Probability in the Engineering and Informational Sciences (2025), **39**:4 605–618
doi:10.1017/S0269964825100053

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

On a new family of *r*-modified reliability systems

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Keywords: consecutive-k-out-of-n: F system, k-out-of-n: F system, monotone system, parallel system, series system, sustainability

Abstract
In this work, we focus on stochastic modeling for sustainable systems and introduce the family of r-modified reliability systems. This new family generalizes classical reliability systems studied in the literature by considering the components in the system to exhibit a kind of dependence that relaxes the component operating requirements and provides energy and resource efficiency. From a theoretical viewpoint, such a dependence is modeled with the use of a modified binary sequence. We then derive the reliability of two members of the family, i.e., the r-modified-k-out-of-n:F system and the r-modified-consecutive-k-out-of-n:F system, under different assumptions on the component reliabilities by using a variety of approaches, including Markov chains, combinatorial methods, and simple probabilistic arguments. We finally give some examples of real-life systems wherein the developed models and results are applicable and present the corresponding numerical results.

1. Introduction

A classical framework for modeling real-life engineering systems is by considering a reliability system consisting of n independent components, with each of which being at a specific instance in one of two states: operational (working) or failed (non-working). The state of the system itself (operational or failed) is completely determined by the state of its components according to a pre-fixed criterion. A variety of such criteria have been discussed in the literature, depending on the real-life engineering system under consideration. Each criterion can be imposed under two different (frequently dual) scopes: either determining the system's operational condition (which corresponds to G: systems) or determining the system's failure (which corresponds to F: systems). In this work, we study all reliability systems under the scope of their failure condition (i.e., we study F: systems).

The most fundamental system reliability structures are the series and parallel structures. An engineering system which is modeled by a series structure fails if and only if a single (among all n) component fails. On the other hand, a parallel structure fails if and only if all n components fail. Series and parallel structures are prominent members of two families of reliability systems which have been studied extensively in the literature. We now provide a brief review of these systems.

An n-component system that fails if and only if at least k of the n components fail is called a k-out-of-n: F system. The family of k-out-of-n systems was first introduced by [3] and has been studied extensively since then (see e.g., [4]). For more recent contributions in this area, we refer to [5, 7, 11–14, 34]. A series system is equivalent to an 1-out-of-n: F system, while a parallel system is equivalent to an n-out-of-n: F system.

Let us now recall the second family of systems of interest, introduced by [19]. A consecutive k-out-of-n: F system consists of an ordered sequence of n components for which the existence of at least k consecutive failed components causes system's failure. For $k = 1$, a consecutive 1-out-of-n: F system reduces to a series system, while for $k = n$, a consecutive n-out-of-n: F system reduces to a parallel system. For detailed reviews of consecutive k-out-of-n systems, we refer to [6, 23, 29, 32]. Recent contributions in this area can be found in [9, 21, 25, 33].

Both these families of reliability systems have attracted considerable attention from researchers due to their wide applicability. The k-out-of-n structure is extensively used to model, among others, fault-tolerant industrial and military systems. Consecutive k-out-of-n: F system models have been proposed for real-life engineering systems such as integrated circuits, microwave relay stations in communications, oil pipeline systems, vacuum systems in accelerators, computer ring networks, and so on; see [23]. Next, we describe a real-life engineering system, which can be modeled by using the aforementioned systems.

Let us consider a system of street lights, which are equally spaced on a highway. Each street light (i.e., the system's components) at a specific instance is either operational or failed. There could be different criteria determining the failure of such a system. A first criterion could be that the system fails when k out of n lights fail. On the other hand, the system's failure could be determined by the existence of k consecutive failed lights, a condition which results in a longer part of the street being not lighted sufficiently. Obviously, the first case corresponds to a k-out-of-n:F system, while the second case corresponds to a consecutive-k-out-of-n:F system.

In both cases, the system fails either because there exist several parts of the street or a longer part of the street in which the electric light does not comply with the initial design. On the other hand, an important consideration these days is also to do with sustainability and the construction of more sustainable models, being able to reduce the natural resources and costs. Bearing this in mind, there are cases when the optimal design could be relaxed by thinking that a working system's component can compensate for a number of (probably non-working) adjacent components. In the case of the system of street lights under consideration, this assumption could mean that in some cases (e.g., in daytime when there is limited visibility and the electric lights are required), an operational street light can sufficiently cover a larger area (than the one foreseen by the optimal design) next to it, allowing some of the next lights not to work. Therefore, when a street light works, some of the next street lights are automatically turned off. It is worth noting that probabilistic modeling of smart light systems is of special interest in the literature (see [24]).

Looking for theoretical tools to model reliability systems of this type, we may recall the modified binary sequence introduced by [10] (see also [8, 26, 31]). We shall briefly recall the concept of modified binary sequence. Let us consider a typical sequence of Bernoulli trials with the additional assumption that whenever a success occurs, the sequence locks the next $r-1$ trials (with r being an integer greater than or equal to 1). Therefore, no matter if one is interested in registering successes or failures, she/he will not be able to register in these $r-1$ trials. This idea originates from Geiger counters, which are discrete analogues of the so-called counters of type I and are used for cosmic rays and a-particles (see [15, p. 306]). In all these works, the authors have used the modified binary sequence for registering (runs of) successes. However, in the case of current study, a variation is needed, and we need to register failures since we wish to study F: systems. The combination of the new modified binary sequence, along with the classical reliability systems, will provide the new theoretical framework for modeling sustainable systems of interest.

We, therefore, consider reliability systems consisting of n-ordered components, each of which can either be operational or failed. The occurrence of an operational component will guarantee that the next $r-1$ components can also be considered as non-failed ones when it comes to the evaluation of the system's reliability (since it is only affected by the occurrence of failed components). This assumption will allow the operators to automatically force these $r-1$ components to a non-working state without affecting the system's reliability.

It is then clear that by replacing the classical theoretical framework of independent components in reliability systems with components that exhibit dependency according to the modified sequence described above, a new family of reliability systems gets generated. We refer to this new family as the family of r-modified reliability systems. It is worth noting that the study of systems with components that exhibit stochastic dependency is of great interest in the reliability literature; recent contributions in this regard can be found in [1, 27].

In Section 2, we formally define the family of r-modified reliability systems and discuss its general properties. In Section 3, we obtain the reliability of two members of this family, namely, the r-modified-k-out-of-n:F system and the r-modified-consecutive-k-out-of-n:F system. We show that the newly developed results reduce to the well-known results for the corresponding classical reliability systems. In Section 4, we present some illustrative numerical results. Finally, in Section 5, we discuss the main findings of this work and present some concluding remarks.

2. Family of *r*-modified reliability systems

The discussion in the last section for more sustainable operation of a classical lighting system reveals that if one replaces the assumption of independent components in the classical reliability framework by components that exhibit dependency according to the modified sequence, the family of r-modified reliability systems will be generated.

Hence, any system will belong to the family of r-modified reliability systems if and only if it obeys the following basic assumptions:

(a) The system can be seen as a linear system with n ($n \geq 1$) ordered components;
(b) Each component has three possible states, operational (working), failed (non-working), or locked;
(c) After the occurrence of an operational component, the system locks the next $r-1$ ($r \geq 1$) components, i.e., a working component guarantees that the next $r-1$ components can also be considered as non-failed ones. This assumption would permit the operator to automatically force these $r-1$ components to a non-working state without affecting the system's reliability;
(d) At a specific instance, the reliability (working probability) of the t-th component that is not locked is equal to p_t ($q_t = 1 - p_t$ is the component's failure probability);
(e) The system fails according to a pre-fixed criterion, similar to the ones in the case of classical reliability systems. Typical criteria considered are as follows: the system fails if
    (i) a single component fails, or
    (ii) all n components fail, or
    (iii) at least k of the n components fail, or
    (iv) at least consecutive k of the n components fail.

Depending on the exact form of system's failure criterion (i.e., the different cases of assumption (e)), we will have a different member of the family of r-modified reliability systems. Specifically,

- When in assumption (e), (i) is valid, then we will have an r-modified series system. For $r = 1$, an 1-modified series system, reduces to a typical series system;
- When in assumption (e), (ii) is valid, then we will have an r-modified parallel system. For $r = 1$, an 1-modified parallel system, corresponds to a typical parallel system;
- When in assumption (e), (iii) is valid, then we will have an r-modified-k-out-of-n:F reliability system. For $r = 1$, an 1-modified-k-out-of-n:F system, reduces to a typical k-out-of-n:F system;
- When in assumption (e), (iv) is valid, then we will have an r-modified-consecutive-k-out-of-n:F reliability system. For $r = 1$, an 1-modified-consecutive-k-out-of-n:F system, reduces to a typical consecutive-k-out-of-n:F system.

An r-modified-(consecutive-)k-out-of-n:F system has three design parameters, n, k, and r. The extra parameter r (in comparison to the typical two-parameter (consecutive-)k-out-of-n:F system) can be seen

https://doi.org/10.1017/S0269964825100053 Published online by Cambridge University Press

as a "sustainability" parameter. However, both r-modified series and r-modified parallel systems are special cases of r-modified-k-out-of-n:F system or r-modified-consecutive-k-out-of-n:F system, since an r-modified-(consecutive-)1-out-of-n:F system corresponds to an r-modified series system and an r-modified-(consecutive-)n-out-of-n:F system corresponds to an r-modified parallel system. Therefore, in this work, we will focus on the family of r-modified-(consecutive-)k-out-of-n:F systems and the results developed here will naturally cover their special cases as well. We shall denote the reliability of an r-modified-(consecutive-)k-out-of-n:F system by $R_{n,k,r}(\mathbf{p})$.

For a clear understanding, let us consider the following example. Let us consider the case $r=2$ (a system locks $r-1=1$ component after the occurrence of a working one) and assume that at a specific instance, the state of an r-modified reliability system with $n=10$ ordered components can be represented as follows:

$$S * FS * S * FFS,$$

where S stands for a working component, F stands for a failed component, and $*$ stands for a component (following a working one) when the system is locked. We can then readily observe the following:

- If the above representation corresponds to a 2-modified(-consecutive)-1-out-of-10:F system (2-modified series system), then the system fails;
- If the above representation corresponds to a 2-modified(-consecutive)-10-out-of-10:F system (2-modified parallel system), then the system does not fail;
- If the above representation corresponds to a 2-modified-3-out-of-10:F system, then the system fails;
- If the above representation corresponds to a 2-modified-4-out-of-10:F system, then the system does not fail;
- If the above representation corresponds to a 2-modified-consecutive-2-out-of-10:F system, then the system fails;
- If the above representation corresponds to a 2-modified-consecutive-3-out-of-10:F system, then the system does not fail.

The above representation could, for example, model the system of street lights, described in the Introduction section, at a specific instance during daytime when there is limited visibility. This would mean that the first working street light can guarantee that there is sufficient light in the area next to it and, therefore, the operator can turn off the second street light (from a reliability viewpoint, the second street light is considered as an operating one). This is also the case for the pair of the 4-th and 5-th street-lights, and the 6-th and 7-th ones. It is worth noting that at different time instances, different system components are locked. If, for example, at a different time instance the first street light failed, the second one would not be locked being either a working or failed one.

Having formally introduced the family of r-modified reliability systems, it should be noted that each component of the system has three possible states, denoted by S, F, and $*$. As the systems' failure criteria are dependent only on the occurrence of failed components (i.e., we study F: systems), the states S and $*$ can be considered, from a reliability viewpoint, as non-failed components. Having this in mind, we can see that the following conditions hold:

(a) Any r-modified reliability system works when there are no failures (all its components are S's or $*$'s);
(b) Any r-modified reliability system fails when all its components are "failures" (F's);
(c) The structure function of any r-modified reliability system is nondecreasing.

The validity of the above (a), (b), and (c) conditions classifies r-modified reliability systems in the class of monotone systems (see [22, p. 299]). It is worth noting that examples of monotone systems are found in different fields such as communication, multiprocessor and transportation systems (see [28]).

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In the following two sections, we study the reliability of two members of this family of r-modified:F systems. For this purpose, we will use two different approaches. First, we will take advantage of the fact that the sequence that models any r-modified system exhibits a particular kind of dependence (as described in the current section) which can be modeled by the use of higher-order Markov chains. Therefore, we shall construct such chains and classify the systems under study as Markov chain imbeddable structures (MIS's; [20]). For more information on the finite Markov chain imbedding technique, we refer to [17]; see also [2, 16] and the references therein. Our second approach is combinatorial, taking also advantage of the structure of the sequence under study and the specific failure-patterns of interest.

3. Reliability of r-modified-k-out-of-n:F system

In this section, we study the reliability of r-modified-k-out-of-n:F system, i.e., any system that obeys assumptions (a)–(d) and part (iii) of (e) listed in Section 2.

Proposition 3.1. *The reliability $R_{n,k,r}(\mathbf{p})$ of r-modified-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$) is given by*

$$R_{n,k,r}(\mathbf{p}) = \boldsymbol{\pi}_0 \left(\prod_{t=1}^{n} \Lambda_t \right) \mathbf{1}', \quad n \geq k, \tag{1}$$

where $\boldsymbol{\pi}_0$ denotes an $1 \times k(r+1)$ vector with all its entries being 0 except for the r-th entry which is 1, $\mathbf{1}'$ denotes a $k(r+1) \times 1$ vector with all its entries being 1, and Λ_t is a $k(r+1) \times k(r+1)$ matrix which has all its entries 0 except for the following entries:

- $(r, 1)$*, which is* p_t*;*
- $(1 + ir + j, 2 + ir + j)$*,* $i = 0, \ldots, \min\{k-1, 1\}, j = 0, \ldots, r-2$*, which are all* 1*;*
- $(2r + 2 + (r+1)i + j, 2r + 3 + (r+1)i + j)$*, for* $k \geq 3, i = 0, \ldots, k-3, j = 0, \ldots, r-2$*, which are all* 1*;*
- $(2r + i, r + 1)$*, for* $k \geq 2$*,* $i = 0, 1$*, which are all* p_t*;*
- $(3r + 1 + (r+1)j + i, 2r + 2 + (r+1)j)$*, for* $k \geq 3$*,* $i = 0, 1, j = 0, \ldots, k-3$*, which are all* p_t*;*
- $(r, \min\{k, 2\}r + 1)$*, which is* q_t*;*
- $(2r + i, 3r + 2)$*, for* $k \geq 3$*,* $i = 0, 1$*, which are all* q_t*;*
- $(3r + 1 + (r+1)j + i, 4r + 3 + (r+1)j)$*, for* $k \geq 4$*,* $i = 0, 1, j = 0, \ldots, k-4$*, which are all* q_t*;*
- $(k(r+1) - i, k(r+1))$*, for* $k \geq 2$*,* $i = 1, 2$*, which are all* q_t*;*
- $(k(r+1), k(r+1))$*, which is* 1*.*

Proof. Let us introduce a Markov chain $\{Y_t, t \geq 0\}$, with a finite state space $S = \{0, 0^{(1)}, \ldots, 0^{(r-1)}, 1^{(1)}, \ldots, 1^{(r-1)}, 1^{(S)}, \ldots, (k-1)^{(S)}, 1^{(F)}, \ldots, (k-1)^{(F)}, k\}$ and state k representing an absorbing state, as it corresponds to the failure of the system, as follows:

- $Y_t = 0$ if the t-th component is in a working state and there is no failed component among the preceding components;
- $Y_t = i^{(S)}$ ($Y_t = i^{(F)}$), $i = 1, \ldots, k-1$, if the t-th component is in a working (resp. failed) state and among trials 1 to t, there are i failed components;
- $Y_t = i^{(j)}$, $i = 0, \ldots, k-1$, $j = 1, \ldots, r-1$, if the t-th component is the j-th consecutive locked component and at time $t - j$, the system is either at state $i^{(S)}$ for $i > 0$ or at state 0 for $i = 0$;
- $Y_t = k$ if the t-th component is the k-th failed one (absorbing state).

Under this setup, the system under study is an MIS and the $k(r+1) \times k(r+1)$ one-state transition probability matrix Λ_t has the entries as presented in Proposition 3.1. Therefore, a direct application of

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Theorem 3.1 of [20] readily yields (1), where $\boldsymbol{\pi}_0$ is the vector of initial probabilities of the Markov chain $\{Y_t, t \geq 0\}$. □

It is worth noting that when the Markov chain imbedding technique is applied, the typical convention is that $\boldsymbol{\pi}_0 = (1, 0, \ldots, 0)$. However, in the present case, such a choice would result in a Markov chain $\{Y_t, t \geq 0\}$ that would be locked for $t = 1, \ldots, r-1$. Instead, we do assume that at time $t = 0$, the Markov chain is at state $0^{(r-1)}$ (which ensures that the system is not locked at time $t = 1$) and use the vector $\boldsymbol{\pi}_0$ of Proposition 3.1. This is also the case later on in Proposition 4.1.

In order to get a clear understanding of Proposition 3.1, we now present two examples of the transition probability matrix Λ_t for two different parametrizations. For $k = 3$ and $r = 2$, Λ_t takes on the form

$$\Lambda_t = \left(\begin{array}{cccccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_t & 0 & 0 & 0 & q_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_t & 0 & 0 & 0 & 0 & q_t & 0 \\ 0 & 0 & p_t & 0 & 0 & 0 & 0 & q_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_t & 0 & 0 & q_t \\ 0 & 0 & 0 & 0 & 0 & p_t & 0 & 0 & q_t \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

while for $k = 2$ and $r = 3$, we have

$$\Lambda_t = \left(\begin{array}{ccccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_t & 0 & 0 & 0 & 0 & 0 & q_t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p_t & 0 & 0 & 0 & q_t \\ 0 & 0 & 0 & p_t & 0 & 0 & 0 & q_t \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

We should note that a careful observation of the one-state transition probability matrix Λ_t of Proposition 3.1 reveals that each state $i^{(F)}$, $i = 1, \ldots, k-1$, of the Markov chain can be accumulated at state $i^{(r-1)}$. Such a procedure would reduce the dimension of the square matrix Λ_t by $k-1$ (i.e., it would result in a $(kr+1) \times (kr+1)$ matrix Λ_t). Although we maintained all states of the Markov chain for the construction of the Markov chain to be clearer, such a procedure could be useful for numerical calculations when the values of the design parameters k and r are large.

It is clear that for $r = 1$, Proposition 3.1 provides the reliability of a classical k-out-of-n:F system. Moreover, for $k = 1$, it provides the reliability of an r-modified series system, while, for $k = n$, it provides the reliability of an r-modified parallel system.

Proposition 3.1 provides a closed and numerically efficient formula for the calculation of the reliability of an r-modified reliability system through the use of the transition probability matrix in the general case when its non-locked components do not necessarily have equal reliabilities. In the special case when the reliabilities of the non-locked components are equal, different formulas for the calculation of the reliability of the system, which do not involve matrices, can also be derived. We shall first derive a closed-form expression for the reliability of the system based on combinatorial arguments.

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Proposition 3.2. *The reliability $R_{n,k,r}(p)$ of r-modified-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$), when the non-locked components have constant and equal reliabilities $p_t = p$, is given, for $n \geq k$, by*

$$R_{n,k,r}(p) = \sum_{j=0}^{r-1} \sum_{y=\lceil \frac{n-k-j}{r} \rceil}^{\lfloor \frac{n-1-j}{r} \rfloor} \binom{n-1-j-y(r-1)}{y} p^{y+1} q^{n-1-j-yr} + \sum_{i=1}^{k-1} \sum_{y=\lceil \frac{n-r-k+1}{r} \rceil}^{\lfloor \frac{n-r-i}{r} \rfloor} \binom{n-r-i-y(r-1)}{y} p^{y+1} q^{n-(y+1)r}, \tag{2}$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the greatest (least) integer less (greater) than or equal to x.

Proof. The states of the system's components at a specific time instance can be considered as a realization of a sequence of n trials. Let j be the number of trials following the last S in the sequence. We may consider the cases $0 \leq j \leq r-1$ and $j \geq r$.

The case $0 \leq j \leq r-1$ corresponds to the event that there are no Fs following the last S of the sequence. Let y be the number of Ss preceding the last S, i.e., the number of Ss in the sequence of the first $n-1-j$ trials. The number of ways that the y Ss and the $n-1-j-yr$ Fs can appear in the sequence of the first $n-1-j$ trials is given by $\binom{(n-1-j-yr)+y}{y}$. Then, a k-out-of-n: F system works if and only if $n-1-j-yr \leq k-1$. Moreover, because of the internal structure of the sequence, we have $yr \leq n-1-j$. Fix y and j. Then, the probability for the sequence of n trials to appear is equal to $p^{y+1}q^{n-1-j-yr}$. Summing now with respect to j and y, we get the first term of (2).

On the other hand, for $j \geq r$, there are i, $i = 1, 2, \ldots, k-1$, Fs following the last S in the sequence ($j = r-1+i$). Again, let y be the number of Ss preceding the last S in the sequence. Next, the number of different ways in which the $n-r-i-yr$ Fs can appear in the sequence of the first $n-r-i$ trials is given by $\binom{n-r-i-yr+y}{y}$. Note that $yr \leq n-r-i$ and for the total number of Fs in the sequence, we need to have $(n-r-i-yr)+i \leq k-1$. It is also clear that for a fixed y, such a sequence has probability $p^{y+1}q^{n-(y+1)r}$ to appear. Then, summing with respect to i and y, we obtain the second term of (2). Hence, the proposition. □

In the case of equal (non-locked) component reliabilities, we can also derive the following simple recursive relation for $R_{n,k,r}(p)$ by conditioning on the number of failed components before the first occurrence of a working one in the system of n components.

Proposition 3.3. *The reliability $R_{n,k,r}(p)$ of the r-modified-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$), when the non-locked components have constant and equal reliabilities $p_t = p$, satisfies the recursive scheme*

$$R_{n,k,r}(p) = p \sum_{i=0}^{k-1} q^i R_{n-i-r,k-i,r}(p), \quad n \geq k, \tag{3}$$

with initial conditions $R_{n,k,r}(p) = 1$, for $n < k$.

4. Reliability of r-modified-consecutive-k-out-of-n:F system

In this section, we study the reliability of r-modified-consecutive-k-out-of-n:F system, i.e., any system that obeys the assumptions (a)–(d) and part (iv) of (e) listed earlier in Section 2.

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Proposition 4.1. *The reliability $R_{n,k,r}(\mathbf{p})$ of r-modified-consecutive-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$) is given by*

$$R_{n,k,r}(\mathbf{p}) = \boldsymbol{\pi}_0 \left(\prod_{t=1}^{n} \Lambda_t \right) \mathbf{1}', \tag{4}$$

where $\boldsymbol{\pi}_0$ denotes an $1 \times (k+r)$ vector with all its entries being 0 except for the r-th entry which is 1, $\mathbf{1}'$ denotes a $(k+r) \times 1$ vector with all its entries being 1 and Λ_t is a $(k+r) \times (k+r)$ matrix which has all its entries as 0 except for the following entries:

- $(i, i+1)$, $i = 1, \ldots, r-1$, *which are all* 1*;*
- $(i, 1)$, $i = r, \ldots, k+r-1$, *which are all* p_t*;*
- $(i, i+1)$, $i = r, \ldots, k+r-1$, *which are all* q_t*;*
- $(k+r, k+r)$, *which is* 1.

Proof. Let us introduce a Markov chain $\{Y_t, t \geq 0\}$, with a finite state space $S = \{0, 0^{(1)}, \ldots, 0^{(r-1)}, 1, \ldots, k\}$ and state k representing an absorbing state as it corresponds to the failure of the system, as follows:

- $Y_t = 0$ if the t-th component is in a working state;
- $Y_t = 0^{(i)}$, $i = 1, \ldots, r-1$, if the t-th component is the r-th consecutive component which is locked and the $t-i$-th component is in a working state;
- $Y_t = i$, $i = 1, \ldots, k$, if the t-th component is the i-th consecutive component which is in a failed state and the $t-i$-th component (if there is one) is a locked one.

Then, the one-state transition probability matrix can be written as

$$\Lambda_t = \left(\begin{array}{cccccccc|c} 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_t & 0 & 0 & \ldots & 0 & q_t & 0 & \ldots & 0 \\ p_t & 0 & 0 & \cdot & 0 & 0 & q_t & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_t & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & q_t \\ \hline 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{array} \right)_{(k+r)\times(k+r)},$$

which has the entries as described above. Under this setup, Theorem 3.1 of [20] for an MIS readily yields (4), where $\boldsymbol{\pi}_0$ is the vector of initial probabilities of the Markov chain $\{Y_t, t \geq 0\}$. □

For illustrative purposes, let us consider a specific configuration of the one-state transition probability matrix Λ_t. For $k = 3$ and $r = 2$, Λ_t takes on the form

$$\Lambda_t = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ p_t & 0 & q_t & 0 & 0 \\ p_t & 0 & 0 & q_t & 0 \\ p_t & 0 & 0 & 0 & q_t \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

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Special cases of the r-modified-consecutive-k-out-of-n:F system are also of interest. For $r = 1$, Proposition 4.1 provides the reliability of a classical consecutive-k-out-of-n:F system. For $k = 1$, it provides the reliability of an r-modified series system and for $k = n$, it provides the reliability of an r-modified parallel system.

We now focus on the special case in which the reliabilities of the non-locked components of an r-modified-consecutive-k-out-of-n:F system are constant and equal to p. A combinatorial approach can also be applied in this case in order to derive a closed-form expression for the reliability of the system not involving matrices. For this purpose, we first recall the following well-known result.

Corollary 4.2. [30]. *The number of allocations of α indistinguishable balls into m distinguishable cells, so that each cell is occupied by at most k balls, is given by*

$$C(\alpha, m, k) = \sum_{j=0}^{\lfloor \frac{\alpha}{k+1} \rfloor} (-1)^j \binom{m}{j} \binom{\alpha - (k+1)j + m - 1}{\alpha - (k+1)j}.$$

With Corollary 4.2, we now proceed to obtain the following proposition.

Proposition 4.3. *The reliability $R_{n,k,r}(p)$ of r-modified-consecutive-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$), when the non-locked components have constant and equal reliabilities $p_t = p$, is given, for $n \geq k$, by*

$$\begin{aligned} R_{n,k,r}(p) \quad = \quad & \sum_{j=0}^{r-1} \sum_{y=1}^{1+\lfloor \frac{n-1-j}{r} \rfloor} C(n-(y-1)r-j-1, y, k-1) p^y q^{n-1-(y-1)r-j} \\ & + \sum_{j=1}^{k-1} \sum_{y=1}^{\lfloor \frac{n-j}{r} \rfloor} C(n-yr-j, y, k-1) p^y q^{n-yr}. \end{aligned} \tag{5}$$

Proof. As in the proof of Proposition 3.2, the states of the components of the system can be considered as a realization of a sequence of n trials. Let j be the number of trials following the last S in the sequence.

For $0 \leq j \leq r-1$, let us denote by y the total number of Ss in the sequence. We consider that among the y ($1 \leq y \leq 1 + \lfloor \frac{n-1-j}{r} \rfloor$) Ss, y cells are created in which the $n-(y-1)r-1-j$ Fs are allocated so that no cell receives more than $k-1$ Fs. By using Corollary 4.2, there are $C(n-(y-1)r-1-j, y, k-1)$ such different allocations. Fix y and j. Then, the probability of occurrence of each sequence, which has been created, is equal to $p^y q^{n-(y-1)r-j-1}$.

For $j \geq r$, let us again denote by y the number of Ss in the sequence. Then, the last $i = j - r + 1$ trials are Fs. Among the y Ss, $1 \leq y \leq \lfloor \frac{n-i}{r} \rfloor$, y cells are created in which the $n - yr - i$ Fs are allocated so that no cell receives more than $k-1$ Fs. This can be accomplished in $C(n-ry-i, y, k-1)$ ways and each created sequence has probability $p^y q^{n-yr}$. Summing the probabilities of all possible sequences, we obtain the reliability of the system, as given in (5). □

We mention that $\binom{n}{m}$ here denotes the extended binomial coefficient; see [15, p. 50].

Moreover, the following proposition (analogous to Proposition 3.3) can be established by conditioning on the number of failed components before the first occurrence of a working one in the sequence of n components.

Proposition 4.4. *The reliability $R_{n,k,r}(p)$ of r-modified-consecutive-k-out-of-n:F system ($k \geq 1$ and $r \geq 1$), when the non-locked components have constant and equal reliabilities $p_t = p$, satisfies the recursive scheme*

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Table 1. *Exact reliabilities of an r-modified-k-out-of-n:F system with $n = 32$ components and equal component reliabilities $p_t = p = 0.90$ (when the components are not locked).*

k	r	System type	$R_{32,k,r}(0.90)$
1	1	Series system	0.034337
	2	2-modified series system	0.185302
	3	3-modified series system	0.313811
2	1	2-out-of-32:F system	0.156423
	2	2-modified-2-out-of-32:F system	0.481785
	3	3-modified-2-out-of-32:F system	0.659002
4	1	4-out-of-32:F system	0.600306
	2	2-modified-4-out-of-32:F system	0.901803
	3	3-modified-4-out-of-32:F system	0.965839
8	1	8-out-of-32:F system	0.988315
	2	2-modified-8-out-of-32:F system	0.999584
	3	3-modified-8-out-of-32:F system	0.999939
16	1	16-out-of-32:F system	1
	2	2-modified-16-out-of-32:F system	1
	3	3-modified-16-out-of-32:F system	1
32	1	Parallel system	1
	2	2-modified parallel system	1
	3	3-modified parallel system	1

$$R_{n,k,r}(p) = p \sum_{i=0}^{k-1} q^i R_{n-i-r,k,r}(p), \quad n \geq k, \tag{6}$$

with initial conditions $R_{n,k,r}(p) = 1$, for $n < k$.

5. Numerical results

In this section, we present some illustrative numerical examples for the members of the family of r-modified reliability systems.

Let us first consider a system consisting of $n = 32$ micro fans along its surface that protect it from overheating. Under specific environmental circumstances (e.g., when the temperature is below a prefixed threshold), an operating micro fan can substitute the operation of the next $r - 1$ micro fans and, therefore, their operation is automatically terminated. We assume that the reliability of each micro fan is constant and equal to $p = 0.90$, unless it is automatically forced to a non-working state. The system fails if and only if at least k of the n micro fans fail. It is then evident that this system corresponds to an r-modified-k-out-of-n:F system.

As a second example, let us recall the lighting system described in the Introduction. Let us assume that it also consists of $n = 32$ components (street lights). When there is restricted visibility during daytime, an operational street light can substitute the operation of the next $r - 1$ lights, which are automatically turned off. We assume again that the reliability of each street light is constant and equal to $p = 0.90$, unless it is automatically turned off. The system fails if quite a large area of the road is not sufficiently lighted, say, there are k consecutive failed lights. This system corresponds to an r-modified-consecutive-k-out-of-n:F system.

We now provide illustrative numerical results for the whole family of r-modified reliability systems by using the results established in Sections 3 and 4. Tables 1 and 2 provide the reliabilities of an r-modified-k-out-of-n:F system and an r-modified-consecutive-k-out-of-n:F system, respectively, for

Table 2. *Exact reliabilities of an r-modified-consecutive-k-out-of-n:F system with $n = 32$ components and equal component reliabilities $p_t = p = 0.90$ (when the components are not locked).*

k	r	System type	$R_{32,k,r}(0.90)$
1	1	Series system	0.034337
	2	2-modified series system	0.185302
	3	3-modified series system	0.313811
2	1	consecutive-2-out-of-32:F system	0.751062
	2	2-modified-consecutive-2-out-of-32:F system	0.859152
	3	3-modified-consecutive-2-out-of-32:F system	0.900949
4	1	consecutive-4-out-of-32:F system	0.997382
	2	2-modified-consecutive-4-out-of-32:F system	0.998598
	3	3-modified-consecutive-4-out-of-32:F system	0.999027
8	1	consecutive-8-out-of-32:F system	1
	2	2-modified-consecutive-8-out-of-32:F system	1
	3	3-modified-consecutive-8-out-of-32:F system	1
16	1	consecutive-16-out-of-32:F system	1
	2	2-modified-consecutive-16-out-of-32:F system	1
	3	3-modified-consecutive-16-out-of-32:F system	1
32	1	Parallel system	1
	2	2-modified parallel system	1
	3	3-modified parallel system	1

the specific configurations of the aforementioned examples and different choices of design parameters k and r.

The numerical results presented in Tables 1 and 2 verify the following:

- For n, r, and p fixed, the reliabilities of r-modified-k-out-of-n:F system and r-modified-consecutive-k-out-of-n:F system are both increasing functions of k;
- For n, k, and p fixed, the reliabilities of r-modified-k-out-of-n:F system and r-modified-consecutive-k-out-of-n:F system are both increasing functions of r;
- For n, k, r, and p fixed, the reliability of r-modified-consecutive-k-out-of-n:F system is greater than or equal to the reliability of r-modified-k-out-of-n:F system.

A graphical display of the last statement is provided in Figure 1, which plots the reliability of two different members of the family of r-modified reliability systems for a specific configuration of the design parameters n, k, and r with respect to the constant component reliability p (when the component is not locked).

On the other hand, Figures 2 and 3 depict the effect of the increase in the sustainability parameter r on the increase in the reliability of the system. In each plot, the lower line depicts the reliability of the classical 4-out-of-16 and the consecutive-4-out-of-16 reliability systems, respectively. Therefore, one may easily observe that the new modified systems outperform the classical non-modified ones. Last observations do pose another interesting question: Could a guidance be given to practitioners about the appropriate value of the sustainability parameter in every practical application? Undoubtedly, the answer has to do with the specific system reliability one wishes to achieve. To make that statement clearer, we provide in Table 3 the reliabilities of r-modified-4-out-of-16 and r-modified-consecutive-4-out-of-16 systems for constant component reliability $p = 0.75$, as r increases. The entries of Table 3 reveal that if, for example, a practitioner is interested in achieving a system reliability of at least 98% when modeling a real-life engineering system by the use of an r-modified-4-out-of-16 system (e.g., the micro fans systems

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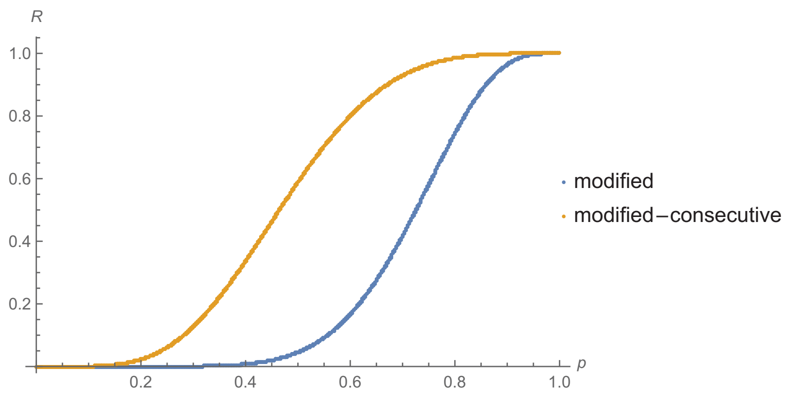

Figure 1. *The reliability of the* 3*-modified-consecutive-*4*-out-of-*32 *reliability system (upper yellow line) and the reliability of the* 3*-modified-*4*-out-of-*32 *reliability system (lower blue line) with respect to constant component reliability p (when the component is not locked).*

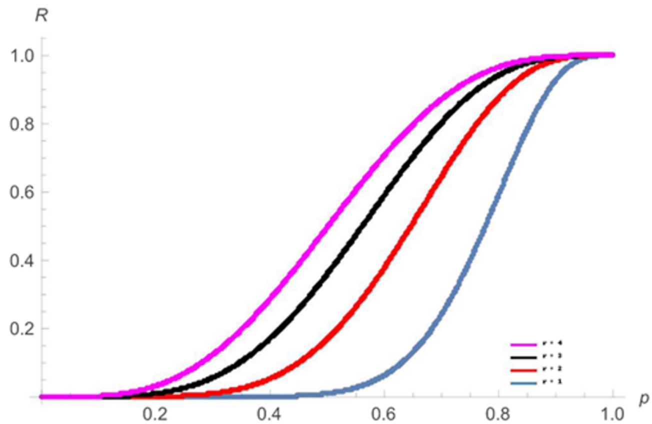

Figure 2. *The reliability of the r-modified-*4*-out-of-*16 *reliability system for* $r = 1$*,* $r = 2$*,* $r = 3$*, and* $r = 4$ *(from lower to the upper line, respectively) with respect to constant component reliability p (when the component is not locked).*

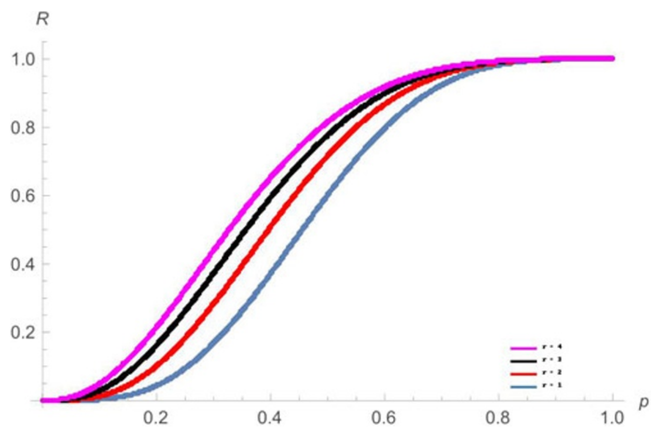

Figure 3. *The reliability of the r-modified-consecutive-*4*-out-of-*16 *reliability system for* $r = 1$*,* $r = 2$*,* $r = 3$*, and* $r = 4$ *(from lower to the upper line, respectively) with respect to constant component reliability p (when the component is not locked).*

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Table 3. *Exact reliabilities of an r-modified-4-out-of-16:F system and an r-modified-consecutive-4-out-of-16:F system with equal component reliabilities $p_t = p = 0.75$ (when the components are not locked) for various values of r.*

r	r-modified-4-out-of-16:F system	r-modified-consecutive-4-out-of-16:F
1	0.404987	0.961269
2	0.775875	0.977037
3	0.886185	0.983332
4	0.929443	0.986684
5	0.962402	0.988495
6	0.962402	0.990005
7	0.984375	0.992203

described in the beginning of Section 5), then she/he should use a value of r of at least 7. On the other hand, if the real-life engineering system was modeled by the use of an r-modified-consecutive-4-out-of-16 system (e.g., the lighting system described before in the current section), then using $r = 3$ would achieve the practitioner's goal. It should be stressed that, theoretically speaking, larger values could be assigned to sustainability parameter r (than the ones presented in Table 3), but the results do not get significantly affected.

A careful observation to be noted from both Figures 2 and 3 is that the interval in which each system's reliability is greater than the common reliability of its components increases as r increases.

6. Concluding remarks

In this paper, we have introduced and studied the family of r-modified reliability systems, which generalizes the (consecutive-)k-out-of-n:F system by adding an extra "sustainability" parameter r to the set of the system's design parameters n and k. We have derived the reliability of two basic members of this family by constructing suitable Markov chains or by using a direct combinatorial approach. We have presented examples of "sustainable" systems, which can be modeled using this new theoretical framework, along with illustrative numerical results, which reveal that the members of the new family outperform the classical reliability systems studied in the literature.

This is the first and introductory work on the family of r-modified reliability systems. Different traditional reliability systems can also be studied by assuming that their components exhibit dependence generated by the modified sequence. A straightforward example is the introduction and study of the r-modified-m-within-consecutive-k-out-of-n: F systems (for the m-within-consecutive-k-out-of-n: F systems, one may see [18]). Furthermore, the present work can also be extended to the case of circular r-modified reliability systems. We are currently working on these problems and hope to report the findings in a future paper.

Acknowledgements. The authors would like to thank the editor and the referees for their many helpful suggestions and comments on an earlier version of this manuscript, which resulted in this much improved version.

Conflict of interest. The authors declare no competing interests.

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Cite this article: Balakrishnan N., Dafnis SD. and Makri FS. (2025). On a new family of r-modified reliability systems. *Probability in the Engineering and Informational Sciences* 39(4): 605–618. https://doi.org/10.1017/S0269964825100053