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MULTIPLICATIVE PROPERTIES OF JENSEN MEASURES

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1. Let A be a uniform algebra on the compact set X and let ψ be a non-trivial linear functional on A. A finite non-negative measure μ on X is called a Jensen measure for ψ if

$$(1) \qquad |\psi(f)| \le \exp\left(\int_X \log |f| \, d\mu\right), \qquad f \in A.$$

It is a well-known theorem of Bishop [1] that if ψ is multiplicative on A, then there exists a Jensen representing measure μ for ψ (i.e. μ is a probability measure such that (1) holds and $\psi(f)=\int_X f\,d\mu, f \in A$). Complementing this result, Ito and Schreiber [2] have proved a theorem which can be restated as follows:

THEOREM. *Let ψ be a linear functional on a uniform algebra A. Then ψ is multiplicative if and only if $\psi(1)=1$ and there exists a Jensen measure for ψ. Furthermore this Jensen measure is a representing measure for ψ.*

The object of this note is to give a measure-theoretic proof of this theorem which unlike the one given in [2] avoids the use of complex function theory.

2. **Proof of the theorem.** It follows from (1) that

$$e^t = e^t |\psi(1)| \le \exp(t\mu(X))$$

for all real t. Hence μ is a probability measure and consequently $\|\psi\|=1$. Let α be a complex number and $\eta_1, \ldots, \eta_r$ the rth roots of unity. Then, for $f \in A$ such that $\psi(f)=0$, we have

$$\begin{aligned} 1 &= |\psi(1-\alpha\eta_1 f) \cdots \psi(1-\alpha\eta_r f)| \\ &\le \exp\left(\int \sum_1^r \log |1-\alpha\eta_k f| \, d\mu\right) \\ &\le \int |1-\alpha^r f^r| \, d\mu. \end{aligned}$$

Thus for every $\alpha \in \mathbf{C}$ and for every $t>0$,

$$\int \frac{1}{t}(|1+t\alpha f^r| - 1) \, d\mu \ge 0.$$

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If $z \in \mathbf{C}$ and $0<t\leq 1$, then

$$\frac{1}{t}(|1+tz|-1)=\frac{1}{t}\frac{|1+tz|^2-1}{|1+tz|+1}$$
$$=\frac{z+\bar{z}+t\,|z|^2}{|1+tz|+1}\to \operatorname{Re} z$$

as $t\to 0$. Also

$$\left|\frac{1}{t}(|1+tz|-1)\right|\leq \frac{1}{t}\big||1+tz|-1\big|\leq |z|.$$

Applying Lebesgue's bounded convergence theorem, we get that for all $\alpha \in \mathbf{C}$,

$$\int \operatorname{Re} \alpha f^r\,d\mu \geq 0.$$

Hence

$$\int f^r\,d\mu = 0.$$

If $f \in A$ is arbitrary, then

$$\int f^r\,d\mu=\int [f-\psi(f)+\psi(f)]^r\,d\mu$$
$$=\sum_{k=0}^{r}\binom{r}{k}(\psi(f))^{r-k}\int (f-\psi(f))^k\,d\mu$$
$$=(\psi(f))^r.$$

Thus $\psi(f)=\int f\,d\mu$ and $\psi(f^2)=\int f^2\,d\mu=(\psi(f))^2$. Now a routine argument shows that ψ is multiplicative. This proves the sufficiency part of the theorem, the necessity part being precisely Bishop's theorem.

References

1. Bishop, E. *Holomorphic completions, analytic continuation and the interpolation of semi-norms*, Ann. of Maths (2), **78** (1963), 468–500. MR **27** #4958.

2. Ito, T. and Schreiber, B. M. *Multiplicative properties of Jensen measures*, Proc. Amer. Math. Soc. **26** (1970), 305–306.

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