ASYMPTOTIC FORMULAE FOR LINEAR EQUATIONS

by DON B. HINTON

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Numerous formulae have been given which exhibit the asymptotic behaviour as $t \to \infty$ of solutions of

$$x'' + F(t)x = 0,$$

where $F(t)$ is essentially positive and $\int^{\infty} tF(t)\,dt = \infty$. Several of these results have been unified by a theorem of F. V. Atkinson [1]. It is the purpose of this paper to establish results, analogous to the theorem of Atkinson, for the third order equation

$$x''' + F(t)x = 0, \tag{1}$$

and for the fourth order equation

$$x^{(\mathrm{iv})} + F(t)x = 0. \tag{2}$$

However, rather than assume that $F(t)$ is essentially positive, we shall instead assume that $F(t)$ is essentially of one sign. We assume that F has a decomposition for either $l = 1$ or $l = 2$,

$$(-1)^{l+1}F(t) = f(t) + m(t), \tag{3}$$

where $f(t)$ is a " smooth " part of $F(t)$ and $m(t)$ is a " small " part. It is assumed throughout that $f(t)$ and $m(t)$ are continuous on a ray $[a, \infty)$ with $f(t) > 0$ and continuously differentiable. The analysis is similar to that used in [4] for a two term nth order equation with sufficiently smooth coefficients.

It is convenient to express (1) and (2) in the vector forms

$$S' = MS \tag{4}$$

and

$$T' = NT \tag{5}$$

where S, M, T, and N are, respectively

$$\begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -F & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x \\ x' \\ x'' \\ x''' \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -F & 0 & 0 & 0 \end{bmatrix}.$$

THEOREM 1. *Let*

$$f'f^{-4/3} = h(t) + h_1(t) \tag{6}$$

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and suppose that the following conditions hold:

$$\int_a^\infty f^{-2/3}|m|\,dt < \infty, \tag{7}$$

$$\int_a^\infty f^{1/3}|h_1|\,dt < \infty, \tag{8}$$

and

$$\int_a^\infty |h'|\,dt < \infty \quad \textit{with} \quad h(\infty)^2 \neq 27.4^{-1/3}. \tag{9}$$

Then there is a fundamental matrix $S(t)$ *of* (4) *and a* t_0 *such that, as* $t \to \infty$,

$$\begin{bmatrix} f^{1/3}(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f^{-1/3}(t) \end{bmatrix} S(t) \begin{bmatrix} \exp(-\Lambda_1(t)) & 0 & 0 \\ 0 & \exp(-\Lambda_2(t)) & 0 \\ 0 & 0 & \exp(-\Lambda_3(t)) \end{bmatrix} \to L, \tag{10}$$

where, for $i = 1, 2, 3$,

$$\Lambda_i(t) = \int_{t_0}^t f^{1/3}(\tau)\lambda_i(\tau)\,d\tau,$$

$$\lambda_i(\tau) = w_i(\tau) + h(\tau)^2/27w_i(\tau),$$

$$w_i(\tau) = \mu_i[(1+\{1-4h(\tau)^6/3^9\}^{1/2})/2]^{1/3},$$

μ_1, μ_2, μ_3 *are the cube roots of* $(-1)^l$, *and* $L = \{l_{ij}\}$ *is the* 3×3 *matrix given by* $l_{1j} = 1$, $l_{2j} = \lambda_j(\infty) - h(\infty)/3$, *and* $l_{3j} = \lambda_j(\infty) l_{2j}$ *for* $j = 1, 2, 3$.

Proof. We first transform (4) by defining $Z = QS$, where Q is the diagonal matrix $Q = \text{diag}[f^{1/3}, 1, f^{-1/3}]$. Then

$$\begin{aligned} Z' &= [QMQ^{-1} + Q'Q^{-1}]Z \\ &= f^{1/3}(A+B+C)Z, \end{aligned} \tag{11}$$

where

$$A = \begin{bmatrix} h(\infty)/3 & 1 & 0 \\ 0 & 0 & 1 \\ (-1)^l & 0 & -h(\infty)/3 \end{bmatrix}, \quad B = [h(t)-h(\infty)] \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} h_1(t)/3 & 0 & 0 \\ 0 & 0 & 0 \\ (-1)^l m(t)/f(t) & 0 & -h_1(t)/3 \end{bmatrix}.$$

If $\int_a^\infty f^{1/3}\,dt < \infty$, it follows from (6), (8), and (9) that $\int_a^\infty |f'/f|\,dt < \infty$; hence $\log f$ has a limit at ∞. Thus f has a positive lower bound, contrary to $\int^\infty f^{1/3}\,dt < \infty$. Let $k(t) = \int_a^t f^{1/3}\,d\tau$, and denote the inverse of k by g. The change of variable $W(s) = Z(g(s))$ in (11) yields for W,

$$W'(s) = [A + \tilde{B}(s) + \tilde{C}(s)]W(s), \tag{12}$$

where $\tilde{B}(s) = B(g(s))$ and $\tilde{C}(s) = C(g(s))$. By conditions (7) and (8),

$$\int_0^\infty |\tilde{C}(s)|\,ds = \int_a^\infty f^{1/3}(t)\,|C(t)|\,dt < \infty.$$

By condition (9), $\tilde{B}(s) \to 0$ as $s \to \infty$ and $\int_0^\infty |\tilde{B}'(s)|\,ds = \int_a^\infty |B'(t)|\,dt < \infty$. Hence, if the characteristic roots of A are distinct and the real parts of the roots of $A + \tilde{B}(s)$ are well-behaved, we may apply the asymptotic theorem due to Levinson [3, Chap. 3, Theorem 8.1]. A calculation shows that the roots $\tilde{\lambda}(s)$ of $A + \tilde{B}(s)$ satisfy the equation

$$\lambda^3 - \lambda h(g(s))^2/9 - (-1)^l = 0. \tag{13}$$

By recalling that, if $w \neq 0$ satisfies the equation

$$w^3 = -q/2 \pm \sqrt{q^2/4 + p^3/27}, \tag{14}$$

then $z = w - p/3w$ satisfies the equation $z^3 + pz + q = 0$ (cf. [2, p. 112]), the roots of (13) may be written (for $l = 1$ use the $-$ sign in (14) and for $l = 2$ use the $+$ sign), for $i = 1, 2, 3$,

$$\tilde{\lambda}_i(s) = \tilde{w}_i(s) + h(g(s))^2/27\tilde{w}_i(s), \tag{15}$$

where

$$\tilde{w}_i(s) = \mu_i[(1 + \{1 - 4h(g(s))^6/3^9\}^{1/2})/2]^{1/3}, \tag{16}$$

and μ_1, μ_2, and μ_3 are the cube roots of $(-1)^l$. For $4h(g(s))^6 > 3^9$ in (16), the exponent 1/2 denotes the root in the upper half plane and the exponent 1/3 denotes the first quadrant root. Thus the characteristic roots of A are distinct and a short calculation shows that the columns of L are characteristic vectors of A. From (13) and $h(\infty)^2 \neq 27.4^{-1/3}$, it follows that the roots $\tilde{\lambda}_i(s)$ (s sufficiently large and $\leqq \infty$) must occur in one of the following combinations: (i) one negative root and a pair of complex conjugate roots with positive real part, (ii) one positive root and a pair of complex conjugate roots with negative real part, or (iii) three distinct real roots. In either case, we have, for each i, j, that either $\operatorname{Re}[\tilde{\lambda}_i(s) - \tilde{\lambda}_j(s)] \equiv 0$ or $\operatorname{Re}[\tilde{\lambda}_i(s) - \tilde{\lambda}_j(s)] \to$ a nonzero constant as $s \to \infty$; thus the theorem of Levinson applies. There is then a number s_0 and a fundamental matrix W of (12) such that, as $s \to \infty$,

$$W(s)\begin{bmatrix} \exp\left(-\int_{s_0}^s \tilde{\lambda}_1(u)\,du\right) & 0 & 0 \\ 0 & \exp\left(-\int_{s_0}^s \tilde{\lambda}_2(u)\,du\right) & 0 \\ 0 & 0 & \exp\left(-\int_{s_0}^s \tilde{\lambda}_3(u)\,du\right) \end{bmatrix} \to L.$$

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Since $W(k(t)) = Q(t)S(t)$ and (for $t_0 = g(s_0)$)

$$\int_{s_0}^{k(t)} \tilde{\lambda}_i(u)\,du = \int_{t_0}^{t} f^{1/3}(\tau)\lambda_i(\tau)\,d\tau,$$

the above asymptotic behaviour for W yields (10) for S.

For the perturbed Euler equation

$$x''' + (K/t^3 + \xi)x = 0,$$

Theorem 1 is applicable if $\int^\infty t^2 |\xi|\,dt < \infty$ and $K^2 \neq 4/27$. Also applicable to perturbations of the Euler equation is the result that, if $m(t) = 0$ and $\int_a^\infty |(tf^{1/3})'|\,dt < \infty$, with $tf^{1/3}$ tending as $t \to \infty$ to a positive limit L, $L^6 \neq 4/27$, then the hypothesis of Theorem 1 is satisfied with $h(t) = -3/(tf^{1/3})$ and $h_1(t) = 3(tf^{1/3})'/(tf^{2/3})$.

THEOREM 2. *Let*

$$f'f^{-5/4} = h(t) + h_1(t)$$

and suppose that the following conditions hold:

$$\int_a^\infty f^{-3/4}|m|\,dt < \infty,$$

$$\int_a^\infty f^{1/4}|h_1|\,dt < \infty,$$

and

$$\int_a^\infty |h'|\,dt < \infty,$$

with $h(\infty)^2 \neq 16$ *for* $l = 1$, $h(\infty)^2 \neq 64/3$ *for* $l = 2$. *Then there is a number* t_0 *and a fundamental matrix* T *of* (5) *such that, as* $t \to \infty$,

$$Q(t)T(t)E(t) \to K, \tag{17}$$

where $Q(t)$ *and* $E(t)$ *are the diagonal matrices*

$$Q(t) = \mathrm{diag}\,[f(t)^{3/8}, f(t)^{1/8}, f(t)^{-1/8}, f(t)^{-3/8}] \tag{18}$$

and

$$E(t) = \mathrm{diag}\,[\exp(-\Lambda_1(t)), \ldots, \exp(-\Lambda_4(t))],$$

in which, for $i = 1, 2, 3, 4$,

$$\Lambda_i(t) = \int_{t_0}^{t} f^{1/4}(\tau)\lambda_i(\tau)\,d\tau,$$

and the $\lambda_i(\tau)$ *are the* 4 *roots of the equation*

$$\lambda^2 = 5h(\tau)^2/64 \pm \{(-1)^l + h(\tau)^4/4^4\}^{1/2}.$$

Also $K = \{k_{ij}\}$ *is given by* $k_{1j} = 1$, $k_{2j} = \lambda_j(\infty) - 3h(\infty)/8$, $k_{3j} = \lambda_j(\infty)k_{2j} - h(\infty)/8$, *and* $k_{4j} = \lambda_j(\infty)k_{3j} + h(\infty)/8$ *for* $j = 1, 2, 3, 4$.

Proof. The proof is similar to that of Theorem 1. The transformation $Z = QT$ with Q as in (18) yields

$$Z' = f^{1/4}(A+B+C)Z$$

where

$$A = \begin{bmatrix} 3h(\infty)/8 & 1 & 0 & 0 \\ 0 & h(\infty)/8 & 1 & 0 \\ 0 & 0 & -h(\infty)/8 & 1 \\ (-1)^l & 0 & 0 & -3h(\infty)/8 \end{bmatrix},$$

$$B = [h(t) - h(\infty)] \operatorname{diag}[3/8, 1/8, -1/8, -3/8],$$

and

$$C = \begin{bmatrix} 3h_1(t)/8 & 0 & 0 & 0 \\ 0 & h_1(t)/8 & 0 & 0 \\ 0 & 0 & -h_1(t)/8 & 0 \\ (-1)^l m(t)/f(t) & 0 & 0 & -3h_1(t)/8 \end{bmatrix}.$$

Define $k(t) = \int_a^t f^{1/4}\,d\tau$, and let g be the inverse of k. As in Theorem 1, $k(t) \to \infty$ as $t \to \infty$ and, if $W(s) = Z(g(s))$, then

$$W'(s) = [A + \tilde{B}(s) + \tilde{C}(s)]W(s),$$

where $\tilde{B}(s) = B(g(s))$ and $\tilde{C}(s) = C(g(s))$. The characteristic roots $\tilde{\lambda}(s)$ of $A + \tilde{B}(s)$ satisfy the equation

$$\lambda^4 - 5\lambda^2 h(g(s))^2/32 + 9h(g(s))^4/8^4 = (-1)^l. \tag{19}$$

From (19) it follows that

$$\lambda^2 = 5h(g(s))^2/64 \pm \{(-1)^l + h(g(s))^4/4^4\}^{1/2}. \tag{20}$$

For $l = 1$, the condition $h(\infty)^2 \neq 16$ implies that (20) has, for s sufficiently large and $\leqq \infty$ and $h(\infty)^2 > 16$, four distinct real roots; for $h(\infty)^2 < 16$, roots of the form $\alpha \pm i\beta$ and $-\alpha \pm i\beta$ with $\alpha > 0$, $\beta > 0$. For $l = 2$, the condition $h(\infty)^2 \neq 64/3$ implies that (20) has, for s sufficiently large and $\leqq \infty$ and $h(\infty)^2 > 64/3$, four distinct real roots; for $h(\infty)^2 < 64/3$, roots of the form $\pm\alpha$ and $\pm i\beta$ with $\alpha > 0$, $\beta > 0$. Thus, as in Theorem 1, we have, for each i, j, that either $\operatorname{Re}[\tilde{\lambda}_i(s) - \tilde{\lambda}_j(s)] \equiv 0$ or $\operatorname{Re}[\tilde{\lambda}_i(s) - \tilde{\lambda}_j(s)]$ tends to a nonzero constant as $s \to \infty$. Application of the theorem of Levinson as in Theorem 1 yields (17).

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UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE