,C'}\colon C \to C'$ by $h_{C,C'}(x) = (\pi|_{C'})^{-1} \circ (\pi|_C)(x)$, for any $x \in C$. See Figure 3.1. Since $\pi|_{C'}$ and $\pi|_C$ are both isometries, we know that $h_{C,C'}$ is an isometry.

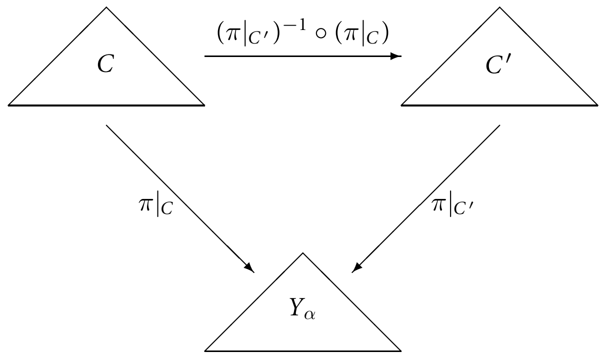

Figure 3.1: The definition of $h_{C,C'}$, where $Y_\alpha = \pi(C) = \pi(C')$.

Definition 3.1 Let C and C' are two 0-cells with the same label. f_ω is called *geometrically equal* on C and C', denoted by $f_\omega|_C \overset{g.e.}{\sim} f_\omega|_{C'}$, if

$$f_\omega(h_{C,C'}(x)) = f_\omega(x), \quad \forall x \in C.$$

It is easy to check that $\overset{g.e.}{\sim}$ is an equivalence relation.

Given two 0-cells C, C' with the same label, we know by the definition of $h_{C,C'}(x)$ that for any $x \in C$, $\pi(h_{C,C'}(x)) = \pi(x)$. Thus if $f_\omega \in \mathrm{Per}(K_\omega)$, we have $f_\omega|_C \overset{g.e.}{\sim} f_\omega|_{C'}$. In fact, we have the following theorem.

Theorem 3.2 *$f_\omega \in \mathrm{Per}(K_\omega)$ if and only if $f_\omega|_C \overset{g.e.}{\sim} f_\omega|_{C'}$ for any two 0-cells C and C' with $L(C) = L(C')$.*

Proof By the above discussion, we only need to prove the "if" part. Suppose $f_\omega|_C \overset{g.e.}{\sim} f_\omega|_{C'}$ for any two 0-cells C and C' with same label. We select $n+2$ 0-cells $C^{(1)}, C^{(2)}, \dots, C^{(n+2)}$ with $L(C^{(i)}) = i$ and define $f_i \colon Y_i \to \mathbb{R}$ by

$$f_i(x) = f_\omega \circ (\pi|_{C^{(i)}})^{-1}(x), \quad \forall x \in Y_i.$$

Then for any cell C with $L(C) = i$ and any $x \in C$, by $f_\omega|_C \overset{g.e.}{\sim} f_\omega|_{C'}$, we have

$$f_\omega(x) = f_\omega \circ h_{C,C^{(i)}}(x) = f_\omega \circ (\pi|_{C^{(i)}})^{-1} \circ (\pi|_C)(x) = f_i(\pi(x)). \tag{3.1}$$

For $\{x_{ij}\} = Y_i \cap Y_j$, we will show

$$f_i(x_{ij}) = f_j(x_{ij}). \tag{3.2}$$

Given two 0-cells C and C' with $L(C) = i$, $L(C') = j$ and $C \cap C'$ is a singleton $\{x^*\}$, then $\{\pi(x^*)\} = Y_i \cap Y_j = \{x_{ij}\}$. By (3.1), $f_i(x_{ij}) = f_i(\pi(x^*)) = f_\omega(x^*)$ and $f_j(x_{ij}) = f_j(\pi(x^*)) = f_\omega(x^*)$ so that (3.2) holds. Thus, we can define

$$f(x) = f_i(x), \quad \forall x \in Y_i.$$

By (3.1), $f_\omega = f \circ \pi$ so that $f_\omega \in \mathrm{Per}(K_\omega)$. ∎

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In the following, we will define the notion of *central cycle* to characterize periodic functions.

If $n = 2$, the central cycle of a $(-n)$-cell C can be easily defined [S4]: *A central cycle of a $(-k)$-cell C is a union of $3 \cdot 2^{k-1}$ 0-cells surrounding the inner deleted triangle in C.* For example, if $ord(C) = -2$, the cell sequence $C_{13}, C_{12}, C_{21}, C_{23}, C_{32}, C_{31}$ is a central cycle of C, see Figure 2.2.

Theorem A ([S4]) *A continuous function f_ω on K_ω is in $Per(K_\omega)$ if and only if its restriction to every central cycle is invariant under ρ^3, where ρ is defined to move each cell to its neighbor, counterclockwise, translating along each of the three sides of the cycle, and rotating around the three corners, conveyer-belt style.*

Now the following question arises: How can we define the central cycle for higher dimensional cases? Does the analog of Theorem A still hold for higher dimensional cases?

Define $(\Omega^1)_0 = \Omega_0 = \Omega$. For $m \geq 2$, we define

$$(\Omega^m)_0 = \{i_1 i_2 \cdots i_m | \, i_1 \in \Omega, i_j \neq i_1 \text{ for any } j \geq 2\}.$$

Define $(\Omega^*)_0 = \cup_{m=1}^{\infty} (\Omega^m)_0$. It is natural that the set of all elements of a central cycle of $(-m)$-cell C should be $\{C'_\omega\}_{\omega' \in (\Omega^m)_0}$. But how can we arrange the order for these 0-cells such that it looks like a cycle?

In order to answer this question, we first define a label for each word in $\{\varnothing\} \cup \Omega^*$.

Step 1. We define $L(\varnothing) = 0$ and $L(i) = i$ for any $i \in \Omega$.

Step 2. For any $\omega' \in \Omega^* \cup \{\varnothing\}$ and any $i, j \in \Omega$, we define

$$L(\omega' i j) = \begin{cases} L(\omega' j), & \text{if } i \neq j, \\ L(\omega'), & \text{if } i = j. \end{cases} \tag{3.3}$$

For any $(-m)$-cell C with $m \geq 0$, if we define a bijection $\sigma_C \colon \Omega' \to \Omega'$ by $\sigma_C(L(C)) = L(\varnothing)$ and $\sigma_C(L(C_i)) = L(i)$, then by the definition of label of words in Ω^* and Lemma 2.5, we know that $\sigma_C(L(C'_\omega)) = L(\omega')$ for any ω' with $|\omega'| \leq m$. This implies the following lemma.

Lemma 3.3 *Let C be a $(-m)$-cell with $m > 0$. Let $C_{\omega'}$ and $C_{\omega''}$ be two subcells of C with the same nonpositive order. Then $L(C_{\omega'}) = L(C_{\omega''})$ if and only if $L(\omega') = L(\omega'')$.* ∎

By Lemma 3.3, we know from the properties of labels of cells that

$$L(\omega' i) \neq L(\omega' j) \quad \text{for any } i \neq j \tag{3.4}$$

and the singleton

$$\{L(\omega')\} = \Omega' \setminus \{L(\omega' i) | \, i \in \Omega\}. \tag{3.5}$$

Lemma 3.4 *For each $\omega' \in (\Omega^m)_0$, we have $L(\omega') \in \Omega$.*

Proof Fix any $i_1 i_2 \cdots i_{m-1} \in (\Omega^{m-1})_0$. If $i_m = i_1$, by (3.3), we have

$$L(i_1 i_2 \cdots i_{m-1} i_m) = L(i_1 i_2 \cdots i_{m-2} i_m) = \cdots$$
$$= L(i_1 i_m) = L(i_1 i_1) = L(\varnothing) = 0.$$

By (3.4), if $i_m \neq i_1$, $L(i_1 i_2 \cdots i_{m-1} i_m) \neq 0$. ■

From Lemma 3.4 with (3.4), and (3.5), we know that

$$\{L(\omega' jk) | \, k \in \Omega \setminus \{i_1\}\} = \Omega \setminus \{L(\omega' j)\},$$

for any fixed $\omega' = i_1 i_2 \cdots i_{m-1} \in (\Omega^{m-1})_0$ and $j \neq i_1$. Using this property, we define the word cycle with order m for any positive m as follows.

Step 1. If $m = 1$, we define the word sequence $1, 2, \ldots, n+1$ to be the word cycle with order 1. Define $\omega_i^{(1)} = i$ for any $i \in \Omega$.

Step 2. Suppose for $m = k \geq 1$, we have the word cycle with order k as follows:

$$\omega_1^{(k)}, \omega_2^{(k)}, \ldots, \omega_{\#k}^{(k)}, \tag{3.6}$$

where $\#k = \#(\Omega^k)_0 = (n+1)n^{k-1}$ and $\{\omega_i^{(k)} | \, 1 \leq i \leq \#k\} = (\Omega^k)_0$.

Now we construct the word cycle with order $k+1$. For each $1 \leq i \leq \#k$, suppose $\omega_i^{(k)} = \omega'$, then we replace $\omega_i^{(k)}$ in (3.6) by $\omega' j_1, \omega' j_2, \ldots, \omega' j_n$, where for $1 \leq t \leq n$, $j_t \in \Omega$ is defined by $L(\omega' j_t) \in \Omega$ and

$$L(\omega' j_t) \equiv \begin{cases} L(\omega') + t \mod n+1 & \text{if } k \text{ is even,} \\ L(\omega') - t \mod n+1 & \text{if } k \text{ is odd.} \end{cases}$$

Note that from $\omega' \in \Omega^k$ and $L(\omega' j_t) \in \Omega$, we have $\omega' j_t \in (\Omega^{k+1})_0$. Thus, after the replacement, (3.6) will be changed to a word sequence with form

$$\omega_1^{(k+1)}, \omega_2^{(k+1)}, \ldots, \omega_{\#(k+1)}^{(k+1)}, \tag{3.7}$$

where $\#(k+1) = \#(\Omega^{k+1})_0$ and $\{\omega_i^{(k+1)} | \, 1 \leq i \leq \#(k+1)\} = (\Omega^{k+1})_0$. We call the word sequence in (3.7) the word cycle with order $k+1$.

By induction, we have defined the word cycle with order m for any positive integer m.

Remark 3.5. It is easy to check that labels of words in the word cycle with order m are:

$$1, 2, \ldots, n+1, 1, 2, \ldots, n+1, \ldots, 1, 2, \ldots, n+1, \quad \text{if } m \text{ is odd,}$$
$$n+1, n, \ldots, 1, n+1, n, \ldots, 1, \ldots, n+1, n, \ldots, 1, \quad \text{if } m \text{ is even.}$$

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Definition 3.6 Let C be a $(-m)$-cell in K_ω with $m \geq 1$. We call the sequence $C_{\omega_1^{(m)}}, C_{\omega_2^{(m)}}, \ldots, C_{\omega_{\#m}^{(m)}}$ of 0-subcells of C the *central cycle* of C.

The following proposition shows that any two successive 0-cells in the central cycle intersect. Thus Definition 3.6 really makes sense.

Proposition 3.7 *Let C be a $(-m)$-cell in K_ω with $m \geq 1$. Then $C_{\omega_i^{(m)}}$ intersects $C_{\omega_{i+1}^{(m)}}$ for any $1 \leq i \leq \#m$, where we define $\omega_{\#m+1}^{(m)} = \omega_1^{(m)}$.*

Proof We will prove a stronger claim by induction on m:

(3.8) *Let C be a $(-m)$-cell in K_ω with $m \geq 1$. Then $C_{\omega_i^{(t)}}$ intersects $C_{\omega_{i+1}^{(t)}}$ for any $1 \leq t \leq m$ and $1 \leq i \leq \#t$.*

It is clear that when $m = 1$, C_i intersects C_{i+1} for any $1 \leq i \leq n+1$, where we define $C_{n+2} = C_1$.

Assume that (3.8) holds for some $m = k \geq 1$. We will prove (3.8) holds for $m = k+1$. If $t = 1$, we know that $C_{\omega_i^{(t)}}$ intersects $C_{\omega_{i+1}^{(t)}}$ for any $1 \leq i \leq n+1$ as above. Thus we can suppose $2 \leq t \leq k+1$ so that $1 \leq t-1 \leq k$.

If $C_{\omega_i^{(t)}}$ and $C_{\omega_{i+1}^{(t)}}$ have the same parent cell, then $C_{\omega_i^{(t)}}$ intersects $C_{\omega_{i+1}^{(t)}}$. Otherwise, suppose ω' and ω'' are two successive cells in word cycle with order $t-1$. Let $\omega' i$ be the last word of word sequence to replace ω' and $\omega'' j$ be the first word of the word sequence to replace ω'', that is, $i, j \in \Omega$ are defined by

$$L(\omega' i) \equiv \begin{cases} L(\omega') + n \mod n+1 & \text{if } t \text{ is even,} \\ L(\omega') - n \mod n+1 & \text{if } t \text{ is odd.} \end{cases} \tag{3.9}$$

and

$$L(\omega'' j) \equiv \begin{cases} L(\omega'') + 1 \mod n+1 & \text{if } t \text{ is even,} \\ L(\omega'') - 1 \mod n+1 & \text{if } t \text{ is odd.} \end{cases} \tag{3.10}$$

It suffices to prove that $C_{\omega' i}$ intersects $C_{\omega'' j}$.

Without loss of generality, we suppose t is even. By Remark 3.5, we have $L(\omega'') \equiv L(\omega') - 1 \mod n+1$. Using (3.9) and (3.10), we know that $L(\omega' i) = L(\omega'')$ and $L(\omega'' j) = L(\omega')$. It follows from Lemma 3.3 that $L(C_{\omega' i}) = L(C_{\omega''})$ and $L(C_{\omega'' j}) = L(\omega')$. By inductive assumption, we know that $C_{\omega'}$ intersects $C_{\omega''}$. Thus by the intersection property, we know that $C_{\omega' i}$ intersects $C_{\omega'' j}$. ■

Definition 3.8 A rotation R on any finite real number sequence $\alpha_1, \alpha_2, \ldots, \alpha_m$ is defined by $R(\alpha_i) = \alpha_{i+1}$ for any $1 \leq i \leq m-1$ and $R(\alpha_m) = \alpha_1$.

Let C be a $(-k)$-cell of K_ω with $k > 0$. Suppose

$$C_{\omega_1^{(k)}}, C_{\omega_2^{(k)}}, \ldots, C_{\omega_{\#k}^{(k)}}, \tag{3.11}$$

is the central cycle of a cell C. We define the rotation ρ on the cell sequence (3.11) by $\rho(C_{\omega_i^{(k)}}) = C_{\omega_{i+1}^{(k)}}$ for any $1 \le i \le \#k$. We say that the restriction of f_ω to (3.11), the central cycle of the cell C, is invariant under ρ^{n+1}, if

$$f_\omega|_{C_{\omega_i^{(k)}}} \overset{g.e.}{\sim} f_\omega|_{C_{\omega_{i+n+1}^{(k)}}}$$

for any $1 \le i \le \#k$, where we define $C_{\omega_{\#k+j}^{(k)}} = C_{\omega_j^{(k)}}$ for $1 \le j \le n+1$.

Theorem 3.9 *A continuous function f_ω is in* $\mathrm{Per}(K_\omega)$ *if and only if for any cell C with* $\deg(C) < 0$*, the restriction of f_ω to the central cycle of C is invariant under ρ^{n+1}.*

Remark 3.10. When $n > 2$, the cells in a central cycle will also have intersections other that those described in Proposition 3.7. The mappings ρ and ρ^{n+1} will not preserve these intersections, and so will not be local isometries on the subset of C corresponding to the cells in the central cycle.

Proof First we prove the "only if" part. Given a $(-k)$-cell C with $k \ge 1$, it follows from Remark 3.5 that the labels of the word sequence

$$L(\omega_1^{(k)}), L(\omega_2^{(k)}), \dots, L(\omega_{\#k}^{(k)})$$

are invariant under R^{n+1}. Thus by Lemma 3.3, the labels of cell sequence

$$L(C_{\omega_1^{(k)}}), L(C_{\omega_2^{(k)}}), \dots, L(C_{\omega_{\#k}^{(k)}})$$

are invariant under R^{n+1}. By Theorem 3.2, the restriction of f_ω to the central cycle of C is invariant under ρ^{n+1}.

Next we prove the "if" part. Suppose for any cell C with $\mathrm{ord}(C) < 0$, the restriction of f_ω to the central cycle of C is invariant under ρ^{n+1}. Then, by the proof of the "only if" part, we know that $f_\omega|_{C'} \overset{g.e.}{\sim} f_\omega|_{C''}$ for any 0-cells C', C'' with the same label if they are in the central cycle of some cell with negative degree.

Definition 3.11 Two 0-cells C', C'' of K_ω are called *centrally linked*, denoted by $C' \overset{c.l.}{\sim} C''$, if there exists a 0-cell sequence

$$C' = C^{(1)}, C^{(2)}, \dots, C^{(t)} = C''$$

such that $L(C^{(i)})$ are all equal, and for any $1 \le i < t$, $C^{(i)}$ and $C^{(i+1)}$ are in the same central cycle.

We will prove the following lemma in the rest of the section.

Lemma 3.12 *$C' \overset{c.l.}{\sim} C''$ for any two distinct 0-cells C', C'' in K_ω with the same label.*

By Lemma 3.12, for any two distinct 0-cells C', C'' in K_ω with $L(C') = L(C'')$, we have $f_\omega|_{C'} \overset{g.e.}{\sim} f_\omega|_{C''}$. Using Theorem 3.2, we know that $f_\omega \in Per(K_\omega)$. ■

In order to prove Lemma 3.12, we first show some properties of the notion "centrally linked".

Definition 3.13 Suppose $\omega', \omega'' \in \Omega^*$ have the same length and the same label. If there exists a word $\mathcal{I} \in \{\varnothing\} \cup \Omega^*$ such that $\omega' = \mathcal{I}\omega^{(1)}$ and $\omega'' = \mathcal{I}\omega^{(2)}$ with $\omega^{(1)}, \omega^{(2)} \in (\Omega^*)_0$, then ω' and ω'' are said to be *strongly centrally linked with respect to* $\mathcal{I}$, denoted by $\omega' \overset{s.c.l.}{\sim} \omega''$ with respect to $\mathcal{I}$, or $\omega' \overset{s.c.l.}{\sim} \omega''$ for short.

Let $\omega', \omega'' \in \Omega^*$ with $|\omega'| = |\omega''|$. ω' and ω'' are called *centrally linked*, denoted by $\omega' \overset{c.l.}{\sim} \omega''$, if there exists a word sequence $\omega' = \omega^{(1)}, \omega^{(2)}, \dots, \omega^{(t)} = \omega''$ such that $\omega^{(i)} \overset{s.c.l.}{\sim} \omega^{(i+1)}$ for any $1 \le i < t$.

It is clear that if $C_{\omega'}$ and $C_{\omega''}$ are two 0-subcells of C with $\omega \overset{c.l.}{\sim} \omega''$, then $C_{\omega'} \overset{c.l.}{\sim} C_{\omega''}$.

Definition 3.14 Let $\omega' \in \Omega^*$. If

$$\omega' = \underbrace{\alpha_1 \cdots \alpha_1}_{m_1} \underbrace{\alpha_2 \cdots \alpha_2}_{m_2} \cdots \underbrace{\alpha_r \cdots \alpha_r}_{m_r},$$

where all m_i, $1 \le i \le r$, are positive integers and $\alpha_i \ne \alpha_{i+1}$ for $1 \le i < r$, then r is called the complexity of the word ω', denoted by $\operatorname{cpl}(\omega')$. We also denote ω' by $(\alpha_1 : m_1; \alpha_2 : m_2; \dots; \alpha_r : m_r)$ for simplicity.

Lemma 3.15 *Let $\omega' \in \Omega^*$ with $\operatorname{cpl}(\omega') \ge 4$. Then, there exists a word $\omega'' \in \Omega^*$ with $\operatorname{cpl}(\omega'') = 3$ such that $\omega' \overset{c.l.}{\sim} \omega''$.*

Proof It suffices to prove for any $\omega' \in \Omega^*$ with $\operatorname{cpl}(\omega') \ge 4$, there exists ω'' with $\operatorname{cpl}(\omega'') = \operatorname{cpl}(\omega') - 1$ such that $\omega' \overset{c.l.}{\sim} \omega''$.

Let $\omega' = (\alpha_1 : m_1; \dots; \alpha_r : m_r)$. For $0 \le i \le r-1$, we define

$$\omega^{(i)} = (\alpha_1 : m_1; \dots; \alpha_{r-i} : m_{r-i}).$$

Define $M = m_{r-1} + m_r$. Note that if m_r is even, we can select $\ell \in \Omega \setminus \{\alpha_{r-1}, \alpha_{r-2}\}$ and define $\omega'' = \omega^{(2)}(\alpha_{r-1}, M)$, $\omega''' = \omega^{(1)}(\ell, m_r)$. Then $\omega' \overset{s.c.l.}{\sim} \omega'''$ with respect to $\omega^{(2)}(\alpha_{r-1} : m_{r-1} - 1)$ and $\omega''' \overset{s.c.l.}{\sim} \omega''$ with respect to $\omega^{(3)}(\alpha_{r-2} : m_{r-2} - 1)$ so that $\omega' \overset{c.l.}{\sim} \omega''$. Hence, in the following, we always suppose m_r is odd.

Case I. If $\alpha_{r-2} \ne \alpha_r$. Then in the case that m_{r-1} is even, $\omega' \overset{s.c.l.}{\sim} \omega^{(2)}(\alpha_r : M)$ with respect to $\omega^{(3)}(\alpha_{r-2} : m_{r-2} - 1)$. In the case that m_{r-1} and m_{r-2} are odd,

$$\begin{aligned}
\omega' &\overset{s.c.l.}{\sim} \omega^{(3)}(\alpha_{r-2} : m_{r-2} - 1; \alpha_r : 1; \alpha_{r-1} : M) \\
&\overset{s.c.l.}{\sim} \omega^{(3)}(\alpha_{r-2} : m_{r-2} - 2; \alpha_{r-1} : 1; \alpha_r : M + 1) \overset{s.c.l.}{\sim} \cdots \\
&\overset{s.c.l.}{\sim} \omega^{(3)}\alpha_{r-2}\alpha_{r-1}(\alpha_r : m_{r-2} + M - 2) \overset{s.c.l.}{\sim} \omega^{(3)}(\alpha_{r-2} : 2; \alpha_r : m_{r-2} + M - 2).
\end{aligned}$$

Similarly, in the case that m_{r-1} is odd and m_{r-2} is even,

$$\omega' \overset{c.l.}{\sim} \omega^{(3)}\alpha_{r-2}\alpha_r(\alpha_{r-1}: m_{r-2}+M-2) \overset{s.c.l.}{\sim} \omega^{(3)}\alpha_{r-2}(\alpha_r: m_{r-2}+M-1).$$

Case II. If $\alpha_{r-2}=\alpha_r$ and $\alpha_{r-3}=\alpha_{r-1}$, we select $\ell \in \Omega \setminus \{\alpha_{r-1},\alpha_{r-2}\}$. Then in the case that M is even,

$$\begin{aligned}\omega' &\overset{s.c.l.}{\sim} \omega^{(2)}(\alpha_{r-1}: m_{r-1}-1\,; \ell: 1\,; \alpha_r: m_r)\\ &\overset{s.c.l.}{\sim} \omega^{(2)}(\alpha_{r-1}: m_{r-1}-2\,; \alpha_r: 1\,; \ell: m_r+1) \overset{s.c.l.}{\sim} \cdots\\ &\overset{s.c.l.}{\sim} \omega^{(2)}(\ell: 1\,; \alpha_r: M-1) \overset{s.c.l.}{\sim} \omega^{(3)}(\ell: 1\,; \alpha_{r-2}: m_{r-2}+M-1).\end{aligned}$$

where the last $\overset{s.c.l.}{\sim}$ is respect to the word $\omega^{(4)}(\alpha_{r-3}: m_{r-3}-1)$. Similarly, in the case that M is odd, $\omega' \overset{c.l.}{\sim} \omega^{(2)}(\alpha_r: 1\,; \ell: M-1) = \omega^{(3)}(\alpha_{r-2}: m_{r-2}+1\,; \ell: M-1)$ since $\alpha_r=\alpha_{r-2}$.

Case III. If $\alpha_{r-2}=\alpha_r$ and $\alpha_{r-3}\neq\alpha_{r-1}$.

(III.1) If m_{r-2} is even, we can define $\omega''=\omega^{(3)}(\alpha_{r-1}: m_{r-1}\,; \alpha_{r-2}: m_{r-2}+m_r)$.
(III.2) If m_{r-2} is odd and m_{r-1} is even, we can define ω'' just as in (III.1).
(III.3) If m_{r-2} and m_{r-1} are all odd, we select $\ell \in \Omega \setminus \{\alpha_{r-3},\alpha_{r-2}\}$. Define

$$\omega''=\begin{cases}\omega^{(4)}(\alpha_{r-3}: m_{r-3}-1\,; \alpha_{r-2}: 1\,; \alpha_{r-3}: m_{r-2}+M) & \text{if } m_{r-3}>1,\\ \omega^{(4)}(\alpha_{r-2}: 1\,; \ell: M-1\,; \alpha_{r-3}: m_{r-2}) & \text{if } m_{r-3}=1.\end{cases}$$

Then $\omega' \overset{s.c.l.}{\sim} \omega''$. We remark that in the case of $m_{r-3}=1$, $\omega' \overset{s.c.l.}{\sim} \omega''$ w.r.t. $\omega^{(4)}$. ■

Lemma 3.16 *Let i, j, k be distinct elements in Ω. Let m_0, m_1, m_2, m_3 be positive integers. Let $\mathfrak{I}=(i: m_1\,; j: m_2\,; k: m_3)$ and $\mathfrak{J}=(i: m_1\,; j: m_2\,; i: m_3)$. Then each word with one of the following forms can be centrally linked to a word in $(\Omega^*)_0$.*

(i) *$\mathfrak{I}$, where m_3 is odd;*
(ii) *$\mathfrak{I}$ or $\mathfrak{J}$, where m_3 is even and m_2 is odd;*
(iii) *$\gamma\mathfrak{I}$, where m_3 is odd and $\gamma \in \Omega \setminus \{i,k\}$;*
(iv) *$\gamma\mathfrak{I}$ or $\gamma\mathfrak{J}$, where m_2, m_3 are even and $\gamma \in \Omega \setminus \{i\}$;*
(v) *$\gamma\mathfrak{J}$, where m_3 is odd and $\gamma \in \Omega \setminus \{i\}$;*
(vi) *$\gamma(k: m_0)\mathfrak{I}$, where m_3 is odd and $\gamma \in \Omega \setminus \{k\}$;*
(vii) *$\gamma(j: m_0)\mathfrak{I}$, where m_3 is even, m_2 is odd and $\gamma \in \Omega \setminus \{j\}$.*

Proof Let $\ell = m_1+m_2+m_3$.

(i) Note that m_3 is odd. If m_2 is even, then

$$\mathfrak{I} \overset{s.c.l.}{\sim} (i: m_1-1\,; j: 1\,; k: m_2+m_3) \overset{s.c.l.}{\sim} (i: m_1-2\,; k: 1\,; j: m_2+m_3+1) \overset{s.c.l.}{\sim} \cdots$$
$$\overset{s.c.l.}{\sim} \begin{cases} k(j: \ell-1) & \text{if } \ell \text{ is odd},\\ j(k: \ell-1) & \text{if } \ell \text{ is even}.\end{cases}$$

https://doi.org/10.4153/CJM-2009-054-5 Published online by Cambridge University Press

Similarly, if m_2 is odd,

$$\mathcal{I} \overset{s.c.l.}{\sim} (i:m_1-1\,;k:1\,;j:m_2+m_3) \overset{s.c.l.}{\sim} \cdots \overset{s.c.l.}{\sim} \begin{cases} k(j:\ell-1) & \text{if } \ell \text{ is odd,}\\ j(k:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

(ii) Note that $\mathcal{I} \overset{s.c.l.}{\sim} \mathcal{J}$. The lemma holds for this case because

$$\mathcal{I} \overset{s.c.l.}{\sim} (i:m_1-1\,;k:1\,;j:m_2+m_3) \overset{s.c.l.}{\sim} \cdots \overset{s.c.l.}{\sim} \begin{cases} j(k:\ell-1) & \text{if } \ell \text{ is odd,}\\ k(j:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

(iii) By the proof of (i),

$$\gamma\mathcal{I} \overset{c.l.}{\sim} \begin{cases} \gamma k(j:\ell-1) \overset{s.c.l.}{\sim} \gamma k(i:\ell-1) & \text{if } \ell \text{ is odd,}\\ \gamma j(k:\ell-1) \overset{s.c.l.}{\sim} \gamma i(k:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

(iv) Note that $\gamma\mathcal{I} \overset{s.c.l.}{\sim} \gamma\mathcal{J}$. The lemma holds for this case because

$$\gamma\mathcal{I} \overset{s.c.l.}{\sim} \begin{cases} \gamma(i:m_1\,;j:m_2+m_3) & \text{if } \gamma \neq j,\\ \gamma(i:m_1\,;k:m_2+m_3) & \text{if } \gamma \neq k,\end{cases}$$

(v) If $\gamma \neq j$, then $\gamma\mathcal{J} \in (\Omega^*)_0$. If $\gamma = j$, we select $j' \in \Omega \setminus \{i, j\}$ and define $\mathcal{K} = \gamma(i:m_1)$. The lemma holds in this case because

$$\begin{aligned}&\mathcal{J} \overset{s.c.l.}{\sim} \mathcal{K}(j:m_2-1\,;j':1\,;i:m_3) \overset{s.c.l.}{\sim} \mathcal{K}(j:m_2-2\,;i:1\,;j':m_3+1) \overset{s.c.l.}{\sim} \cdots\\ &\overset{s.c.l.}{\sim} \begin{cases} \mathcal{K}j'(i:m_2+m_3-1) & \text{if } m_2+m_3 \text{ is even,}\\ \mathcal{K}i(j':m_2+m_3-1) & \text{if } m_2+m_3 \text{ is odd,}\end{cases}\end{aligned}$$

(vi) If $\gamma \neq j$, by the proof of (i),

$$\gamma(k:m_0)\mathcal{I} \overset{c.l.}{\sim} \begin{cases} \gamma(k:m_0)k(j:\ell-1) & \text{if } \ell \text{ is odd,}\\ \gamma(k:m_0)j(k:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

If $\gamma = j$, by the proof of (i),

$$\gamma(k:m_0)\mathcal{I} \overset{c.l.}{\sim} \begin{cases} j(k:m_0)k(j:\ell-1) \overset{s.c.l.}{\sim} j(k:m_0)k(i:\ell-1) & \text{if } \ell \text{ is odd,}\\ j(k:m_0)j(k:\ell-1) \overset{s.c.l.}{\sim} j(k:m_0)i(k:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

(vii) Select $k' \in \Omega \setminus \{\gamma, j\}$. By the proof of (ii),

$$\gamma(j:m_0)\mathcal{I} \overset{c.l.}{\sim} \begin{cases} \gamma(j:m_0)j(k:\ell-1) \overset{s.c.l.}{\sim} \gamma(j:m_0)j(k':\ell-1) & \text{if } \ell \text{ is odd,}\\ \gamma(j:m_0)k(j:\ell-1) \overset{s.c.l.}{\sim} \gamma(j:m_0)k'(j:\ell-1) & \text{if } \ell \text{ is even.}\end{cases}$$

■

Using Lemmas 3.15 and 3.16, we can prove Lemma 3.12 as follows.

Proof of Lemma 3.12 Define $C^m = B_{[\omega]_m}(K)$ for any $m \geq 1$. Let C', C'' be two distinct 0-cells in K_ω. Since ω is nondegenerate, we can select m large enough such that $C' = C^m_{\mathcal{I}}$, and $C'' = C^m_{\mathcal{J}}$ for some $\mathcal{I}, \mathcal{J} \in \Omega^m$ with $\operatorname{cpl}(\mathcal{I}), \operatorname{cpl}(\mathcal{J}) \geq 4$. Furthermore, by selecting m bigger, we can also suppose $L(C') = L(C'') \neq L(C^m)$. By Lemma 3.15, there exist $\omega', \omega'' \in \Omega^m$ such that $\operatorname{cpl}(\omega') = \operatorname{cpl}(\omega'') = 3$ and $C' \overset{c.l.}{\sim} C^m_{\omega'}, C'' \overset{c.l.}{\sim} C^m_{\omega''}$. Thus it suffices to prove that

$$C_{\omega'} \overset{c.l.}{\sim} C_{\omega''} \textit{ for any } (-\ell)\textit{-cell } C \textit{ in } K_\omega \textit{ with } \ell > 0 \textit{ and any } \omega', \omega'' \in \Omega^\ell \textit{ with } \operatorname{cpl}(\omega') = \operatorname{cpl}(\omega'') = 3 \textit{ and } L(C_{\omega'}) = L(C_{\omega''}) \neq L(C). \tag{3.12}$$

Define $i_0 \in \Omega$ such that $L(C_{i_0}) = L(C_{\omega'})$. Suppose

$$\omega' = (\alpha_1 : m_1 ; \alpha_2 : m_2 ; \alpha_3 : m_3), \quad \omega'' = (\beta_1 : m_1' ; \beta_2 : m_2' ; \beta_3 : m_3'),$$

where $\alpha_2 \neq \alpha_1, \alpha_3$, $\beta_2 \neq \beta_1, \beta_3$ and m_i, m_i', $1 \leq i \leq 3$, are all positive integers with $\sum_{i=1}^3 m_i = \sum_{i=1}^3 m_i'$.

From $L(C_{i_0}) = L(C_{\omega'})$, we know that ω' has four possible forms:

(F1) m_3 is odd and $\alpha_1 \neq \alpha_3$. In this case, $\alpha_3 = i_0$.
(F2) m_3 is odd and $\alpha_1 = \alpha_3$. In this case, m_1 is even and $\alpha_1 = \alpha_3 = i_0$.
(F3) m_3 is even and m_2 is odd. In this case, $\alpha_2 = i_0$.
(F4) m_3 and m_2 are even. In this case, m_1 is odd and $\alpha_1 = i_0$.

Similarly, ω'' has four possible forms as above, where all m_i are replaced by m_i' and all α_i are replaced by β_i, respectively.

Case (I.1) If ω' and ω'' both have form (F1) or (F3), then ω' and ω'' both have form (i) or (ii) in Lemma 3.16 so that $C_{\omega'} \overset{c.l.}{\sim} C_{\omega''}$.

Case (I.2) If ω' and ω'' both have form (F2), then

$$\omega' = (i_0 : m_1 ; \alpha_2 : m_2 ; i_0 : m_3), \quad \omega'' = (i_0 : m_1' ; \beta_2 : m_2' ; i_0 : m_3'),$$

where m_3 and m_3' are odd. Since ω is nondegenerate, there exist $\gamma \neq i_0$ and nonnegative integer t such that $C = \widetilde{C}_{\gamma(i_0:t)}$. Note that $\gamma(i_0 : t)\omega'$ and $\gamma(i_0 : t)\omega''$ both have form (v) in Lemma 3.16, we have

$$C_{\omega'} = \widetilde{C}_{\gamma(i_0:t)\omega'} \overset{c.l.}{\sim} \widetilde{C}_{\gamma(i_0:t)\omega''} = C_{\omega''}. \tag{3.13}$$

Case (I.3) If ω' and ω'' both have form (F4), then

$$\omega' = (i_0 : m_1 ; \alpha_2 : m_2 ; \alpha_3 : m_3), \quad \omega'' = (i_0 : m_1' ; \beta_2 : m_2' ; \beta_3 : m_3'),$$

where m_2, m_3, m_2', m_3' are even and m_1, m_1' are odd. Let $\widetilde{C}, \gamma$ and t be defined as in Case (I.2). Note that $\gamma(i_0 : t)\omega'$ and $\gamma(i_0 : t)\omega''$ both have form (iv) in Lemma 3.16; (3.13) holds.

Case (II.1) If one of ω' and ω'' has form (F1) and the other has form (F2). Without of loss generality, we suppose

$$\omega' = (\alpha_1 : m_1 ; \alpha_2 : m_2 ; i_0 : m_3), \quad \omega'' = (i_0 : m_1' ; \beta_2 : m_2' ; i_0 : m_3'),$$

where m_3 and m_3' are odd and $\alpha_1 \neq i_0$. Let $\widetilde{C}, \gamma$ and t be defined as in Case (I.2). Note that $\gamma(i_0 : t)\omega''$ has form (v) in Lemma 3.16, and $\gamma(i_0 : t)\omega'$ has form (vi) if $t > 0$, has form (iii) if $t = 0$ and $\gamma \neq \alpha_1$, has form (i) if $t = 0$ and $\gamma = \alpha_1$. Thus, (3.13) holds.

Case (II.2) If one of ω' and ω'' has form (F1) and the other has form (F4). Similarly as Case (II.1), we can prove $C_{\omega'} \overset{c.l.}{\sim} C_{\omega''}$.

Case (II.3) If one of ω' and ω'' has form (F3) and the other has form (F2). Without of loss generality, we suppose

$$\omega' = (\alpha_1 : m_1 ; i_0 : m_2 ; \alpha_3 : m_3), \quad \omega'' = (i_0 : m_1' ; \beta_2 : m_2' ; i_0 : m_3'), \tag{3.14}$$

where m_3 is even, m_2 and m_3' is odd. If $\alpha_3 = \alpha_1$, we select $\ell \in \Omega \setminus \{\alpha_1, i_0\}$, then $C_{\omega'} \overset{c.l.}{\sim} C_{(\alpha_1 : m_1 ; i_0 : m_2 ; \ell : m_3)}$. Thus we can suppose $\alpha_3 \neq \alpha_1$ in (3.14). Let $\widetilde{C}, \gamma$ and t be defined as in Case (I.2). Note that $\gamma(i_0, t)\omega''$ has form (v) in Lemma 3.16, and $\gamma(i_0 : t)\omega'$ has form (vii) if $t > 0$, has form (iii) if $t = 0$ and $\gamma \neq \alpha_1$, has form (i) if $t = 0$ and $\gamma = \alpha_1$. Thus, (3.13) holds.

Case (II.4) If one of ω' and ω'' has form (F3) and the other has form (F4). Similarly as Case (II.3), we can prove $C_{\omega'} \overset{c.l.}{\sim} C_{\omega''}$.

Case (II.5) If one of ω' and ω'' has form (F2) and the other has form (F4). Without of loss generality, we suppose

$$\omega' = (i_0 : m_1 ; \alpha_2 : m_2 ; i_0 : m_3), \quad \omega'' = (i_0 : m_1' ; \beta_2 : m_2' ; \beta_3 : m_3'),$$

where m_3 is odd, m_2', m_3' are even. Let $\widetilde{C}, \gamma$ and t be defined as in Case (I.2). Note that $\gamma(i_0 : t)\omega'$ has form (v) in Lemma 3.16, and $\gamma(i_0 : t)\omega''$ has form (iv). Thus, (3.13) holds.

Hence, (3.12) always holds, so the proof of Lemma 3.12 is complete. ■

More generally, we can define periodic functions in $\mathrm{Per}_k(K_\omega)$ for any $k \geq 0$ as lifts of continuous functions on $\mathcal{F}_k$ via the covering map π_k (so $\mathrm{Per}(K_\omega) = \mathrm{Per}_0(K_\omega)$). We have the containments $\mathrm{Per}_0(K_\omega) \subset \mathrm{Per}_1(K_\omega) \subset \mathrm{Per}_2(K_\omega) \subset \cdots$. The functions in $\mathrm{Per}_k(K_\omega)$ for larger k have larger periods. It is easy to formulate analogs of Theorem 3.2 and 3.9 to characterize functions in $\mathrm{Per}_k(K_\omega)$, and the proofs are essentially the same. We leave the details to the reader.

4 Symmetries and Other Covering Maps

Let σ be any permutation of Ω. Denote by S_σ the isometry $K \to K$ such that $S_\sigma(q_i) = q_{\sigma(i)}$ for any $i \in \Omega$. Note that $\{q_i\}_{i \in \Omega}$ is the boundary of K and S_σ is an isometry, we know that S_σ is uniquely determined by σ.

Similarly, let θ be any permutation of Ω'. Denote by T_θ the map $\mathcal{F} \to \mathcal{F}$ such that $T_\theta Y_j = Y_{\theta(j)}$ and $T_\theta|_{Y_j}$ is an isometry for any $j \in \Omega'$. For any $i \neq j$, we define x_{ij} as above to be the unique point in the set $Y_i \cap Y_j$. It is clear that $T_\theta(x_{ij}) = x_{\theta(i)\theta(j)}$. Furthermore, if we fix $i \in \Omega'$, then $\{x_{ij}\}_{j\in\Omega'\setminus\{i\}}$ is the boundary of Y_i, so this tells us how T_θ maps ∂Y_i to $\partial Y_{\theta(i)}$ and so determines a unique isometry $Y_i \to Y_{\theta(i)}$. As a result, T_θ is uniquely determined by θ.

All T_θ's compose an isometry group which is isomorphic to the permutation group on Ω'. For any group of permutations G, we denote by $\widetilde{G}$ the corresponding group of isometries $\{T_\theta \mid \theta \in G\}$. If $\widetilde{G}$ acts without fixed points on $\mathcal{F}$ (except for the identity element), then $\mathcal{F}/\widetilde{G}$ is a fractafold, and there is a natural covering map $\mathcal{F} \to \mathcal{F}/\widetilde{G}$. By composition we obtain a covering map $K_\omega \to \mathcal{F}/\widetilde{G}$. Similarly, if $\widetilde{G}_k$ denotes the corresponding group acting on $\mathcal{F}_k$, we have covering maps $K_\omega \to \mathcal{F}_k \to \mathcal{F}_k/\widetilde{G}_k$. We do not know if these constitute all covering maps to compact fractafolds even in the case $n = 2$ [S4]. Still, it would be interesting to classify (up to conjugacy) all such groups $\widetilde{G}$. Although we do not solve this problem here, we reduce it to a classification problem for subgroups of the permutation group. Incidentally, we observe that there can exist pairs of groups $\widetilde{G}_1$ and $\widetilde{G}_2$ that are not conjugate and yet the fractafolds $\mathcal{F}/\widetilde{G}_1$ and $\mathcal{F}/\widetilde{G}_2$ are isometric.

We begin by answering the following question.

Question 4.1 When does $T_\theta^m \colon \mathcal{F} \to \mathcal{F}$ have fixed points for some m with $T_\theta^m \neq id$?

Let W be a finite subset of $\mathbb{Z}$ and $\tau \colon W \to W$ be a permutation. It is clear that τ may be decomposed, in an essentially unique way, into disjoint cycles:

$$\tau = (\alpha_1\alpha_2\cdots\alpha_i)(\beta_1\beta_2\cdots\beta_j)\cdots(\cdots).$$

This notation means that $\tau(\alpha_1) = \alpha_2$, $\tau(\alpha_2) = \alpha_3$, $\tau(\alpha_i) = \alpha_1$, and so on. See [Arm, Pa] for details.

Definition 4.2 Let W be a finite subset of $\mathbb{Z}$ and $\tau \colon W \to W$ be a permutation. We say $\tau \in$ Type (I) if there exist two non-trivial cycles with different length, *i.e.,*

$$\tau = (\alpha_1\alpha_2\cdots\alpha_i)(\beta_1\beta_2\cdots\beta_j)\cdots(\cdots) \quad \text{with } 2 \leq i < j.$$

We say $\tau \in$ Type (II) if every letter belongs to a non-trivial cycle and all cycle are of equal length, *i.e.,*

$$\tau = (\alpha_1\alpha_2\cdots\alpha_s)(\alpha_{s+1}\cdots\alpha_{2s})\cdots(\alpha_{s(r-1)+1}\cdots\alpha_{sr}) \quad \text{with } sr = \#W \text{ and } 2 \leq s.$$

We remark that if $\tau \colon W \to W$ has no fixed point, then τ must belongs to Type (I) or Type (II).

Proposition 4.3 *Suppose $S_\sigma \colon K \to K$ is an isometry and σ has no fixed point. Then S_σ has a fixed point if and only if σ contains a 2-cycle,* i.e., $\sigma = (ab)\cdots(\cdots)$.

Proof By the definition of S_σ, we know that S_σ is an isometry and $S_\sigma(q_i) = q_{\sigma(i)}$. Thus S_σ maps $F_i(K)$ to $F_{\sigma(i)}(K)$. Since σ has no fixed point, $F_i(K) \cap F_{\sigma(i)}(K)$ is the singleton $\{q_{i,\sigma(i)}\} = \{\frac{1}{2}q_i + \frac{1}{2}q_{\sigma(i)}\}$. Thus all possible fixed points of S_σ must belong to $\{q_{i,\sigma(i)}\}_{i\in\Omega}$. Since

$$S_\sigma(\{q_{i,\sigma(i)}\}) = S_\sigma(F_i(K) \cap F_{\sigma(i)}(K)) = F_{\sigma(i)}(K) \cap F_{\sigma(\sigma(i))}(K) = \{q_{\sigma(i),\sigma(\sigma(i))}\},$$

we know that $q_{i,\sigma(i)}$ is a fixed point of S_σ if and only if $\sigma(\sigma(i)) = i$, which is equivalent to $\sigma = (i, \sigma(i)) \cdots (\cdots)$. ■

Proposition 4.4 *Suppose $T_\theta\colon \mathcal{F} \to \mathcal{F}$ and θ has no fixed point. Then T_θ has a fixed point if and only if $\theta = (ab)\cdots(\cdots)$.*

Proof Since θ has no fixed point, each fixed point of T_θ must be one of the vertices of Y_a for some a, *i.e.*, x_{ab} for some a, b. Thus T_θ has a fixed point if and only if $x_{\theta(a),\theta(b)} = x_{ab}$ for some a, b. Again, note that since θ has no fixed point, this is equivalent to $\theta = (ab)\cdots(\cdots)$ for some a, b. ■

Now, we can answer Question 4.1.

Case I. Suppose θ has two fixed points a, b, then x_{ab} is a fixed point of T_θ.

Case II. If θ has exactly one fixed point, say $\theta(a) = a$. Then $T_\theta(Y_a) = Y_a$ and

$$\theta = \begin{pmatrix} a & \alpha_1 & \alpha_2 & \cdots & \alpha_{n+1} \\ a & \beta_1 & \beta_2 & \cdots & \beta_{n+1} \end{pmatrix}$$

for some $\alpha_1, \dots, \alpha_{n+1}, \beta_1, \dots, \beta_{n+1} \in \Omega' \setminus \{a\}$. Define $\sigma = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n+1} \\ \beta_1 & \beta_2 & \cdots & \beta_{n+1} \end{smallmatrix}\right)$, then σ has no fixed point. We will call $T_\theta|_{Y_a} \in$ Type (I)(or Type (II)) if $\sigma \in$ Type (I)(or Type (II)). It is easy to see that Proposition 4.3 still holds if we replace S_σ and K with $T_\theta|_{Y_a}$ and K_a, respectively.

(II.1) If $T_\theta|_{Y_a} \in$ Type (I), then $T_\theta^i|_{Y_a}$ has a fixed point, hence so does T_θ^m.

(II.2) If $T_\theta|_{Y_a} \in$ Type (II) and s is even, then $\sigma^{s/2} = (\alpha_1\alpha_{s/2+1})\cdots(\cdots)$. By Proposition 4.3, $T_\theta^{s/2}|_{Y_a}$, hence $T_\theta^{s/2}$ has a fixed point.

(II.3) If $T_\theta|_{Y_a} \in$ Type (II) and s is odd, then $T_\theta^m|_{Y_a}(\neq id)$ has no fixed point. On the other hand, it is impossible for T_θ^m to have the form $(bc)\cdots(\cdots)$. Similarly as in Proposition 4.4, $T_\theta^m(\neq id)$ has no fixed point for any m.

Case III. If θ has no fixed point.

(III.1) If $\theta \in$ Type (I), then θ^i has at least $i(\geq 2)$ fixed points. Thus T_π^i has fixed points.

(III.2) If $\theta \in$ Type (II) and s is even, then $\theta^{s/2} = (\alpha_1\alpha_{s/2+1})\cdots(\cdots)$. By Proposition 4.4, $T_\theta^{s/2}$ has a fixed point.

(III.3) If $\theta \in$ Type (II) and s is odd, then $T_\theta^m(\neq id)$ has no fixed point for any m.

Combining these results, we have the following theorem.

Theorem 4.5 *Let $T_\theta\colon \mathcal{F} \to \mathcal{F}$ be an isometry. Then, $T_\theta^m(\neq id)$ has no fixed point for all positive integer m if and only if one of the following conditions holds:*

(a) *θ has exactly one fixed point a, and $T_\theta|_{Y_a} \in$ Type (II) with s odd,*
(b) *θ has no fixed point and $\theta \in$ Type (II) with s odd.*

By Theorem 4.5, we define notions *permissible permutation* and *permissible permutation group* as follows.

Definition 4.6 A permutation $\theta\colon \Omega' \to \Omega'$ is called *permissible* if the following two conditions are satisfied:

(1) θ has at most 1 fixed point;
(2) $\theta = (\alpha_1\alpha_2 \cdots \alpha_s)(\alpha_{s+1} \cdots \alpha_{2s}) \cdots (\alpha_{s(r-1)+1} \cdots \alpha_{sr})$ with $s \geq 2$ and s odd.

A permutation group G on Ω' is called *permissible* if θ is permissible for all $\theta \in G$ except $\theta = id$.

It is clear that if θ is permissible, then $G(\theta)$, the permutation group generated by θ, is permissible. Two permutations θ_1, θ_2 are called *independent* if $\theta_1 \notin G(\theta_2)$ and $\theta_2 \notin G(\theta_1)$. It is natural to ask following question.

Question 4.7 When will two independent permissible permutations θ_1, θ_2 generate a permissible group $G(\theta_1, \theta_2)$? More generally, what is the structure of a general permissible group?

We will not discuss the question in detail in this paper. However, we will present some examples.

Example 1 Suppose n is even. Then $\theta = (123 \cdots n(n+1))$ is permissible. The isometry T_θ maps Y_0 to itself and cyclicly permutes the other 0-cells Y_i, $i \neq 0$. A fundamental domain for the action of the group $\widetilde{G}(\theta)$ consists of the union of Y_1 and Y_{01} (the 1-cell in Y_0 that intersects Y_1), with the identification of the other boundary points of Y_1 (not equal to $Y_0 \cap Y_{01}$) in pairs, and a similar identification of boundary points of Y_{01}. Since n is even, there are even number of boundary points in each cell after one is deleted. We may subdivide Y_1 into 1-cells Y_{1j} ($j \neq 1$). Then we may describe the 1-cell structure of the quotient fractafold $\mathcal{F}/\widetilde{G}(\theta)$ as follows: the $n+1$ 1-cells Y_{1j} are connected to each other, with an additional connection between Y_{1j} and $Y_{1,n+1-j}$ for $j \geq 2$. The cell Y_{10} is connected to Y_{01}, and there are $n/2$ edges connecting Y_{01} to itself. Figure 4.1 shows the case $n = 4$ with each 1-cell represented by a point, and Figure 4.2 shows the same case with each 1-cell represented by a pentagon.

Example 2 Suppose n is odd. Then $\theta = (012 \cdots n(n+1))$ is permissible. The isometry T_θ cyclicly permutes the 0-cells Y_i. For a fundamental domain we may take the 0-cell K_0, with boundary points identified in pairs (x_{0j} with $x_{0,n+2-j}$). If we split Y_0 into 1-cells Y_{0j} ($j \neq 0$), then all are connected by one edge, with a second edge connecting Y_{0j} and $Y_{0,n+2-j}$. The case $n = 3$ is shown in Figures 4.3 and 4.4.

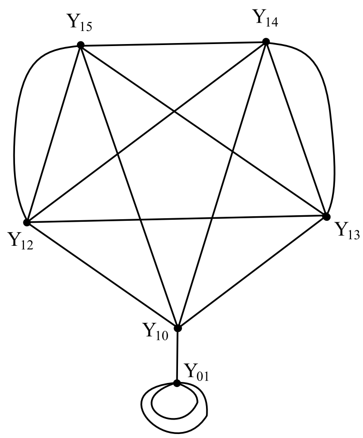

Figure 4.1: $\mathcal{F}/\widetilde{G}(\theta)$, $n = 4$. 1-cell represented by a point.

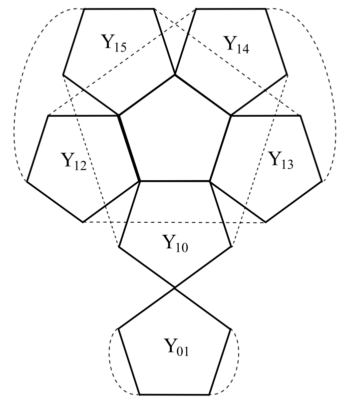

Figure 4.2: $\mathcal{F}/\widetilde{G}(\theta)$, $n = 4$. 1-cell represented by a pentagon.

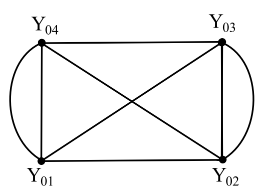

Figure 4.3: $\mathcal{F}/\widetilde{G}(\theta)$, $n = 3$. 1-cell represented by a point.

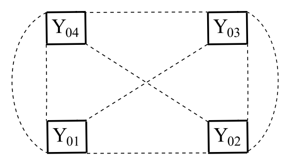

Figure 4.4: $\mathcal{F}/\widetilde{G}(\theta)$, $n = 3$. 1-cell represented by a rectangle.

Example 3 Let $n = 7$. Then $\Omega' = \{0, 1, \dots, 8\}$. Let $\theta_1 = (012)(345)(678)$, $\theta_2 = (036)(147)(258)$, $\theta_3 = (046)(137)(258)$.

Note that $\theta_1\theta_2 = \theta_2\theta_1 = (048)(156)(237)$, and that all elements in $G(\theta_1, \theta_2)$ can be written as $\theta_1^i\theta_2^j$ $(0 \le i, j \le 2)$. Since

$$\theta_1^2 = (021)(354)(687), \quad \theta_2^2 = (063)(174)(285), \quad \theta_1^2\theta_2 = (057)(138)(246),$$
$$\theta_1\theta_2^2 = (075)(183)(264), \quad \theta_1^2\theta_2^2 = (084)(165)(273),$$

we know that $\#G(\theta_1, \theta_2) = 9$ and $G(\theta_1, \theta_2)$ is permissible.

We may identify the fractalfold $\mathcal{F}/\widetilde{G}(\theta_1, \theta_2)$ as follows. The 0-cell Y_0 gets mapped to each 0-cell Y_j for $j \neq 0$ by one of the 8 elements of $\widetilde{G}(\theta_1, \theta_2)$ not equal to the identity. Moreover, there are 4 pairs of boundary points of Y_0 that become identified. For example, T_{θ_1}, maps y_{02} to y_{01}, and its inverse $T_{\theta_1^2}$ maps y_{01} to y_{02}. Thus y_{01} and y_{02} are identified. Similarly, T_{θ_2} identifies y_{03} and y_{04}, $T_{\theta_1^2\theta_2}$ identifies y_{05} and y_{07}, and $T_{\theta_1^2\theta_2^2}$ identifies y_{04} and y_{08}. Thus the quotient fractafold consists of a single 0-cell with 4 identified pairs of boundary points, or equivalently 8 1-cells, each joined to the other 7 as in SG_7, with an additional 4 pairs of boundary points identified.

We note that this is isometric to the quotient fractafold described in Example 2 ($n = 7$). However, since the groups of isometries are clearly not conjugate (they are not isomorphic as groups), the covering maps from $\mathcal{F}$ to the isometric fractafolds are not conjugate. The same is true for the covering map from K_ω to the quotient fractafolds. In other words, the uniqueness up to conjugacy in Theorem 2.7 does not extend to all fractafolds.

On the other hand,

$$\theta_1\theta_3 = (036157248),\ \theta_3\theta_1 = (056147238),\ (\theta_1\theta_3)^2 = (065283174)$$

and $\theta_3(\theta_1\theta_3)^2 = (345)$. That means $\theta_1\theta_3 \neq \theta_3\theta_1$, and $\theta_3(\theta_1\theta_3)^2$, hence $G(\theta_1, \theta_2)$ is not permissible.

Example 4 Let $n = 5$. Let $\theta_1 = (0123456)$, $\theta_2 = (124)(365)$. Then θ_1 has no fixed point while θ_2 has exactly 1 fixed point. It follows from $\theta_1\theta_2 = (026)(143)$ and $\theta_2\theta_1 = (013)(254)$ that $\theta_1\theta_2 \neq \theta_2\theta_1$.

For any $0 \leq i, k \leq 6$, we define an integer $a_{i,k} \in [0, 6]$ by $a_{i,k} \equiv ki \mod 7$. We can check that for any $1 \leq i \leq 6$,

$$\theta_2\theta_1^i = \theta_1^{4i}\theta_2 = (a_{i,0}a_{i,1}a_{i,3})(a_{i,2}a_{i,5}a_{i,4}),\quad \theta_2^2\theta_1^i = \theta_1^{2i}\theta_2^2 = (a_{i,0}a_{i,1}a_{i,5})(a_{i,3}a_{i,6}a_{i,4}).$$

From this we know that $G(\theta_1, \theta_2) = \{\theta_1^i\theta_2^j : 0 \leq i \leq 6, 0 \leq j \leq 2\} = \{\theta_2^i\theta_1^j : 0 \leq i \leq 2, 0 \leq j \leq 6\}$ is permissible and $\#G(\theta_1, \theta_2) = 21$.

Now $G(\theta_1, \theta_2)$ contains 7 pairs of permutations of the form $(abc)(def)$, $(cba)(fed)$ where one number in $\{0, 1, \ldots, 6\}$ is missing.

We can take a fundamental domain as $Y_{01} \cup Y_{03}$ since $(124)(365)$ and its inverse determine values on Y_{0k} hence on Y_0, and (0123456) then determines values on other Y_j. On Y_{01} we have one identification of vertices $y_{012} \leftrightarrow y_{014}$ (action of T_{θ_2}) and 4 vertices are identified with vertices of Y_{03}:

$$y_{013} = y_{031}, \quad \text{(automatic)},$$
$$y_{016} \leftrightarrow y_{034} \ \text{ and } \ y_{015} \leftrightarrow y_{032}, \quad (\text{action of } T_{\theta_2}),$$
$$y_{010} \leftrightarrow y_{030}, \quad (\text{action of } T_{\theta_1}).$$

Similarly, we have $y_{035} \leftrightarrow y_{036}$. See Figures 4.5 and 4.6.

5 Eigenfunction Expansions

We can expand functions on $\mathcal{F}$ in eigenfunctions of the Laplacian on $\mathcal{F}$, and then lift this to an eigenfunction expansion of periodic functions on K_ω. In fact we can explicitly describe all the eigenvalues and eigenfunctions using the spectral decimation method of Fukushima and Shima [FS] together with some ideas from [S2] and [Shi]. The same is true for any of the fractafolds obtained from $\mathcal{F}$ by subdivision.

We will briefly describe the spectrum for $\mathcal{F}$. We may write it as a disjoint union

$$\Lambda = \Lambda^{(0)} \cup \Lambda^{(n+2)} \cup \Lambda^{(n+3)} \cup \Lambda^{(2n+2)},$$

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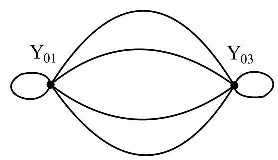

Figure 4.5: $n = 5$. 1-cell represented by a point.

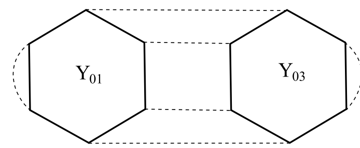

Figure 4.6: $n = 5$. 1-cell represented by a hexagon.

where $\Lambda^{(0)}$ consists of the single eigenvalue $\lambda = 0$ with multiplicity one, with the eigenspace consisting of the constant functions. Eigenvalues in the other series have a "generation of birth" m_0, and are determined by a sequence $\{\lambda_k\}$ for $k \geq m_0$ of eigenvalues of the graph Laplacian Δ_k on the graph of vertices V_k, with $\lambda_{m_0} = n + 2, n+3$ or $2n+2$ depending on the series. The eigenvalues in the sequence are related by the equation

$$\lambda_k = P(\lambda_{k+1}), \tag{5.1}$$

where P is the quadratic polynomial $P(x) = x(n+3-x)$. Note that we can solve (5.1) explicitly to obtain

$$\lambda_{k+1} = \frac{n+3+\varepsilon_k\sqrt{(n+3)^2 - 4\lambda_k}}{2}$$

for $\varepsilon_k = \pm 1$, so the value of λ_{m_0} and the choice of ε_k uniquely determines the sequence $\{\lambda_k\}$, and then

$$\lambda = c_n \lim_{k\to\infty} (n+3)^k \lambda_k \tag{5.2}$$

(the normalization factor $c_n = \frac{n+1}{n}$ arise from the normalization of the Laplacian on K, which in turn arises from the relationship between the counting measure on V_k and the probability measure on K). In order for the limit in (5.2) to exist we must have $\varepsilon_k = -1$ for all but a finite number of values. We also have the following conditions:

(i) if $\lambda \in \Lambda^{(2n+2)}$, then $\varepsilon_{m_0} = +1$ so that $\lambda_{m_0+1} = n+1$;
(ii) if $\lambda \in \Lambda^{(n+3)}$, then $m_0 \geq 1$;
(iii) if $\lambda \in \Lambda^{(n+2)}$, then $m_0 = 0$.

We note that $\Lambda^{(n+3)} \cup \Lambda^{(2n+2)}$ is contained in the set of eigenvalues of the spectrum of Δ on $L^2(K_\omega)$, which all have infinite multiplicity, and each eigenspace has a basis of compactly supported eigenfunctions. This follows by a simple extension of the arguments in [T] for the case $n = 2$. If we periodize these compactly supported eigenfunctions, we obtain periodic eigenfunctions with the same eigenvalue.

The most interesting part the spectrum is thus $\Lambda^{(n+2)}$, since the corresponding periodic eigenfunctions cannot be obtained by periodization. There are two interesting properties of the value $\lambda_0 = n + 2$. One is that it is a fixed point of the polynomial $P(x)$. The other is that it is an eigenvalue of the Laplacian on the cell graph of $\mathcal{F}$ (the complete graph on $n+2$ vertices). Indeed if φ is any function on Ω' with $\sum_{i=0}^{n+1} \varphi(i) = 0$, then $-\Delta\varphi = (n+2)\varphi$ on Ω', and then $\widetilde{\varphi}(x_{ij}) = \varphi(i)+\varphi(j)$ on V_0 satisfies $-\Delta_0\widetilde{\varphi} = (n+2)\widetilde{\varphi}$ on V_0. In this way we explicitly exhibit $\lambda_0 = n+2$ as an eigenvalue of Δ_0 with multiplicity $n+1$.

It is possible to compute the multiplicities of all the eigenvalues in Λ explicitly, but we will not do so here. The case $n = 2$ is completely analyzed in [S4].

6 Nonexistence of Covering Maps on Other Fractals

In this section we will show that covering maps do not exist for many other fractals, even those with enough symmetry to admit spectral decimation. Let K denote a self-similar fractal, K_ω a blowup, and $\mathcal{F}$ a compact fractafold without boundary. We will assume that K is PCF, but a similar argument can be made for the Sierpinski carpet. We will assume that the blowup is nondegenerate, and $\mathcal{F}$ has no boundary. We also assume that two different 0-cells in K_ω can intersect in at most one point.

Definition 6.1 Let C denote any m-cell in K_ω. Then the *valence* $\nu(C)$ is defined to be the number of distinct m-cells (not equal to C) in K_ω with nonempty intersection with C.

In the case of SG_n, the valences are all equal to $n+1$. We conjecture that the existence of covering maps is only possible when all valences are equal.

Lemma 6.2 *Suppose there exists a covering map $\pi\colon K_\omega \to \mathcal{F}$, and suppose C', C'' are 0-cells in K_ω such that $\pi(C') = \pi(C'') = C$ is a 0-cell in $\mathcal{F}$. Then $\nu(C') = \nu(C'')$.*

Proof By the definition of (large, locally isometric) covering map, there exist connected neighborhoods U, U', U'' of C, C', C'' such that $\pi\colon U' \to U$ and $\pi\colon U'' \to U$ are (onto) isometries. Then $(\pi|_{U''})^{-1} \circ \pi\colon U' \to U''$ is an (onto) isometry. Since $\nu(C') = \#\{\text{connected components of } U' \setminus C'\}$, and similarly for $\nu(C'')$, they must be equal. ∎

In order to use the lemma freely, we will need to know that the only subsets of $\mathcal{F}$ that are isometric to K are the 0-cells. This is easy to check in most fractals, but it is not true in general (the interval and the von Koch curve are counterexamples). Under this assumption, for each 0-cell C in $\mathcal{F}$, the 0-cells in $\pi^{-1}(C)$ all have the same valence. But we can say more: if C', C'' are two such 0-cells of valence ν, and $C'_1, \dots, C'_\nu$ and $C''_1, \dots, C''_\nu$ are their neighboring 0-cells in K_ω, then we can arrange the ordering so that $\nu(C'_j) = \nu(C''_j)$ for all $j = 1, \dots, \nu$. Indeed, let $C_1, \dots, C_\nu$ denote 0-cells in $\mathcal{F}$ intersecting C at distinct points (the cells $\{C_j\}$ do *not* have to be distinct, however). We then apply the lemma to C'_j, C''_j, C_j. As a consequence we obtain a lower bound for the number of cells in $\mathcal{F}$, namely the number of different values taken on by $(\nu(C'), \{\nu(C'_1), \dots, \nu(C'_\nu)\})$. (The term in braces is an unordered set.)

By iterating the above argument, considering neighbors of neighbors, and so on, we obtain other, perhaps larger, lower bounds, for the number of 0-cells in $\mathcal{F}$. If these lower bounds grow without limit, then we obtain a contradiction, since for $\mathcal{F}$

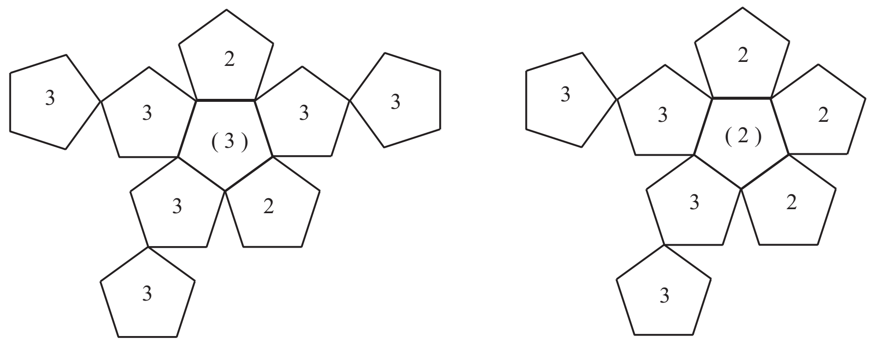

Figure 6.1: Cells of order (-1) split into 0-cells, together with neighboring 0-cells. The valences are labeled for both (-1)-cell and 0-cells.

to be compact it can have only a finite number of 0-cells. The estimates for these lower bounds will depend on the fractal, and are sometimes quite complicated to obtain. So we prefer a different iterative procedure involving cells of different orders. (Incidently, the main idea used in this argument is not valid for SG_n).

For simplicity we describe the argument for the case of the pentagasket. Let C', C'' be as in the lemma, and suppose $C' \subset C'_{(-1)}$ and $C'' \subset C''_{(-1)}$ where $C'_{(-1)}$ and $C''_{(-1)}$ are (-1)-cells in K_ω. The idea is that the isometry $(\pi|_{U''})^{-1} \circ \pi \colon U' \to U''$ extends to an isometry of $C'_{(-1)}$ onto $C''_{(-1)}$, and in particular $\nu(C'_{(-1)}) = \nu(C''_{(-1)})$. On the pentagasket, valences may assume the values 2 and 3. In Figure 6.1 we show the configurations of (-1)-cells with these valences, showing the 0-cells inside and neighboring the (-1)-cell. Suppose $\nu(C') = 2$. If $\nu(C'_{(-1)}) = 3$, then the two neighbors of C' have valence 3, whereas if $\nu(C'_{(-1)}) = 2$, then at least one neighbor of C' has valence 2. This already implies $\nu(C''_{(-1)}) = \nu(C'_{(-1)})$ because we must have the same pattern of valences among neighbors of C' and C''. In the case $\nu(C'_{(-1)}) = \nu(C''_{(-1)}) = 3$, the isometry could map C' onto either of the two 0-cells of valence 2 in $C''_{(-1)}$. But in that case the isometry is determined by the fact that neighboring 0-cells to C' have different valence patterns ($(3,3,2)$ versus $(3,2,2)$) for their neighbors. It is then clear how to extend the isometry to $C'_{(-1)} \to C''_{(-1)}$. In the case $\nu(C'_{(-1)}) = \nu(C''_{(-1)}) = 2$, we need to consider two cases. Case I has C' with two neighbors of valence 2, and case II has C' with one neighbor of valence 2 and one neighbor of valence 3. Whichever case we are in, the same has to hold for C''. In the first case there are two isometries $C' \to C''$ that are possible, and in the second case there is a unique isometry possible. Again it is easy to see how to extend any of the possible isometries from $C' \to C''$ to $C'_{(-1)} \to C''_{(-1)}$.

We also have to consider the possibility that $\nu(C') = 3$. We see from Figure 6.1 that every such cell has a neighbor of valence 2 lying in $C'_{(-1)}$. We can then repeat the argument above starting with that neighbor.

By iterating the above argument we obtain extensions of the isometry to $C'_{(-m)} \to C''_{(-m)}$ and the identity $\nu(C'_{(-m)}) = \nu(C''_{(-m)})$ for chains of cells $C' \subset C'_{(-1)} \subset C'_{(-2)} \subset \cdots$, where $C'_{(-m)}$ has order $-m$, and similarly for C''. In particular, all

0-cells C' in $\pi^{-1}(C)$ must have the same values for the valence sequence

$$(\nu(C'), \nu(C'_{(-1)}), \nu(C'_{(-2)}), \dots, \nu(C'_{(-m)})).$$

It is easy to check that all 2^{m+1} valences sequences occur for 0-cells in K_ω, so $\mathcal{F}$ contains at least 2^{m+1} 0-cells. Since m is arbitrary, the covering map does not exist.

Acknowledgments We are grateful to Martin Kassabov for discussions in connection with the material in Section 4, and to Kwong-Wing Ng in connection with the material in Section 6.

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