Fourth Meeting, February 14th, 1896.

Dr PEDDIE, President, in the Chair.

Note on a Certain Harmonical Progression.
Note on Continued Fractions.
On Methods of Election.

BY PROFESSOR STEGGALL.

A Simple Method of Finding any Number of Square Numbers whose Sum is a Square.

BY ARTEMAS MARTIN, LL.D.

I.—Take the well-known identity

$$(w+z)^2 = w^2 + 2wz + z^2 = (w-z)^2 + 4wz \quad - \quad - \quad (1).$$

Now if we can transform $4wz$ into a square we shall have *two* square numbers whose sum is a square. This will be effected by taking $w = p^2$, $z = q^2$, for then $4wz = 4p^2q^2 = (2pq)^2$ and we have

$$(p^2+q^2)^2 = (p^2-q^2)^2 + (2pq)^2 \quad - \quad - \quad - \quad (2).$$

See *Mathematical Magazine*, Vol. II., No. 5, p. 69.

In (2) the values of p and q may be chosen at pleasure, but to have numbers that are prime to each other p and q must also be prime to each other and one odd and the other even.

Examples.—1. Take $p = 2$, $q = 1$; then we find

$$3^2 + 4^2 = 5^2.$$

2. Take $p = 3$, $q = 2$; then we shall have

$$5^2 + 12^2 = 13^2.$$

3. Take $p = 4$, $q = 1$; then we get

$$8^2 + 15^2 = 17^2.$$

And so on, *ad lib.*

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II.—We can obtain from (1) any number of squares whose sum is a square by simply substituting for w the sum of two, of three, of four, etc., other quantities.

In (1) put $x+y$ for w and we have

$$(x+y+z)^2 = (x+y-z)^2 + 4(x+y)z,$$
$$= (x+y-z)^2 + 4xz + 4yz,$$
$$= (x+z-y)^2 + 4xy + 4yz,$$
$$= (y+z-x)^2 + 4xy + 4xz.$$

Assume $x=p^2$, $y=q^2$, $z=r^2$, and we have

$$(p^2+q^2+r^2)^2 = (p^2+q^2-r^2)^2 + (2pr)^2 + (2qr)^2,$$
$$= (p^2+r^2-q^2)^2 + (2pq)^2 + (2qr)^2,$$
$$= (q^2+r^2-p^2)^2 + (2pr)^2 + (2pq)^2. \quad - \quad (3),$$

three sets of *three* squares, the sum of each of which is $(p^2+q^2+r^2)^2$, where p, q, r may have any integer values.

See *Mathematical Magazine*, Vol. II., No. 5, p. 72.

Examples.—1. Take $p=4$, $q=2$, $r=1$; then we have

$$21^2 = 19^2 + 8^2 + 4^2 = 16^2 + 13^2 + 4^2 = 16^2 + 11^2 + 8^2.$$

2. Take $p=5$, $q=3$, $r=1$, and we get

$$35^2 = 33^2 + 10^2 + 6^2 = 30^2 + 17^2 + 6^2 = 30^2 + 15^2 + 10^2.$$

3. Take $p=5$, $q=4$, $r=2$, and we find

$$45^2 = 37^2 + 20^2 + 16^2 = 40^2 + 16^2 + 13^2 = 40^2 + 20^2 + 5^2.$$

And so on, *ad lib.*

III.—In (1) put $v+x+y$ for w, and we have

$$(v+x+y+z)^2 = (v+x+y-z)^2 + 4(v+x+y)z,$$
$$= (v+x+y-z)^2 + 4zv + 4zx + 4zy,$$
$$= (v+x+z-y)^2 + 4yz + 4yx + 4yv,$$
$$= (v+y+z-x)^2 + 4xz + 4xy + 4xv,$$
$$= (x+y+z-v)^2 + 4vx + 4vy + 4vz.$$

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Now take $v = p^2$, $x = q^2$, $y = r^2$, $z = s^2$, and we have

$$\begin{aligned}(p^2+q^2+r^2+s^2)^2 &= (p^2+q^2+r^2-s^2)^2+(2ps)^2+(2qs)^2+(2rs)^2,\\ &= (p^2+q^2+s^2-r^2)^2+(2pr)^2+(2qr)^2+(2rs)^2,\\ &= (p^2+r^2+s^2-q^2)^2+(2pq)^2+(2qr)^2+(2qs)^2,\\ &= (q^2+r^2+s^2-p^2)^2+(2pq)^2+(2pr)^2+(2ps)^2. \quad (4),\end{aligned}$$

four sets of *four* square numbers, the sum of each set being $(p^2+q^2+r^2+s^2)^2$, where p, q, r, s may have any integer values.

Examples.—1. Take $p = 5$, $q = 3$, $r = 2$, $s = 1$, and we find

$$\begin{aligned}39^2 &= 37^2+10^2+6^2+4^2 = 31^2+20^2+12^2+4^2\\ &= 30^2+21^2+12^2+6^2 = 30^2+20^2+11^2+10^2\end{aligned}$$

2. Take $p = 5$, $q = 4$, $r = 3$, $s = 1$; then we shall have

$$\begin{aligned}51^2 &= 49^2+10^2-8^2+6^2 = 33^2+30^2+24^2+6^2\\ &= 40^2+24^2+19^2+8^2 = 40^2+30^2+10^2+1^2\end{aligned}$$

3. Take $p = 7$, $q = 5$, $r = 2$, $s = 1$, and we will get

$$\begin{aligned}79^2 &= 77^2+14^2+10^2+4^2 = 71^2+28^2+20^2+4^2\\ &= 70^2+29^2+20^2+10^2 = 70^2+28^2+19^2+14^2\end{aligned}$$

And so on, *ad lib.*

IV.—In the same way we might find formulas for five squares whose sum is a square, *six* squares whose sum is a square, and so on; but from what has been done above it is obvious that we may write at once

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+\ldots+a_n^2)^2 &= (a_1^2+a_2^2+a_3^2+\ldots+a_{n-1}^2-a_n^2)^2+(2a_1a_n)^2\\ &\quad +(2a_2a_n)^2+(2a_3a_n)^2+\ldots\ldots+(2a_{n-1}a_n)^2 \quad (5),\end{aligned}$$

one set of n square numbers whose sum is a square; and we can obtain by cyclic permutation $(n-1)$ other sets, the sum of each of which is equal to $(a_1^2+a_2^2+a_3^2+\ldots\ldots+a_n^2)^2$, where a_1, a_2, $a_3 \ldots\ldots a_n$ may have any integer values chosen at pleasure.