ON SOME PROPERTIES OF FUNCTIONS REGULAR IN THE UNIT CIRCLE

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The space H_p, $1 \leq p \leq \infty$ consists of those analytic functions $f(z)$ regular in the unit circle, for which $M_p(f;r)$ is bounded for $0 \leq r < 1$, where

$$M_p(f;r) = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}, & 1 \leq p < \infty \\ \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|, & p = \infty \end{cases}$$

These spaces have been extensively studied.

One well known result concerning these spaces is that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\{a_n\} \in \ell_p$ for some p, $1 \leq p \leq 2$, then $f \in H_q$, where $p^{-1}+q^{-1} = 1$, and conversely if $f \in H_p$, $1 \leq p \leq 2$, then $\{a_n\} \in \ell_q$. We propose to generalize this result to deal with functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{n^{-\lambda} a_n ; n = 1, 2, \ldots\} \in \ell_p$, where $\lambda \geq 0$. The resulting generalization is contained in the theorems below.

However, in order to make these generalizations we must first generalize the spaces H_p. To this end we make the following definition.

DEFINITION. $H_{o,p} = H_p$. For $\lambda > 0$, $H_{\lambda,p}$ consists of those analytic functions f, regular in the unit circle and such that $M_{\lambda,p}(f)$ is finite, where

$$M_{\lambda,p}(f) = \begin{cases} \int_0^1 (1-r^2)^{q\lambda-1} (M_p(f;r))^q r \, dr, & 1 < p \leq \infty, \ p^{-1}+q^{-1}=1, \\ \sup_{0 \leq r < 1} (1-r^2)^{\lambda} M_1(f;r), & p = 1. \end{cases}$$

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THEOREM 1. If for some p, $1 \leq p \leq 2$, and some $\lambda \geq 0$ $f \in H_{\lambda,p}$ where $f(z) = \sum_0^\infty a_n z^n$, then $\{n^{-\lambda} a_n, n = 1,2,\ldots\} \in \ell_q$ where $p^{-1}+q^{-1} = 1$.

Proof. As mentioned previously, the proof for $\lambda = 0$ is well-known. Let $\lambda > 0$ and suppose first that $p \neq 1$. Then since $M_{\lambda,p}(f) < \infty$, it follows that $M_p(f;r) < \infty$ for almost all r, $0 \leq r < 1$. But by [2], $M_p(f;r)$ is a steadily increasing logarithmicly-convex function of r. Hence $M_p(f;r) < \infty$ for all r, $0 \leq r < 1$. Thus for each r, $0 \leq r < 1$, $f(re^{i\theta}) \in L_p(\theta, 2\pi)$. But

$$f(re^{i\theta}) = \sum_0^\infty a_n r^n e^{in\theta} .$$

Hence by the Hausdorff-Young theorem [3; p. 190], if $0 \leq r < 1$

$$\left(\sum_0^\infty |a_n|^q r^{qn}\right)^{1/q} \leq \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p} = M_p(f;r) ,$$

that is, for $0 \leq r < 1$

$$\sum_1^\infty |a_n|^q r^{qn} \leq (M_p(f;r))^q - |a_0|^q .$$

Multiplying both sides of this last inequality by $r(1-r^2)^{q\lambda-1}$ and integrating from zero to one we obtain

$$\tfrac{1}{2}\Gamma(q\lambda)\sum_1^\infty \frac{\Gamma(1+\frac{1}{2}qn)}{\Gamma(1+q\lambda+\frac{1}{2}qn)} |a_n|^q \leq M_{\lambda,p}(f) - \frac{|a_0|^q}{2q\lambda} < \infty.$$

But from [1; 1.18(4)]

$$\Gamma(1+\tfrac{1}{2}qn)/\Gamma(1+q\lambda+\tfrac{1}{2}qn) \sim (\tfrac{1}{2}qn)^{-q\lambda} \quad \text{as } n \to \infty,$$

so that

$$\sum_1^\infty |n^{-\lambda} a_n|^q < \infty ,$$

and $\{n^{-\lambda} a_n, n = 1,2,\ldots\} \in \ell_q$.

If $p = 1$, we have for $0 < r < 1$ that

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, r^{-n} e^{-in\theta}\, d\theta ,$$

so that

$$|a_n| \leq \frac{r^{-n}}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta = r^{-n} M_1(f;r) .$$

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Hence $(1-r^2)^\lambda r^n |a_n| \leq (1-r^2)^\lambda M_1(f;r) \leq M_{\lambda,1}(f)$.

Thus $\sup_{0 \leq r < 1} (1-r^2)^\lambda r^n |a_n| \leq M_{\lambda,1}(f)$.

But an easy calculation shows that

$$\sup_{0 \leq r < 1} (1-r^2)^\lambda r^n = \left(\frac{2\lambda}{n+2\lambda}\right)^\lambda \left(\frac{n}{n+2\lambda}\right)^{\frac{1}{2}n} \sim e^{-\lambda}(2\lambda)^\lambda n^{-\lambda} \text{ as } n \to \infty$$

so that $n^{-\lambda}|a_n| \leq K$, $n = 1, 2, \ldots$ and

$$\{n^{-\lambda} a_n, \quad n = 1, 2, \ldots\} \in l_\infty .$$

THEOREM 2. If for some p , $1 \leq p \leq 2$, and some $\lambda \geq 0$

$$\{n^{-\lambda} a_n , n = 1, 2, \ldots\} \in l_p , \text{ and } f(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

then $f \in H_{\lambda, q}$ where $p^{-1} + q^{-1} = 1$.

Proof. The series for $f(z)$ clearly converges for $|z| < 1$. The proof for $\lambda = 0$ is well known. Let $\lambda > 0$ and suppose first that $p \neq 1$. Since

$$\sum_1^\infty |n^{-\lambda} a_n|^p < \infty$$

and $$r^{pn} \leq K(r) n^{-\lambda} , \quad 0 \leq r < 1 ,$$

it follows that

$$\sum_1^\infty |a_n|^p r^{pn} < \infty , \quad 0 \leq r < 1 .$$

Hence by [3; p. 190] it follows that there is a function $f(r, \theta)$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) e^{in\theta} \, d\theta = \begin{cases} a_n r^n & n \geq 0 \\ 0 & n < 0 \end{cases} , \quad 0 < r < 1 ,$$

and so that

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(r,\theta)|^q \, d\theta \right\}^{1/q} \leq \left\{ \sum_0^\infty |a_n|^p r^{pn} \right\}^{1/p} .$$

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But clearly if $0 < r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\ e^{in\theta}\,d\theta = \begin{cases} a_n\, r^n & n \geq 0 \\ 0 & n < 0 \end{cases},$$

so that for each such r, $f(r,\theta) = f(re^{i\theta})$ a. e., and our inequality on $f(r,\theta)$ becomes

$$M_q(f;r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q\, d\theta\right\}^{1/q} \leq \left\{\sum_0^{\infty} |a_n|^p\, r^{pn}\right\}^{1/p}.$$

Hence we have

$$(M_q(f;r))^p \leq \sum_0^{\infty} |a_n|^p\ r^{pn}$$

and this inequality remains true for $p = 1$. For then

$$|f(re^{i\theta})| \leq \sum_{n=0}^{\infty} |a_n|\ r^n ,$$

and hence

$$M_{\infty}(f;r) \leq \sum_{n=0}^{\infty} |a_n|\ r^n .$$

Thus we have for any p, $1 \leq p \leq 2$,

$$M_{\lambda,q}(f) = \int_0^1 (1-r^2)^{p\lambda-1}\ (M_q(f;r))^p\ r\ dr$$

$$\leq \tfrac{1}{2}\Gamma(p\lambda)\sum_0^{\infty} \frac{\Gamma(1+\frac{1}{2}pn)}{\Gamma(1+p\lambda+\frac{1}{2}pn)}\ |a_n|^p .$$

But by [1; 1.18(4)],

$$\Gamma(1+\tfrac{1}{2}pn)/\Gamma(1+p\lambda+\tfrac{1}{2}pn) \sim (\tfrac{1}{2}pn)^{p\lambda} ,$$

and thus since

$$\sum_1^{\infty} |n^{-\lambda} a_n|^p < \infty$$

we must have $M_{\lambda,q}(f) < \infty$, and $f \in H_{\lambda,p}$.

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