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MORREY SPACES AND FRACTIONAL OPERATORS

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Abstract

The relation between the fractional integral operator and the fractional maximal operator is investigated in the framework of Morrey spaces. Applications to the Fefferman–Phong and the Olsen inequalities are also included.

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1. Introduction

The purpose of this paper is to study certain estimates related to the fractional integral operator, defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n(1-\alpha)}}\,dy \quad \text{for } 0<\alpha<1,$$

and to the fractional maximal operator, defined by

$$M_\alpha f(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha}}\int_Q |f(y)|\,dy \quad \text{for } 0\le\alpha<1,$$

in the framework of Morrey spaces. Here, the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x with sides parallel to the coordinate axes. Let $0<p_1\le p_0\le\infty$. For an L^{p_1} locally integrable function f on $\mathbb{R}^n$ we set

$$\begin{aligned}\|f\|_{p_0,p_1}&=\sup_Q |Q|^{1/p_0-1/p_1}\left(\int_Q |f(x)|^{p_1}\,dx\right)^{1/p_1}\\&=\sup_Q |Q|^{1/p_0}\left(\frac{1}{|Q|}\int_Q |f(x)|^{p_1}\,dx\right)^{1/p_1},\end{aligned}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We call the Morrey space $\mathcal{M}^{p_0}_{p_1}$ the subset of all L^{p_1} locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{p_0,p_1}$ is finite. Applying Hölder's inequality, we see that

$$\|f\|_{p_0,p_1} \geq \|f\|_{p_0,p_2} \quad \text{whenever } p_0 \geq p_1 \geq p_2 > 0.$$

This tells us that

$$L^{p_0} = \mathcal{M}^{p_0}_{p_0} \subset \mathcal{M}^{p_0}_{p_1} \subset \mathcal{M}^{p_0}_{p_2} \quad \text{whenever } p_0 \geq p_1 \geq p_2 > 0.$$

It is well known that the Hardy–Littlewood maximal operator M, $M = M_0$, is bounded on $\mathcal{M}^{p_0}_{p_1}$ when $1 < p_1 \leq p_0 < \infty$ and the fractional integral operator I_α is bounded from $\mathcal{M}^{p_0}_{p_1}$ to $\mathcal{M}^{q_0}_{q_1}$ when $1 < p_1 \leq p_0 < \infty$, $1 < q_1 \leq q_0 < \infty$, $1/q_0 = 1/p_0 - \alpha$ and $q_1/q_0 = p_1/p_0$ (see [3, Theorems 1, 2]). The Morrey spaces, which were introduced by Morrey in order to study regularity questions which appeared in the calculus of variations, describe local regularity more precisely than Lebesgue spaces and are widely used not only in harmonic analysis but also in partial differential equations (see [6]).

When $1 \leq p_2 < \infty$, an l^{p_2}-valued function $(f_\nu)_{\nu\in\mathbb{N}}$ on $\mathbb{R}^n$ is said to be measurable if each f_ν is a (real- or complex-valued) measurable function and $\sum_\nu |f_\nu(x)|^{p_2} < \infty$ almost everywhere. For $0 < p_1 \leq p_0 \leq \infty$ and $1 \leq p_2 < \infty$, we define the space $\mathcal{M}^{p_0}_{p_1}(l^{p_2})$ consisting of all l^{p_2}-valued measurable functions (f_ν) such that

$$\|(f_\nu)\|_{p_0,p_1,p_2} = \left\| \left(\sum_\nu |f_\nu|^{p_2} \right)^{1/p_2} \right\|_{p_0,p_1} < \infty.$$

The good-λ inequality of Fefferman and Stein motivated the development of the theory of capacities for potentials of functions in the Morrey space. Adams and Xiao observed in [1] the equivalence of the Morrey norms of the fractional integral operator and the fractional maximal operator, which is an extension of an earlier result of the Lebesgue space due to Muckenhoupt and Wheeden [8].

THEOREM 1.1 [1, Theorem 4.2]. *Let* $0 < \alpha < 1$ *and* $1 < q_1 \leq q_0 < \infty$. *Then*

$$C^{-1}\|M_\alpha f\|_{q_0,q_1} \leq \|I_\alpha f\|_{q_0,q_1} \leq C\|M_\alpha f\|_{q_0,q_1},$$

where the constant C *is independent of* f.

It is evident that $M_\alpha f \leq C I_\alpha |f|$ due to the estimate

$$\frac{1}{r^{n(1-\alpha)}} \int_{\{|x-y|\leq r\}} |f(y)|\, dy \leq I_\alpha |f|(x) \quad \forall x \in \mathbb{R}^n,\ r > 0.$$

However, if $f(y) = |y|^{-n\alpha}$ and $x = 0$, then the reverse inequality is false. In view of this, the Morrey norm equivalence of $I_\alpha f$ and $M_\alpha f$ is quite surprising. In this paper, without using the good-λ inequality of Fefferman and Stein, we shall prove the

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following elementary theorems which link $I_\alpha f$ and $M_\beta f$ and contain, as a special case, the Morrey norm equivalence.

Theorem 1.2 is concerned with Morrey spaces whose parameters are small.

THEOREM 1.2. *Let* $0 < \alpha < 1$, $0 < q_1 \le q_0 < \infty$ *and* $0 < r_1 \le r_0 \le \infty$. *Suppose that* $0 < q_1 \le 1$, $q_1 \le r_1$, $q_0 < r_0$ *and* $0 \le \beta = \alpha - (1/r_0) < 1$. *Then, for any locally integrable function* f *such that* $\|M_\beta f\|_{q_0,q_1} < \infty$ *and for any function* g *in* $\mathcal{M}_{r_1}^{r_0}$,

$$\|g \cdot I_\alpha f\|_{q_0,q_1} \le C\|g\|_{r_0,r_1}\|M_\beta f\|_{q_0,q_1},$$

where the constant C *is independent of* f *and* g.

Theorem 1.3 is a vector-valued inequality for the functions in a Morrey space.

THEOREM 1.3. *Let* $0 < \alpha < 1$, $1 < q_1 \le q_0 < \infty$, $1 < q_2 < \infty$ *and* $1 < r_1 \le r_0 \le \infty$. *Suppose that* $q_1, q_2 < r_1$, $q_0 < r_0$ *and* $0 \le \beta = \alpha - (1/r_0) < 1$. *Then, for any locally integrable function* (f_ν) *such that* $\|(M_\beta f_\nu)\|_{q_0,q_1,q_2} < \infty$ *and for any function* (g_ν) *such that* $\sup_\nu \|g_\nu\|_{r_0,r_1} < \infty$,

$$\|(g_\nu \cdot I_\alpha f_\nu)\|_{q_0,q_1,q_2} \le C \sup_\mu \|g_\mu\|_{r_0,r_1}\|(M_\beta f_\nu)\|_{q_0,q_1,q_2},$$

where the constant C *is independent of* (f_ν) *and* (g_ν).

If we let $r_0 = r_1 = \infty$ and $g, g_\nu \equiv 1$ then we have the following.

COROLLARY 1.4. *Let* $0 < q_1 \le q_0 < \infty$. *Then, for any locally integrable function* f *such that* $\|M_\alpha f\|_{q_0,q_1} < \infty$,

$$\|I_\alpha f\|_{q_0,q_1} \le C\|M_\alpha f\|_{q_0,q_1}.$$

Corollary 1.4 is an extension of Theorem 1.1 to small parameters.

COROLLARY 1.5. *Let* $1 < q_1 \le q_0 < \infty$ *and* $1 < q_2 < \infty$. *Then, for any locally integrable functions* (f_ν) *such that* $\|(M_\alpha f_\nu)\|_{q_0,q_1,q_2} < \infty$,

$$\|(I_\alpha f_\nu)\|_{q_0,q_1,q_2} \le C\|(M_\alpha f_\nu)\|_{q_0,q_1,q_2}.$$

Corollary 1.5 is a vector-valued extension of Theorem 1.1.

Theorems 1.2 and 1.3 can also be thought of as weighted inequalities linking the fractional integral operator $I_\alpha f$ and the fractional maximal operator $M_\beta f$ (see [2, 7, 10, 11] and so on). The methods of proof of these results follow a widely used argument. We use a dyadic decomposition of the kernel of I_α and a linearization method. In the last section we will consider some applications of the theorems. The letter C will be used for constants that may change from one occurrence to another.

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2. Proof of Theorem 1.2

We denote by $\mathcal{D}$ the family of all dyadic cubes on $\mathbb{R}^n$. We assume that f and g are nonnegative. First, we discretize the operator I_α as follows:

$$\begin{aligned} I_\alpha f(x) &= \sum_{\mu\in\mathbb{Z}} \int_{2^{\mu-1}<|x-y|\le 2^\mu} \frac{f(y)}{|x-y|^{n(1-\alpha)}}\,dy \\ &\le C \sum_{\mu\in\mathbb{Z}} \frac{1}{(2^\mu)^{n(1-\alpha)}} \int_{\{|x-y|\le 2^\mu\}} f(y)\,dy \\ &\le C \sum_{\mu\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{D}:\\ Q\ni x, |Q|=2^{n\mu}}} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f(y)\,dy \\ &= C \sum_{Q\in\mathcal{D}} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f(y)\,dy\,\chi_Q(x). \end{aligned}$$

To prove the theorem it suffices to show that

$$\left(\int_{Q_0} (g(x) I_\alpha f(x))^{q_1}\,dx\right)^{1/q_1} \le C\|g\|_{r_0,r_1}\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/q_1-1/q_0}$$

for all dyadic cubes Q_0. Hereafter, we let $\mathcal{D}(Q_0)=\{Q\in\mathcal{D} \mid Q\subset Q_0\}$ and $\tilde{\mathcal{D}}(Q_0)=\{Q\in\mathcal{D} \mid Q\supset Q_0\}$.

We decompose $I_\alpha f(x)$, where $x\in Q_0$, according to Q_0, that is,

$$\begin{aligned} I_\alpha f(x) &\le C(F_1(x)+F_2(x)), \\ F_1(x) &= \sum_{Q\in\mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f(y)\,dy\,\chi_Q(x), \\ F_2(x) &= \sum_{Q\in\tilde{\mathcal{D}}(Q_0)} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f(y)\,dy\,\chi_Q(x), \end{aligned}$$

and we evaluate

$$\begin{aligned} &(1)\colon \left(\int_{Q_0} (g(x)F_1(x))^{q_1}\,dx\right)^{1/q_1}, \\ &(2)\colon \left(\int_{Q_0} (g(x)F_2(x))^{q_1}\,dx\right)^{1/q_1}. \end{aligned}$$

Estimate of (1). We need the following crucial observation.

For a nonnegative function h in $L^\infty(Q_0)$ we let

$$\gamma_0 = \frac{1}{|Q_0|}\int_{Q_0} h(x)\,dx, \qquad a = 2^{n+1}.$$

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For $k = 1, 2, \dots$ let

$$D_k = \bigcup_{\substack{Q \in \mathcal{D}(Q_0): \\ (1/|Q|)\int_Q h(x)\,dx > \gamma_0 a^k}} Q.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\} \subset \mathcal{D}(Q_0)$ are nonoverlapping. By the maximality of $Q_{k,j}$ we see that

$$\gamma_0 a^k < \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} h(x)\,dx \le 2^n \gamma_0 a^k. \tag{2.1}$$

We need the following properties. Let

$$E_0 = Q_0 \backslash D_1, \quad E_{k,j} = Q_{k,j} \backslash D_{k+1}.$$

Then $\{E_0\} \cup \{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0 and satisfies

$$|Q_0| \le 2|E_0|, \quad |Q_{k,j}| \le 2|E_{k,j}|. \tag{2.2}$$

Indeed,

$$\begin{aligned} |D_1| &= \sum_j |Q_{1,j}| \\ &\le \frac{1}{\gamma_0 a}\sum_j \int_{Q_{1,j}} h(x)\,dx \le \frac{1}{\gamma_0 a}\int_{Q_0} h(x)\,dx = \frac{1}{a}|Q_0| \le \frac{1}{2}|Q_0| \end{aligned}$$

and

$$\begin{aligned} |Q_{k,j} \cap D_{k+1}| &= \sum_{i:\, Q_{k+1,i} \subset Q_{k,j}} |Q_{k+1,i}| \\ &\le \frac{1}{\gamma_0 a^{k+1}} \sum_{i:\, Q_{k+1,i} \subset Q_{k,j}} \int_{Q_{k+1,i}} h(x)\,dx \\ &\le \frac{1}{\gamma_0 a^{k+1}} \int_{Q_{k,j}} h(x)\,dx \le \frac{2^n}{a}|Q_{k,j}| = \frac{1}{2}|Q_{k,j}|, \end{aligned}$$

where we have used (2.1). Clearly, these imply (2.2).

We set

$$\mathcal{D}_0 = \left\{ Q \in \mathcal{D}(Q_0) : \frac{1}{|Q|}\int_Q h(x)\,dx \le \gamma_0 a \right\}$$

and

$$\mathcal{D}_{k,j} = \left\{ Q \in \mathcal{D}(Q_0) : Q \subset Q_{k,j},\ \gamma_0 a^k < \frac{1}{|Q|}\int_Q h(x)\,dx \le \gamma_0 a^{k+1} \right\}.$$

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Then by the definition we obtain

$$\mathcal{D}(Q_0) = \mathcal{D}_0 \cup \bigcup_{k,j} \mathcal{D}_{k,j}. \tag{2.3}$$

We now return to the proof. It follows that

$$\int_{Q_0} (g(x)F_1(x))^{q_1}\,dx = \sum_{Q\in\mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy \int_Q g(x)^{q_1} F_1(x)^{q_1-1}\,dx.$$

Putting $h = g^{q_1}$, we apply the relations (2.2) and (2.3) to the estimation of this quantity. First, we evaluate

$$\sum_{Q\in\mathcal{D}_{k,j}} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy \int_Q g(x)^{q_1} F_1(x)^{q_1-1}\,dx. \tag{$*$}$$

Noticing that $q_1 - 1 \le 0$ and by the definition of F_1,

$$F_1(x)^{q_1-1} \le \left(\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1-1} \quad \forall x \in Q_{k,j}.$$

This yields

$$\begin{aligned}
(*) &\le \left(\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1-1} \sum_{Q\in\mathcal{D}_{k,j}} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy \int_Q g(x)^{q_1}\,dx \\
&\le \left(\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1-1} \gamma_0 a^{k+1} \sum_{Q\in\mathcal{D}_{k,j}} |Q|^\alpha \int_{3Q} f(y)\,dy \\
&\le C\left(\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1-1} \gamma_0 a^{k+1} |Q_{k,j}|^\alpha \int_{3Q_{k,j}} f(y)\,dy \\
&= C\left(\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1} |Q_{k,j}|\gamma_0 a^{k+1},
\end{aligned}$$

where in the last inequality we have used the support condition and properties of dyadic cubes.

Recalling that $|Q_{k,j}| \le 2|E_{k,j}|$, $\beta = \alpha - (1/r_0)$, $q_1 \le r_1$ and

$$\begin{aligned}
\gamma_0 a^k &< \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} g(x)^{q_1}\,dx \\
&\le \left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} g(x)^{r_1}\,dx\right)^{q_1/r_1} \le \|g\|_{r_0,r_1}{}^{q_1} |Q_{k,j}|^{-q_1/r_0},
\end{aligned}$$

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we conclude that

$$\begin{aligned}(*) &\le C\|g\|_{r_0,r_1}{}^{q_1}\left(\frac{|Q_{k,j}|^\beta}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1}|E_{k,j}|\\ &\le C\|g\|_{r_0,r_1}{}^{q_1}\int_{E_{k,j}} M_\beta f(x)^{q_1}\,dx,\end{aligned}$$

where in the last inequality we have used the fact that

$$\left(\frac{|Q_{k,j}|^\beta}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\right)^{q_1} \le CM_\beta f(x)^{q_1} \quad \forall x\in E_{k,j}.$$

Similarly,

$$\begin{aligned}&\sum_{Q\in\mathcal{D}_0}\frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy\int_Q g(x)^{q_1}F_1(x)^{q_1-1}\,dx\\ &\qquad\le C\|g\|_{r_0,r_1}{}^{q_1}\int_{E_0} M_\beta f(x)^{q_1}\,dx.\end{aligned}$$

Summing up all factors, we conclude that the expression in (1) is less than or equal to

$$C\|g\|_{r_0,r_1}\left(\int_{Q_0} M_\beta f(x)^{q_1}\,dx\right)^{1/q_1} \le C\|g\|_{r_0,r_1}\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/q_1-1/q_0},$$

where we have used (2.3) and the fact that $\{E_0\}\cup\{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0. This is our desired inequality.

Estimate of (2). By a property of dyadic cubes

$$F_2(x)=\sum_{Q\in\tilde{\mathcal{D}}(Q_0)}\frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy \quad \forall x\in Q_0.$$

Notice that $\beta=\alpha-(1/r_0)$. For all $Q\in\tilde{\mathcal{D}}(Q_0)$, it follows that

$$\frac{|Q|^\alpha}{|Q|}\int_{3Q} f(y)\,dy = |Q|^{1/r_0}\left(\frac{|Q|^\beta}{|Q|}\int_{3Q} f(y)\,dy\right) \le C|Q|^{1/r_0}\inf_{y\in Q} M_\beta f(y),$$

$$\inf_{y\in Q} M_\beta f(y) \le \left(\frac{1}{|Q|}\int_Q M_\beta f(y)^{q_1}\,dy\right)^{1/q_1}$$

and

$$\left(\frac{1}{|Q|}\int_Q M_\beta f(y)^{q_1}\,dy\right)^{1/q_1} \le \|M_\beta f\|_{q_0,q_1}|Q|^{-1/q_0}.$$

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These imply, by using the fact that $(1/r_0) - (1/q_0) < 0$,

$$F_2(x) \le C\|M_\beta f\|_{q_0,q_1} \sum_{Q\in\tilde{\mathcal{D}}(Q_0)} |Q|^{1/r_0-1/q_0} = C\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/r_0-1/q_0}.$$

This concludes that the expression in (2) is less than or equal to

$$\begin{aligned}
&C\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/r_0-1/q_0}\left(\int_{Q_0} g(x)^{q_1}\,dx\right)^{1/q_1} \\
&\quad = C\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/q_1-1/q_0+1/r_0}\left(\frac{1}{|Q_0|}\int_{Q_0} g(x)^{q_1}\,dx\right)^{1/q_1} \\
&\quad \le C\|g\|_{r_0,r_1}\|M_\beta f\|_{q_0,q_1}|Q_0|^{1/q_1-1/q_0}.
\end{aligned}$$

This is our desired inequality.

3. Proof of Theorem 1.3

In what follows we shall prove Theorem 1.3. We assume that f_ν and g_ν are non-negative. By the same manipulation as in the previous section, to prove the theorem it suffices to estimate

$$\left(\int_{Q_0}\left(\sum_\nu (g_\nu(x)F_\nu(x))^{q_2}\right)^{q_1/q_2} dx\right)^{1/q_1}$$

for all dyadic cubes Q_0. Here,

$$F_\nu(x) = \sum_{Q\in\mathcal{D}} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy\,\chi_Q(x).$$

We shall estimate this quantity by way of a duality argument. To this end, we take a vector-valued weight (w_ν) supported on Q_0 satisfying

$$\int_{Q_0}\left(\sum_\nu w_\nu(x)^{q_2'}\right)^{q_1'/q_2'} dx = 1, \tag{3.1}$$

and evaluate

$$\begin{aligned}
&(3)\text{:}\ \sum_{\nu\in\mathbb{N}}\sum_{Q\in\mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy \int_Q g_\nu(x)w_\nu(x)\,dx, \\
&(4)\text{:}\ \sum_{\nu\in\mathbb{N}}\sum_{Q\in\tilde{\mathcal{D}}(Q_0)} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy \int_{Q_0} g_\nu(x)w_\nu(x)\,dx.
\end{aligned}$$

Estimate of (3). We compute

$$\sum_{Q\in\mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy \int_Q g_\nu(x)w_\nu(x)\,dx. \tag{3.2}$$

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Putting $h = g_\nu w_\nu$, we apply the relations (2.2) and (2.3) to the estimation of this quantity. First, we evaluate

$$\sum_{Q \in \mathcal{D}_{k,j}} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f_\nu(y)\,dy \int_Q g_\nu(x) w_\nu(x)\,dx. \tag{$**$}$$

It follows from the same argument as in the previous section that

$$(**) \le C \frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|} \int_{3Q_{k,j}} f_\nu(y)\,dy\, \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g_\nu(x) w_\nu(x)\,dx\, |E_{k,j}|.$$

Using Hölder's inequality,

$$\begin{aligned}(**) \le C &\frac{|Q_{k,j}|^\alpha}{|Q_{k,j}|} \int_{3Q_{k,j}} f_\nu(y)\,dy \\ &\times \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g_\nu(x)^{r_1}\,dx \right)^{1/r_1} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w_\nu(x)^{r_1'}\,dx \right)^{1/r_1'} |E_{k,j}|.\end{aligned}$$

Recalling that

$$\left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g_\nu(x)^{r_1}\,dx \right)^{1/r_1} \le \|g_\nu\|_{r_0,r_1} |Q_{k,j}|^{-1/r_0},$$

we see that

$$\begin{aligned}(**) &\le C \|g_\nu\|_{r_0,r_1} \frac{|Q_{k,j}|^\beta}{|Q_{k,j}|} \int_{3Q_{k,j}} f_\nu(y)\,dy \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w_\nu(x)^{r_1'}\,dx \right)^{1/r_1'} |E_{k,j}| \\ &\le C \|g_\nu\|_{r_0,r_1} \int_{E_{k,j}} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{Q \in \mathcal{D}_0} \frac{|Q|^\alpha}{|Q|} \int_{3Q} f_\nu(y)\,dy &\int_Q g_\nu(x) w_\nu(x)\,dx \\ &\le C \|g_\nu\|_{r_0,r_1} \int_{E_0} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx.\end{aligned}$$

Summing up all factors, we conclude that the expression in (3.2) is at most

$$C \|g_\nu\|_{r_0,r_1} \int_{Q_0} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx.$$

Now we conclude that the expression in (3) is less than or equal to

$$\begin{aligned}C \sum_\nu \|g_\nu\|_{r_0,r_1} &\int_{Q_0} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx \\ &\le C \sup_\mu \|g_\mu\|_{r_0,r_1} \sum_\nu \int_{Q_0} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx.\end{aligned}$$

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It follows from using Hölder's inequality that

$$\begin{aligned}\sum_\nu \int_{Q_0} M_\beta f_\nu(x) M {w_\nu}^{r_1'}(x)^{1/r_1'}\,dx &\le \left(\int_{Q_0}\left(\sum_\nu M_\beta f_\nu(x)^{q_2}\right)^{q_1/q_2} dx\right)^{1/q_1}\\ &\quad\times\left(\int_{Q_0}\left(\sum_\nu M{w_\nu}^{r_1'}(x)^{q_2'/r_1'}\right)^{q_1'/q_2'} dx\right)^{1/q_1'}.\end{aligned}$$

Notice that the condition $q_1, q_2 < r_1$ implies that $q_1'/r_1', q_1'/r_1' > 1$. These facts and the boundedness of the vector-valued Hardy–Littlewood maximal operator M (see [5, p. 498, Corollary 4.3]) yield

$$\begin{aligned}&\left(\int_{Q_0}\left(\sum_\nu M{w_\nu}^{r_1'}(x)^{q_2'/r_1'}\right)^{q_1'/q_2'} dx\right)^{r_1'/q_1'}\\ &\qquad\le C\left(\int_{Q_0}\left(\sum_\nu w_\nu(x)^{q_2'}\right)^{q_1'/q_2'} dx\right)^{r_1'/q_1'} \le C,\end{aligned}$$

where in the last inequality we have used (3.1).

Hence, the expression in (3) is less than or equal to

$$C\sup_\mu \|g_\mu\|_{r_0,r_1}\|(M_\beta f_\nu)\|_{q_0,q_1,q_2}|Q_0|^{1/q_1-1/q_0}.$$

This is our desired inequality.

Estimate of (4). In this case we evaluate

$$\sum_\nu \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy \int_{Q_0} g_\nu(x)w_\nu(x)\,dx \quad \text{for } Q\in\tilde{\mathcal{D}}(Q_0). \tag{3.3}$$

It follows that

$$\begin{aligned}\int_{Q_0} g_\nu(x)w_\nu(x)\,dx &\le \left(\int_{Q_0} g_\nu(x)^{r_1}\,dx\right)^{1/r_1}\left(\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{1/r_1'}\\ &\le \|g_\nu\|_{r_0,r_1}|Q_0|^{1/r_1-1/r_0}\left(\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{1/r_1'}\end{aligned}$$

and that

$$\begin{aligned}&\sum_\nu \frac{|Q|^\alpha}{|Q|}\int_{3Q} f_\nu(y)\,dy\left(\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{1/r_1'}\\ &\qquad\le C|Q|^{1/r_0}\left(\sum_\nu\left(\frac{|3Q|^\beta}{|3Q|}\int_{3Q} f_\nu(y)\,dy\right)^{q_2}\right)^{1/q_2}\\ &\qquad\quad\times\left(\sum_\nu\left(\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{q_2'/r_1'}\right)^{1/q_2'}.\end{aligned}$$

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We see that

$$\begin{aligned}
&\left(\sum_\nu\left(\frac{|3Q|^\beta}{|3Q|}\int_{3Q} f_\nu(y)\,dy\right)^{q_2}\right)^{1/q_2}\\
&\quad\le \frac{1}{|Q|}\int_Q\left(\sum_\nu M_\beta f_\nu(x)^{q_2}\right)^{1/q_2}dx\\
&\quad\le \left(\frac{1}{|Q|}\int_Q\left(\sum_\nu M_\beta f_\nu(x)^{q_2}\right)^{q_1/q_2}dx\right)^{1/q_1}\\
&\quad\le \|(M_\beta f_\nu)\|_{q_0,q_1,q_2}|Q|^{-1/q_0}
\end{aligned}$$

and that

$$\begin{aligned}
&\left(\sum_\nu\left(\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{q_2'/r_1'}\right)^{1/q_2'}\\
&\quad= |Q_0|^{1/r_1'}\left(\sum_\nu\left(\frac{1}{|Q_0|}\int_{Q_0} w_\nu(x)^{r_1'}\,dx\right)^{q_2'/r_1'}\right)^{1/q_2'}\\
&\quad\le |Q_0|^{-1/r_1}\int_{Q_0}\left(\sum_\nu M{w_\nu}^{r_1'}(x)^{q_2'/r_1'}\right)^{1/q_2'}dx\\
&\quad\le |Q_0|^{1/q_1-1/r_1}\left(\int_{Q_0}\left(\sum_\nu M{w_\nu}^{r_1'}(x)^{q_2'/r_1'}\right)^{q_1'/q_2'}dx\right)^{1/q_1'}\\
&\quad\le C|Q_0|^{1/q_1-1/r_1},
\end{aligned}$$

where in the last inequality we have used the same argument as in the last part of the previous paragraph.

These imply that the expression in (3.3) is at most

$$C\sup_\mu \|g_\mu\|_{r_0,r_1}\|(M_\beta f_\nu)\|_{q_0,q_1,q_2}|Q_0|^{1/q_1-1/r_0}|Q|^{1/r_0-1/q_0},$$

and hence the expression in (4) is less than or equal to

$$\begin{aligned}
&C\sup_\mu \|g_\mu\|_{r_0,r_1}\|(M_\beta f_\nu)\|_{q_0,q_1,q_2}|Q_0|^{1/q_1-1/r_0}\sum_{Q\in\tilde{\mathcal{D}}(Q_0)}|Q|^{1/r_0-1/q_0}\\
&\quad\le C\sup_\mu \|g_\mu\|_{r_0,r_1}\|(M_\beta f_\nu)\|_{q_0,q_1,q_2}|Q_0|^{1/q_1-1/q_0}.
\end{aligned}$$

This is our desired inequality. Here, we have used the fact that $(1/r_0)-(1/q_0)<0$.

4. Applications to some inequalities

In this section we consider some simple applications of Theorems 1.2 and 1.3. We need some preparations.

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It is known that, for a smooth function f in $C_0^\infty(\mathbb{R}^n)$ (see [13, p. 125]),

$$|f(x)| \leq C \sum_{j=1}^{n} I_{1/n}\left|\frac{\partial f}{\partial x_j}\right|(x). \tag{4.1}$$

Let $0 < p_0 < \infty$. We write the weak-L^{p_0} quasi norm of a function as

$$\|f\|_{L^{p_0,\infty}} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{1/p_0}.$$

LEMMA 4.1 [5, p. 485, Lemma 2.8]. *Let $0 < p_1 < p_0 < \infty$ and, for each measurable function f, define*

$$N_{p_0,p_1}(f) = \sup_E |E|^{1/p_0 - 1/p_1}\left(\int_E |f(x)|^{p_1}\,dx\right)^{1/p_1},$$

where the supremum is taken over all measurable sets E in $\mathbb{R}^n$ with $0 < |E| < \infty$. Then

$$\|f\|_{L^{p_0,\infty}} \leq N_{p_0,p_1}(f) \leq \left(\frac{p_0}{p_0 - p_1}\right)^{1/p_1}\|f\|_{L^{p_0,\infty}}.$$

LEMMA 4.2. *Let $0 < p_1 < p_0 < \infty$. Then*

$$\|f\|_{p_0,p_1} \leq \left(\frac{p_1}{p_0 - p_1}\right)^{1/p_0}\|f\|_{L^{p_0,\infty}}.$$

Fefferman–Phong-type inequalities. We have the following weighted inequalities.

PROPOSITION 4.3. *Let $0 < q_1 \leq q_0 < n$ and*

$$\begin{aligned} q_1 \leq r \leq n \quad &\text{if } q_1 \leq 1, \\ q_1 < r \leq n \quad &\text{if } q_1 > 1. \end{aligned}$$

Then, for $f \in C_0^\infty(\mathbb{R}^n)$ and a weight function w,

$$\|fw\|_{q_0,q_1} \leq C\|w\|_{n,r}\|M|\nabla f|\|_{q_0,q_1}.$$

PROOF. Using (4.1), to prove this proposition we merely check all the conditions of Theorem 1.2 and the scalar-valued case of Theorem 1.3 when $r_1 = r$, $r_0 = n$ and $\alpha = 1/n$. □

Proposition 4.3 is an extension of the so-called Fefferman–Phong condition obtained in [4].

REMARK. For $n > 1$, the weak-L^1 boundedness of M and Lemma 4.2 give us that

$$\|fw\|_{1,q} \leq C\|w\|_{n,q}\||\nabla f|\|_{L^1} \quad \text{for } 0 < q \leq 1.$$

Olsen-type inequalities. It is known (see [12]) that

$$\|(M_\beta f_\nu)\|_{q_0,q_1,p_2} \leq C\|(f_\nu)\|_{p_0,p_1,p_2}, \tag{4.2}$$

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whenever $1 < p_1 \leq p_0 < \infty$, $1 < p_2 < \infty$, $1 < q_1 \leq q_0 < \infty$, $1/q_0 = 1/p_0 - \beta$ and $q_1/q_0 = p_1/p_0$. This gives the following.

PROPOSITION 4.4. *Suppose that* $0 < \alpha < 1$, $1 < p_1 \leq p_0 < \infty$, $1 < p_2 < \infty$, $1 < q_1 \leq q_0 < \infty$ *and* $1 < r_1 \leq r_0 \leq \infty$. *Suppose also that* $q_1, p_2 < r_1$, $1/p_0 > \alpha$, $1/r_0 \leq \alpha$, $1/q_0 = 1/r_0 + 1/p_0 - \alpha$ *and* $q_1/q_0 = p_1/p_0$. *Then*

$$\|(g_\nu \cdot I_\alpha f_\nu)\|_{q_0,q_1,p_2} \leq C \sup_\mu \|g_\mu\|_{r_0,r_1} \|(f_\nu)\|_{p_0,p_1,p_2}.$$

Proposition 4.4 is a vector-valued extension of the theorem of Olsen [9, Theorem 2]. The proof of the scalar-valued case of Theorem 1.3 gives a new and simple proof of Olsen's theorem.

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