FORMATIONS OF π-SOLUBLE GROUPS

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1. Introduction

The theory of formations of soluble groups, developed by Gaschütz [4], Carter and Hawkes [1], provides fairly general methods for investigating canonical full conjugate sets of subgroups in finite, soluble groups. Those methods, however, cannot be applied to the class of all finite groups, since strong use was made of the Theorem of Galois on primitive soluble groups. Nevertheless, there is a possibility to extend the results of the above mentioned papers to the case of π-soluble groups as defined by Čunihin [2]. A finite group G is called *π-soluble*, if, for a given set π of primes, the indices of a composition series of G are either primes belonging to π or they are not divisible by any prime of π. In this paper, we shall frequently use the following result of Čunihin [2]: If π is a non-empty set of primes, π' its complement in the set of all primes, and G is a π-soluble group, then there always exist Hall π-subgroups and Hall π'-subgroups, constituting single conjugate sets of subgroups of G respectively, each π-subgroup of G contained in a Hall π-subgroup of G where each π'-subgroup of G is contained in a Hall π'-subgroup of G. All groups considered in this paper are assumed to be finite and π-soluble. A Hall π-subgroup of a group G will be denoted by G_π.

2. The formation $\mathfrak{F}_\pi$

Let $\mathfrak{F}$ be a saturated formation of soluble groups as defined in [4], $\mathfrak{F}_\pi$ the class of all groups G having a normal π-complement $G_{\pi'}$, and Hall π-subgroups G belonging to $\mathfrak{F}$.

PROPOSITION 2.1. *$\mathfrak{F}_\pi$ is a formation.*

PROOF. (i) Let $G \in \mathfrak{F}_\pi$, $N \lhd G$. Then $G_\pi N/N \cong G_\pi/N \cap G_\pi \in \mathfrak{F}$ and $G_{\pi'}N/N \lhd G/N$.

(ii) Let $N_1 \lhd G_1$, $N_2 \lhd G$ and $G/N_1 \in \mathfrak{F}_\pi$, $G/N_2 \in \mathfrak{F}_\pi$. It follows

$$G_{\pi'}(N_1 \cap N_2)/N_1 \cap N_2 \lhd G/N_1 \cap N_2$$

and since

$$G_\pi N_i/N_i \cong G_\pi/N_i \cap G_\pi \in \mathfrak{F}, \quad \text{for} \quad i = 1, 2,$$

we have $G_\pi/N_1 \cap N_2 \cap G_\pi \in \mathfrak{F}$ whence $G_\pi(N_1 \cap N_2)/N_1 \cap N_2 \in \mathfrak{F}$.

LEMMA 2.2. *If N is a normal π'-subgroup and $G/N \in \mathfrak{F}_\pi$ then $G \in \mathfrak{F}_\pi$.*

PROOF.
$$G/N = (G_\pi N/N) \cdot (G_{\pi'}/N) \in \mathfrak{F}_\pi$$
implies $G_{\pi'} \lhd G$ and $G_\pi \cong G_\pi N/N \in F$. Hence $G \in \mathfrak{F}_\pi$.

By the Theorem of Gaschütz-Lubeseder [8], any saturated formation $\mathfrak{F}$ can be locally defined. This means, that to any prime p, there exists a formation $\mathfrak{F}(p)$, such that $G \in \mathfrak{F}$ if and only if $G/C_G(H/K) \in \mathfrak{F}(p)$ for all p-chief factors H/K of G.

PROPOSITION 2.3. *Let $\mathfrak{S}_\pi$ be the class of all soluble π-groups. If the formation $\mathfrak{F}$ is locally defined by $\mathfrak{F}(p)$, p ranging over all primes, then $G \in \mathfrak{F}_\pi$ if and only if $G/C_G(H/K) \in \mathfrak{F}(p) \cap \mathfrak{S}_\pi$, for all p-chief factors H/K with $p \in \pi$.*

PROOF. Let $G \in \mathfrak{F}_\pi$ and H/K be a p-chief factor of G such that $G_{\pi'} \subset K < H \subset G$. Since $G/G_{\pi'} \in \mathfrak{F}$, it follows $G/C_G(H/K) \in \mathfrak{F}(p) \cap \mathfrak{S}_\pi$. But any p-chief factor H/K with $p \in \pi$ is G-isomorphic to one lying between $G_{\pi'}$ and G.

Conversely, let $G/C_G(H/K) \in \mathfrak{F}(p) \cap \mathfrak{S}_\pi$ for all p-chief factors H/K with $p \in \pi$. By induction and Lemma 2.2 we may assume, that there is a minimal normal p-subgroup N of G with $G/N \in \mathfrak{F}_\pi$, and $p \in \pi$. Then $G_{\pi'}N < G$, and $G/C_G(N)$ is a π-group by assumption. Hence $G_{\pi'} \subset C_G(N)$ and therefore $G_{\pi'}$ char $G_{\pi'}N$ which implies $G_{\pi'} \lhd G$. Also
$$G_\pi/C_{G_\pi}(N) \cong G_\pi C_G(N)/C_G(N) = G/C_G(N) \in \mathfrak{F}(p)$$
whence $G_\pi \in \mathfrak{F}$ and $G \in \mathfrak{F}_\pi$.

DEFINITION. Let $p \in \pi$. Then we denote the formation $\mathfrak{F}(p) \cap \mathfrak{S}_\pi$ by $\mathfrak{F}_\pi(p)$.

PROPOSITION 2.4. *If $G \notin \mathfrak{F}_\pi$ and N is a minimal normal subgroup of G such that $G/N \in \mathfrak{F}_\pi$ then N is complemented and any two complements are conjugate.*

PROOF. By Lemma 2.2, N is a p-group, $p \in \pi$. If $\mathfrak{F}_\pi(p) = \emptyset$ then $p \nmid |G/N|$ and the proposition follows by Schur-Zassenhaus. If $\mathfrak{F}_\pi(p) \neq \emptyset$, then we proceed in a similar manner as in [4]. Let $F^p(G \div N)/N$ be the largest normal p-nilpotent subgroup of G/N. If there were an $x \in N$, $x \neq 1$, which is centralized by $F^p(G \div N)$, then $N \subset Z(F^p(G \div N))$, since N is a minimal normal subgroup. Let $F^p(G)$ be the largest normal p-nilpotent subgroup of G. Then $F^p(G) = F^p(G \div N) \cap C_G(N)$ by [6; VI 5.4.b)]. This implies $F^p(G) = F^p(G \div N)$. Hence $G \in \mathfrak{F}_\pi$ by Proposition 2.3 which is a contradiction. Let L/N be the largest normal p'-subgroup of G/N. Then L

splits over N by Schur-Zassenhaus and any two complements of N in L are conjugate. If there were an $n \in N$ which is centralized by L then $N \subset Z(L)$, since N is a minimal normal subgroup of G. This would imply $L \subset C_{F^p(G \div N)}(N)$. Then $F^p(G \div N)/C_{F^p(G \div N)}(N)$ would be a non-trivial group of p-automorphisms of N, since $N \not\subset Z(F^p(G \div N))$. This is, however, impossible, as $N = N_1 \times N_2 \times \cdots \times N_r$, the N_i's being minimal normal p-subgroups of $F^p(G \div N)$ and thus

$$F^p(G \div N)/C_{F^p(G \div N)}(N) = F^p(G \div N)/\bigcap_{i=1}^{r} C_{F^p(G \div N)}(N_i).$$

Hence $F^p(G \div N)/C_{F^p(G \div N)}(N_i)$ is a non-trivial p-group for at least one i, contradiction. Now let R be an arbitrary complement of N in L. R is self-normalizing in L, otherwise there exists an $n \in N$, $n \neq 1$, such that $R^n = R$, and this n would be centralized by R which is a contradiction. By applying a Frattini argument, the proposition is proved.

Proposition 2.5. *If $\Phi(G)$ is the Frattini subgroup of G and $G/\Phi(G) \in \mathfrak{F}_\pi$ then $G \in \mathfrak{F}_\pi$.*

Proof. Let $G^{\mathfrak{F}_\pi}$ be minimal among the normal subgroups N of G with $G/N \in \mathfrak{F}_\pi$. Then $G^{\mathfrak{F}_\pi} \subset \Phi(G)$. If $G^{\mathfrak{F}_\pi} \neq 1$, then there would exist a chief factor $G^{\mathfrak{F}_\pi}/H$ of G. By Prop. 2.4, $G^{\mathfrak{F}_\pi}/H$ would be complemented which contradicts $G^{\mathfrak{F}_\pi} \subset \Phi(G)$.

3. $\mathfrak{F}_\pi$-covering subgroups

Definition ([4]). A subgroup E of a π-soluble group G is called an *$\mathfrak{F}_\pi$-covering subgroup* if it has the following properties:

(i) $E \in \mathfrak{F}_\pi$

(ii) If $E \subset H \subset G$ and $H_0 \triangleleft H$, such that $H/H_0 \in \mathfrak{F}_\pi$ then $EH_0 = H$.

The following theorem extends a result of Gaschütz [4].

Theorem 3.1. *Every π-soluble group G has $\mathfrak{F}_\pi$-covering subgroups and any two of them are conjugate.*

Proof. We remark that any conjugate and any homomorphic image of an $\mathfrak{F}_\pi$-covering subgroup is an $\mathfrak{F}_\pi$-covering subgroup. We prove the theorem by induction on $|G|$. If $|G| = 1$, then the theorem holds. If $G \in \mathfrak{F}_\pi$ then G is the only $\mathfrak{F}_\pi$-covering subgroup. Assume $G \notin \mathfrak{F}_\pi$.

First Case. There exists a minimal normal subgroup N, such that $G/N \notin \mathfrak{F}_\pi$. In this case, we may use Gaschütz's argument ([4]) in order to prove the theorem. We take an $\mathfrak{F}_\pi$-covering subgroup E/N of G/N (by induction) and $\bar{E} < G$. E has by induction an $\mathfrak{F}_\pi$-covering subgroup

E whence $EN = \bar{E}$. Let $E \subset F \subset G$, $F_0 < F$ and $F/F_0 \in \mathfrak{F}_\pi$. Then $NF/NF_0 \in \mathfrak{F}_\pi$, thus $NEF_0 = NF$ and $(NE \cap F)F_0 = F$. Furthermore

$$NE \cap F/NE \cap F_0 \cong F/F_0 \in \mathfrak{F}_\pi.$$

Therefore

$$F = (NE \cap F)F_0 = E(NE \cap F_0)F_0 = EF_0$$

since E is an $\mathfrak{F}_\pi$-covering subgroup of $\bar{E}$ and so of $NE \cap F$. Hence E is an $\mathfrak{F}_\pi$-covering subgroup of G.

If E_1 and E_2 are $\mathfrak{F}_\pi$-covering subgroups of G then E_1N/N and E_2N/N are $\mathfrak{F}_\pi$-covering subgroups of G/N. By induction there exists $g \in G$, such that $NE_1 = NE_2^g$, and $NE_1 < G$. Hence by induction E_1 and E_2^g are conjugate under NE_1 as they are $\mathfrak{F}_\pi$-covering subgroup of NE_1. Therefore E_1 and E_2 are conjugate under G.

Second Case. $G/N \in \mathfrak{F}_\pi$ for every minimal normal subgroup N of G. Then G has to be monolithic with N as abelian monolith. By Proposition 2.4, N is complemented and any two complements are conjugate. Let M be a complement of N. Then M is a maximal subgroup of G and therefore an $\mathfrak{F}_\pi$-covering subgroup. If $\bar{M}$ is another $\mathfrak{F}_\pi$-covering subgroup, then $\bar{M}N = G$, as $G/N \in \mathfrak{F}_\pi$ and $\bar{M} \cap N = 1$, since N is abelian. By Proposition 2.4, M and $\bar{M}$ are conjugate.

4. $\mathfrak{F}_\pi$-normalizers

Definition. A chief factor H/K of G is called *$\mathfrak{F}_\pi$-central* if $G/C_G(H/K) \in \mathfrak{F}_\pi(p)$ for H/K being a p-chief factor $p \in \pi$, or if H/K is a π'-chief factor. Otherwise H/K is called *$\mathfrak{F}_\pi$-eccentric.*

Definition. A maximal subgroup M of G is called *$\mathfrak{F}_\pi$-normal* if $M/\mathrm{Core}_G(M) \in \mathfrak{F}_\pi(p)$ for M being of p-power index, $p \in \pi$, or if $[G : M]$ has only π'-divisors. Otherwise M is called *$\mathfrak{F}_\pi$-abnormal.*

Remark. Since, in a π-soluble group, a maximal subgroup is either of π'-index or of a p-power index for $p \in \pi$, it follows that a maximal subgroup is either $\mathfrak{F}_\pi$-normal or $\mathfrak{F}_\pi$-abnormal.

Proposition 4.1. *$G \in \mathfrak{F}_\pi$ if and only if every maximal subgroup of G is $\mathfrak{F}_\pi$-normal.*

Proof. Let $G \in \mathfrak{F}_\pi$ and M be a maximal subgroup of π-index. Then $G_{\pi'} \subset \mathrm{Core}_G(M) \subset M$, $G/G_{\pi'} \in \mathfrak{F}$ implies M is $\mathfrak{F}_\pi$-normal. Conversely, let every maximal subgroup of G be $\mathfrak{F}_\pi$-normal. By induction and Lemma 2.2, we may assume $G/N \in \mathfrak{F}_\pi$ for a minimal normal π-subgroup N of G, N being the unique minimal normal subgroup of G. Furthermore, Prop. 2.5 allows us to assume $\Phi(G) = 1$. Hence N has a maximal subgroup M as a complement, M being of π-index with $\mathrm{Core}_G(M) = 1$. Thus

$$G/C_G(N) = G/N \cong M \in \mathfrak{F}_\pi(p),$$

if M is of p-power index. This implies $G \in \mathfrak{F}_\pi$.

DEFINITION. $C^p(G) = \bigcap C_G(H/K)$, the intersection taken over all $\mathfrak{F}_\pi$-central p-chief factors of G, $p \in \pi$. Additionally, we put $C^p(G) = G$ if G has no $\mathfrak{F}_\pi$-central p-chief factors for $p \in \pi$.

DEFINITION. A *Sylow π-system* is a $\bigcap$-closed set of subgroups of G, generated by a complete set of Hall p'-subgroups of G for $p \in \pi$.

The following result is analogous to Ph. Hall's theorem on Sylow systems ([5]).

PROPOSITION 4.2. *Any two Sylow π-systems of a π-soluble group G are conjugate in G.*

PROOF. Let $\mathfrak{K} : K_1, \cdots, K_r$, $\mathfrak{K}^* : K_1^*, \cdots, K_r^*$ be two complete sets of Hall p_k'-subgroups of G, $p_k \in \pi$, and let $K_i = K_i^*$ for $i \leqq s$. Let $K_j \neq K_j^*$. Consider $Q_j = \bigcap_{i \neq j} K_i$. This is a Hall $\{p_j, \pi'\}$-subgroup of G whence $K_j Q_j = G$ (both statements follow from [7; 1.5.5]). Therefore there exists $x \in Q_j$ with $K_j^x = K_j^*$ and $K_i^x = K_i$, for $i \neq j$. Thus $\mathfrak{K}^x$ and $\mathfrak{K}^*$ have $s+1$ elements in common, and induction proves the proposition.

DEFINITION. Let $T = T^p(G) = G_{p'} \cap C^p(G)$, where $G_{p'}$ is a Hall p'-subgroup of G and $C^p(G)$ is as defined above. A subgroup D of G is called an *$\mathfrak{F}_\pi$-normalizer* if $D = \bigcap_{p \in \pi} N_G(T^p)$.

REMARK. Prop. 4.2 implies that all $\mathfrak{F}_\pi$-normalizers of a π-soluble group G are conjugate in G.

For the remainder of the paper, we assume, that $\mathfrak{F}(p) \neq \emptyset$ for all $p \in \pi$.

PROPOSITION 4.3. *If $G \in \mathfrak{F}_\pi$, then G is its own $\mathfrak{F}_\pi$-normalizer.*

PROOF. By Prop. 2.3, any chief factor is $\mathfrak{F}_\pi$-central. Hence, for $p \in \pi$, $C^p(G) = F^p(G)$ ([6; VI, 5.4.6]) which implies T^p char $C^p(G)$. Thus $T^p \lhd G$ and $N_G(T^p) = G$ for all $p \in \pi$.

PROPOSITION 4.4. *Let M be an $\mathfrak{F}_\pi$-abnormal maximal subgroup of G. Then M contains an $\mathfrak{F}_\pi$-normalizer of G.*

PROOF. Since M is $\mathfrak{F}_\pi$-abnormal, it is of p-power index in G, for some $p \in \pi$, and $M/\mathrm{Core}_G(M) \notin \mathfrak{F}_\pi(p)$. Let $\mathfrak{K}_\pi$ be a Sylow π-system, $G_{p'} \in \mathfrak{K}_\pi$, such that $G_{p'} \subset M$. Then $T^p = C^p(G) \cap G_{p'} \subset M$. We will show: $N_G(T^p) \subset M$. Let $K = \mathrm{Core}_G(M)$. Then $M/K \notin \mathfrak{F}_\pi(p)$. By definition of K, there is no

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normal subgroup of G between K and M, hence if H/K is a minimal normal subgroup of G/K, we have $MH = G$. But $|G| = |M||H|/|M \cap H|$ whence $p/|H/K|$ and H/K is an elementary abelian p-group since G is π-soluble. Furthermore $M \cap H = K$ and $G/H \cong M/K \notin \mathfrak{F}_\pi(p)$. H/K is the only minimal normal subgroup of G/K and is self-centralizing by the theorem of Galois ([9; Th. 11.5]). Therefore $H < KC^p(G)$, otherwise $G/H \in \mathfrak{F}_\pi(p)$. Let L/H be a minimal normal subgroup of G/H, lying in $KC^p(G)$. Since $H = C_G(H/K)$, certainly $p \nmid |L/H|$. Hence $L \cap M/K$ is a p-complement of L/K. Now T^p is a p-complement of $C^p(G)$ whence KT^p/K is a p-complement of KC^p/K. But $KC^p \supset L$, therefore $KT^p \cap L/K$ is also a p-complement of L/K. Moreover $KT^p \subset KG_{p'} \subset M$, thus $KT^p \cap L \subset M \cap L$. Hence $KT^p \cap L = M \cap L$. Let $g \in N_G(T^p)$, then $(KT^p \cap L)^g = KT^p \cap L$ and $g \in N_G(KT^p \cap L) = N_G(M \cap L) = M$. Thus $N_G(T^p) \subset M$.

Definition. Denote by $O_{\pi'}(G)$ the largest normal π'-subgroup of a π-soluble group G. Then $F^\pi(G)/O_{\pi'}(G)$ shall be defined to be the Fitting subgroup of $G/O_{\pi'}(G)$ and $\Phi^\pi(G)/O_{\pi'}(G)$ to be the Frattini subgroup of $G/O_{\pi'}(G)$.

Lemma 4.5. *If G is π-soluble, then $G/\Phi^\pi(G)$ has no non-trivial normal π'-subgroup.*

Proof. Assume $A/\Phi^\pi(G)$ is a normal π'-subgroup of $G/\Phi^\pi(G)$. Since $\Phi^\pi(G)/O_{\pi'}(G)$ is a π-group, by Schur-Zassenhaus there exists a subgroup B of A such that $B\Phi^\pi(G) = A$ and $B \cap \Phi^\pi(G) = O_{\pi'}(G)$. B is then a Hall π'-subgroup of A and a Frattini argument yields

$$G = AN_G(B) = B\Phi^\pi(G)N_G(B) = \Phi^\pi(G)N_G(B) = N_G(B).$$

Thus $B \lhd G$ which implies $B = O_{\pi'}(G)$. Hence $A = \Phi^\pi(G)$.

Lemma 4.6. *If G is π-soluble, $O_{\pi'} = O_{\pi'}(G)$, $\Phi^\pi/O_{\pi'}$ is the Frattini subgroup of $G/O_{\pi'}$, and $F^\pi/O_{\pi'}$, is the Fitting subgroup of $G/O_{\pi'}$, then $F^\pi = C_G(F^\pi/\Phi^\pi)$.*

Proof. Put $C = C_G(F^\pi/\Phi^\pi)$. Certainly $F^\pi \subset C$. Assume $F^\pi < C$. By Gaschütz ([3]), there exists a subgroup K of G, such that $F^\pi K = G$, $F^\pi \cap K = \Phi^\pi$. Let $H/O_{\pi'}$ be the largest normal π-subgroup of $G/O_{\pi'}$. Then $F^\pi/O_{\pi'}$ is also the Fitting subgroup of $H/O_{\pi'}$ which is a soluble π-group. Since $H \lhd G$, the Frattini subgroup of H/Φ^π is trivial. Thus [6; III, 4.2b] implies that $C \cap H = C_H(F^\pi/\Phi^\pi) = F^\pi$. Let L/F^π be a minimal normal subgroup of G/F^π in C/F^π. Since

$$L/F^\pi \cap H/F^\pi \subset C/F^\pi \cap H/F^\pi = F^\pi/F^\pi,$$

the chief factor L/F^π of G is a π'-group. Now $F^\pi(K \cap L) = L$ and

$F^\pi \cap (K \cap L) = \Phi^\pi$. But $L \subset C$ implies $L/\Phi^\pi = F^\pi/\Phi^\pi \times (K \cap L)/\Phi^\pi$. It follows that $F^\pi \subset N_G(K \cap L)$. Furthermore $K \cap L \lhd K$ whence $K \cap L \lhd G$. As $(K \cap L)/\Phi^\pi \cong L/F^\pi$, $K \cap L/\Phi^\pi$ is a non-trivial normal π'-subgroup of G/F^π. This contradicts Lemma 4.5.

COROLLARY. If $R/O_{\pi'}$ is the Frattini subgroup of $F^\pi/O_{\pi'}$, then $F^\pi = C_G(F^\pi/R)$.

PROOF. Certainly $F^\pi \subset C_G(F^\pi/R)$, as F^π/R is abelian. But also $R \subset \Phi^\pi$ since $F^\pi \lhd G$. Hence $F^\pi \subset C_G(F^\pi/R) \subset C_G(F^\pi/\Phi^\pi) = F^\pi$ by Lemma 4.6.

LEMMA 4.7. *F^π is the intersection of the centralizers of all π-chief factors of G.*

PROOF. Let N be the intersection of the centralizers of all π-chief factors of G, $O_{\pi'}(N)$ the largest normal π'-subgroup of N, $N_1/O_{\pi'}(N)$ the Fitting subgroup of $N/O_{\pi'}(N)$ and $K/O_{\pi'}(N)$ the Frattini subgroup of $N_1/O_{\pi'}(N)$. By the corollary of Lemma 4.6, $C_N(N_1/K) = N_1$. If $N_1 < N$, then there exists $x \in N$, $x \notin N_1$. N_1/K is a direct product of elementary abelian p-groups for certain primes p in π. Therefore there exists a chief factor L/M of N/K, such that $K \subset M < L \subset N_1$ and $x \notin C_G(L/M)$. This is a contradiction since N centralizes every π-chief factor of N. Hence $N_1 = N$ and $N \subset F^\pi$. Conversely F^π centralizes every π-chief factor of G, thus $N = F^\pi$.

PROPOSITION 4.8. *$G \in \mathfrak{F}_\pi$ if and only if every minimal normal subgroup of G/Φ^π is $\mathfrak{F}_\pi$-central.*

PROOF. If $G \in \mathfrak{F}_\pi$ then any chief-factor of G is $\mathfrak{F}_\pi$-central, particularly the minimal normal subgroups of G/Φ^π. Conversely, let $\bar{G} = G/\Phi^\pi$, then $\bar{F} = F^\pi/\Phi^\pi$ is the Fitting subgroup of $\bar{G}$. $\bar{F}$ is direct sum of certain minimal normal subgroups $\bar{N}_i$ of $\bar{G}$. Let $\bar{C}_i = C_G(\bar{N}_i)$, then $G/\bar{C}_i \in \mathfrak{F}_\pi(p)$, if $\bar{N}_i$ is a p-chief factor. One can choose the $\mathfrak{F}(p)$'s in such a way that $\mathfrak{F}(p) \subset \mathfrak{F}$. Then $G/\bar{C}_i \in \mathfrak{F}_\pi$. But $\bigcap \bar{C}_i = C_G(\bar{F}) = \bar{F} \subset \bigcap \bar{C}_i$ by Lemma 4.6 and Lemma 4.7. Hence $\bar{G}/\bar{F} \in \mathfrak{F}_\pi$ and also $G/O_{\pi'} = \bar{G} \in \mathfrak{F}_\pi$. By Lemma 2.2, $G \in \mathfrak{F}_\pi$.

COROLLARY. *If $G \notin \mathfrak{F}_\pi$, then there exists an $\mathfrak{F}_\pi$-abnormal maximal subgroup M of G with $G = MF^\pi$.*

PROOF. Since $G \notin \mathfrak{F}_\pi$, there exists an $\mathfrak{F}_\pi$-eccentric chief factor N/Φ^π of G and $N \subset F^\pi$. Therefore there exists a maximal subgroup M of G with $G = MN$ and hence $G = MF^\pi$. Furthermore $M \cap N = \Phi^\pi$ as N/Φ^π is an abelian p-group, $p \in \pi$. But $M/\mathrm{Core}_G(M) \cong G/C_G(N/\Phi^\pi) \notin \mathfrak{F}_\pi(p)$.

LEMMA 4.9. *Let G be a π-soluble group, M a maximal subgroup of G*

with $G = MF^\pi$ and H/K a π-chief factor of G. If M covers H/K, then $H \cap M/K \cap M$ is a chief factor of M.

PROOF. M covers H/K whence $H/K \cong H \cap M/K \cap M$ and

$$C_M(H/K) = C_M(H \cap M/K \cap M).$$

Furthermore $G/C_G(H/K) = G/C_{MF^\pi}(H/K)$. But $F^\pi \subset C_G(H/K)$ by Lemma 4.7, thus

$$\begin{aligned} G/C_G(H/K) &= G/F^\pi C_M(H/K) \\ &= F^\pi M/F^\pi C_M(H/K) \cong M/C_M(H/K) \cong M/C_M(H \cap M/K \cap M). \end{aligned}$$

Since H/K is a chief factor of G, this isomorphism shows $H \cap M/K \cap M$ to be a chief factor of M.

COROLLARY. *If $p \in \pi$, then $C^p(M) = M \cap C^p(G)$.*

REMARK. M is of p-power index in G for some $p \in \pi$.

PROPOSITION 4.10. *Let M be defined as in Lemma 4.9. Let $|G : M| = p^\alpha$ for some $p \in \pi$ and $\mathfrak{K} = \{G_{p'}, G_{q'}, \cdots, G_{r'}\}$ be a Sylow π-system of G with $G_{p'} \subset M$. If $\mathfrak{K}' = \mathfrak{K} \cap M$ which is a Sylow π-system of M, then*

$$T^p(G), T^q(G), \cdots, \quad \textit{and} \quad T^p(M), T^q(M), \cdots$$

are determined by $\mathfrak{K}$ and $\mathfrak{K}'$ respectively. They are related by

$$T^q(M) = M \cap T^q(G),$$

particularly $T^p(M) = T^p(G)$.

PROOF. By the corollary of Lemma 4.9, $C^q(M) = M \cap C^q(G)$ for all $q \in \pi$. Then

$$T^q(M) = G_{q'} \cap M \cap C^q(M) = G_{q'} \cap M \cap C^q(G) = M \cap T^q(G).$$

Since $G_{p'} \subset M$, we have in particular $T^p(M) = G_{p'} \cap C^p(G) = T^p(G)$.

PROPOSITION 4.11. *Under the assumption of Prop. 4.10,*

$$N_M(T^q(M)) = N_M(T^q(G))$$

for all $q \in \pi$.

PROOF. If $q = p$, the result follows from Prop. 4.10. Assume $q \neq p$. By Lemma 4.7, $F^\pi \subset C^q(G)$. Let $F^\pi_{q'}/O_{\pi'}$ be the q-complement of $F^\pi/O_{\pi'}$. Then $F^\pi \subset T^q(G)$ and is normal in G. Furthermore $F^\pi_{q'} \not\subset M$ whence $F^\pi_{q'}M = G$. Thus

$$T^q(G) = F^\pi_{q'}(T^q(G) \cap M) = F^\pi_{q'}T^q(M)$$

by Prop. 4.10. Now choose $m \in N_M(T^q(G))$. Then

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$$(T^q(M))^m = (T^q(G) \cap M)^m = T^q(G)^m \cap M = T^q(M).$$

Hence $m \in N_M(T^q(M))$. Choose $x \in N_M((T^q(M))$. Then

$$T^q(G)^x = F^{\pi}_{q'} T^q(M)^x = T^q(G)$$

which proves the proposition.

PROPOSITION 4.12. *Let M of index p^α in G be defined as in Lemma 4.9 for some $p \in \pi$ where M is $\mathfrak{F}_\pi$-abnormal, D_G an $\mathfrak{F}_\pi$-normalizer obtained from a Sylow π-system K of G whose p-complement is contained in M and D_M the $\mathfrak{F}_\pi$-normalizer of M obtained from the Sylow π-system $\mathfrak{K} \cap M$ of M. Then $D_G = D_M$.*

PROOF.

$$D_M = \bigcap_{q \in \pi} N_M(T^q(M)) = \bigcap_{q \in \pi} N_M(T^q(G)) = M \cap \bigcap_{q \in \pi} N_G(T^q(G)) = M \cap D_G,$$

by Proposition 4.11. By the proof of Prop. 4.4, $D_G \subset M$ whence $D_M = D_G$.

THEOREM 4.13. *An $\mathfrak{F}_\pi$-normalizer of G covers every $\mathfrak{F}_\pi$-central chief factor and avoids every $\mathfrak{F}_\pi$-eccentric chief factor.*

PROOF. If $G \in \mathfrak{F}_\pi$, then the theorem is true by Prop. 4.3. Assume $G \notin \mathfrak{F}_\pi$. Then, by the corollary of Prop. 4.8, there exists an $\mathfrak{F}_\pi$-abnormal maximal subgroup M with $G = MF^\pi$ and M is of π-index in G. Let H/K be an $\mathfrak{F}_\pi$-central chief factor of G. If H/K is a π'-chief factor, then D_G covers H/K, since $G_{\pi'} \subset D_G$ (by induction, using Prop. 4.12) and $G_{\pi'}K \supset H$. Now let H/K be a π-chief factor. M covers H/K, otherwise M avoids H/K and $M/\mathrm{Core}_G(M) \cong G/C_G(H/K)$ which would imply that M is $\mathfrak{F}_\pi$-normal. By Lemma 4.9, $H \cap M/K \cap M \cong H/K$ and is $\mathfrak{F}_\pi$-central. Choose D_G as in Prop. 4.12. Then $D_G = D_M$ and by induction $D_G(K \cap M) \supset H \cap M$. Thus $D_G K \supset (H \cap M)K = H$.

If H/K is $\mathfrak{F}_\pi$-eccentric, then H/K is a π-chief factor. Either M avoids H/K, then also D_G avoids H/K, since $D_G \subset M$, or M covers H/K, then by Lemma 4.9, $H/K \cong H \cap M/K \cap M$ which is also not $\mathfrak{F}_\pi$-central. Hence, by induction, $D_M = D_G$ avoids $H \cap M/K \cap M$. Thus $H \cap M \cap D_G \subset K \cap M$ which implies $H \cap D_G \subset K$.

The theorem has been proved for a special $\mathfrak{F}_\pi$-normalizer, but, since all $\mathfrak{F}_\pi$-normalizer are conjugate in G, the theorem is valid for any $\mathfrak{F}_\pi$-normalizer of G.

COROLLARY. *The order of an $\mathfrak{F}_\pi$-normalizer of a π-soluble group G equals the product of all $\mathfrak{F}_\pi$-central chief-factors in a chief series of G.*

Several other theorems which hold in soluble groups can be easily generalized to the π-soluble case. Particularly, one can show, that the

$\mathfrak{F}_\pi$-normalizers of a π-soluble group belong to $\mathfrak{F}_\pi$, and that they are just the minimal members of descending chains of successively $\mathfrak{F}_\pi$-abnormal subgroups. Also, any $\mathfrak{F}_\pi$-covering subgroup contains an $\mathfrak{F}_\pi$-normalizer and any $\mathfrak{F}_\pi$-normalizer is contained in an $\mathfrak{F}_\pi$-covering subgroup.

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