APPROXIMATE INTEGRATION.

To the Editors of the Journal of the Institute of Actuaries.

SIRS,—On reading the recent issue of the Institute's *Text-book* on Calculus and Probability, it occurred to me that results which I had obtained some time ago in the matter of Approximate Integration might be of interest to actuaries generally, and to actuarial students in particular. This feeling was increased when I noticed that the demonstration given of Weddle's rule on p. 121, was that of obtaining the unique formula for the integration of a sixth degree function, and of adding to it an arbitrary expression.

2. In the course of my investigation I ascertained that Weddle's rule is only one of an infinite number which give a correct integration for any function of the fifth or a lower degree, and further that G. F. Hardy's formulæ, generally known as 37 and 39 (a), are members of the same family.

3. The method which I adopted was that of determining the integrating weights necessary for application to $(n+2)$ equidistant ordinates of a nth degree function. To take a simple example:

Let $$f(x) = a + bx + cx^2 + dx^3$$

then $$f(0) = a$$

$$f(1) = a + b + c + d$$

$$f(2) = a + 2b + 4c + 8d$$

$$f(3) = a + 3b + 9c + 27d$$

$$f(4) = a + 4b + 16c + 64d$$

As $$\int_0^4 f(x) = 4a + 8b + \frac{64}{3}c + 64d$$

the problem may be solved by weighting the values of $f(0)$, &c., adding, and equating the coefficients of a, b, c, and d in the sum with those in the integral.

4. This gives the following equations, where w_0, w_1, &c., are the weights for the respective ordinates:

$$w_0 + w_1 + w_2 + w_3 + w_4 = 4$$

$$w_1 + 2w_2 + 3w_3 + 4w_4 = 8$$

$$w_1 + 4w_2 + 9w_3 + 16w_4 = \frac{64}{3}$$

$$w_1 + 8w_2 + 27w_3 + 64w_4 = 64$$

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A solution of this simultaneous equation gives the following results in terms of w_0:

$$\begin{aligned} w_0 &= \phantom{-\frac{8}{3}} \quad w_0 \\ w_1 &= \frac{8}{3} - 4w_0 \\ w_2 &= -\frac{4}{3} + 6w_0 \\ w_3 &= \frac{8}{3} - 4w_0 \\ w_4 &= \phantom{-\frac{8}{3}} \quad w_0 \end{aligned}$$

Here w_0 may be given any value whatever, but if the weights are all to be positive the value of w_0 must lie between $\frac{2}{9}$ and $\frac{2}{3}$.

5. Giving w_0 the values $\frac{2}{9}$, $\frac{3}{9}$, $\frac{4}{9}$, $\frac{5}{9}$, and $\frac{6}{9}$ respectively, the following weights are obtained, all of which give the correct integral for a third degree equation:

$$\frac{2}{9}\{1, 8, 0, 8, 1\}$$

$$\frac{1}{3}\{1, 4, 2, 4, 1\}$$

$$\frac{4}{9}\{1, 2, 3, 2, 1\}$$

$$\frac{1}{9}\{5, 4, 18, 4, 5\}$$

$$\frac{2}{3}\{1, 0, 4, 0, 1\}$$

The symmetry of these results will be more apparent if the results are written as follows:

$$\frac{1}{9}\{2, 16, 0, 16, 2\}$$

$$\frac{1}{9}\{3, 12, 6, 12, 3\}$$

$$\frac{1}{9}\{4, 8, 12, 8, 4\}$$

$$\frac{1}{9}\{5, 4, 18, 4, 5\}$$

$$\frac{1}{9}\{6, 0, 24, 0, 6\}$$

6. It is here evident that the successive terms of the series within the brackets can be obtained by the continuous addition of 1, -4, 6, -4, 1. The justification for this latter procedure is of

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course the fact, that for any function not higher than the third degree,

$$f(0)-4f(1)+6f(2)-4f(3)+f(4)=0.$$

The above gives Simpson's rule in two different forms, namely,

$$\frac{1}{3}\{1,\ 4,\ 2,\ 4,\ 1\} \text{ and } \frac{2}{3}\{1,\ 0,\ 4,\ 0,\ 1\},$$

besides two other simple expressions:

$$\frac{2}{9}\{1,\ 8,\ 0,\ 8,\ 1\} \text{ and } \frac{4}{9}\{1,\ 2,\ 3,\ 2,\ 1\}\text{—not so well-known.}$$

Probably for most purposes the last-mentioned would give as good results as any.

7. It may be noted here that an infinity of expressions could be readily written down for integrating a third degree function. All that is necessary is to write down any multiple whatever of $\{1,\ 0,\ 4,\ 0,\ 1\}$, to attach the appropriate factor, and to subtract from it continuously $\{1,\ -4,\ 6,\ -4,\ 1\}$.

For example:

$$\frac{2}{15}\{5,\ 0,\ 20,\ 0,\ 5\}$$

$$\frac{2}{15}\{4,\ 4,\ 14,\ 4,\ 4\}$$

$$\frac{2}{15}\{3,\ 8,\ 8,\ 8,\ 3\}$$

$$\frac{2}{15}\{2,\ 12,\ 2,\ 12,\ 2\}$$

The second and fourth of these may be written respectively:

$$\frac{4}{15}\{2,\ 2,\ 7,\ 2,\ 2\} \text{ and } \frac{4}{15}\{1,\ 6,\ 1,\ 6,\ 1\}.$$

In the course of such a process it will be found that the simpler forms are deduced over and over again.

8. If the process of par. 3 be applied to determine the weightings for seven equidistant ordinates in the case of a function of the fifth degree, it may be readily shown, as indicated above, that the weights are as follows:

$$\begin{aligned}
w_0 &= \ w_0\\
w_1 &= 3\cdot 3 - 6w_0\\
w_2 &= -4\cdot 2 + 15w_0\\
w_3 &= 7\cdot 8 - 20w_0\\
w_4 &= -4\cdot 2 + 15w_0\\
w_5 &= 3\cdot 3 - 6w_0\\
w_6 &= \ w_0
\end{aligned}$$

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9. In this case, to give positive weights throughout, w_0 must lie between ·28 and ·39. If w_0 be given the value ·28, G. F. Hardy's formula 37 is obtained, namely,

$$\{\cdot 28,\ 1\cdot 62,\ 0,\ 2\cdot 2,\ 0,\ 1\cdot 62,\ \cdot 28\}$$

while a repetition of it gives formula 39 (*a*), or any further extension that may be desired. If w_0 be given the value 3, the result is Weddle's rule, namely,

$$\cdot 3\{1,\ 5,\ 1,\ 6,\ 1,\ 5,\ 1\}.$$

10. As in the case of the third degree weights investigated above, an infinite number of weights could be deduced, but as far as I have examined them there appears no prospect of obtaining another in this series as simple as Weddle's rule.

11. The formulæ obtainable by giving w_0 successively ·28 to ·39 inclusive are shown hereunder, all of which will give the correct integral for a function of the fifth degree. The first of these is G. F. Hardy's Formula 37, and the third is Weddle's rule. The series may be obtained from the first line by the continuous addition of

$$\frac{1}{100}\{1,\ -6,\ 15,\ -20,\ 15,\ -6,\ 1\}$$

·28,	1·62,	0,	2·2,	0,	1·62,	·28
·29,	1·56,	·15,	2·0,	·15,	1·56,	·29
·30,	1·50,	·30,	1·8,	·30,	1·50,	·30
·31,	1·44,	·45,	1·6,	·45,	1·44,	·31
·32,	1·38,	·60,	1·4,	·60,	1·38,	·32
·33,	1·32,	·75,	1·2,	·75,	1·32,	·33
·34,	1·26,	·90,	1·0,	·90,	1·26,	·34
·35,	1·20,	1·05,	·8,	1·05,	1·20,	·35
·36,	1·14,	1·20,	·6,	1·20,	1·14,	·36
·37,	1·08,	1·35,	·4,	1·35,	1·08,	·37
·38,	1·02,	1·50,	·2,	1·50,	1·02,	·38
·39,	·96,	1·65,	0,	1·65,	·96,	·39

12. The third, sixth, ninth and twelfth of these may be somewhat more simply expressed as follows:

$$\cdot 3\{1,\ 5,\ 1,\ 6,\ 1,\ 5,\ 1\}$$
$$\cdot 3\{1\cdot 1,\ 4\cdot 4,\ 2\cdot 5,\ 4,\ 2\cdot 5,\ 4\cdot 4,\ 1\cdot 1\}$$
$$\cdot 3\{1\cdot 2,\ 3\cdot 8,\ 4,\ 2,\ 4,\ 3\cdot 8,\ 1\cdot 2\}$$
$$\cdot 3\{1\cdot 3,\ 3\cdot 2,\ 5\cdot 5,\ 0,\ 5\cdot 5,\ 3\cdot 2,\ 1\cdot 3\}$$

The second of these has a marked resemblance to Simpson's rule applied three times, which would give:

$$\frac{1}{3}\{1, 4, 2, 4, 2, 4, 1\}.$$

13. In my original investigation I deduced further formulæ expressed in terms of two and of three of the weights, but the results, although curious and interesting, appear to have little practical value.

14. In view of the results here shown it is suggested that Weddle's rule might with advantage be more frequently applied in practice than is at present the case. A strong case could also be made for the consideration of the second formula in paragraph 12. Where the weights are to be applied to a function of the fifth or lower degree, it is immaterial, from the point of view of accuracy, which of the formulæ in paragraph 11 is applied; but where, as is usually the case in actuarial practice, the smoothness is only approximate, it is generally desirable that the equidistant ordinates should all be weighted, and that the weights themselves should form a relatively smooth series. From this point of view the formula

$$\cdot 3\{1\cdot 1, 4\cdot 4, 2\cdot 5, 4, 2\cdot 5, 4\cdot 4, 1\cdot 1\}$$

has some advantages over both Hardy's and Weddle's.

Yours faithfully,

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Commonwealth Bureau of Census and Statistics,
Melbourne.
20 *February* 1923.

HEIGHTS AND WEIGHTS.

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—I enclose tables which have been prepared by Mr. V. W. Tyler, F.I.A., from records supplied by seven important British Life Offices.

These records gave the results of 28,697 medical examinations for life assurance made during the years 1921 and 1922. Only male lives were included, and in all cases the examinations were made in Great Britain or Ireland. It was decided that having regard to the