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A Result in Surgery Theory

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Abstract. We study the topological 4-dimensional surgery problem for a closed connected orientable topological 4-manifold X with vanishing second homotopy and $\pi_1(X) \cong A * F(r)$, where A has one end and $F(r)$ is the free group of rank $r \geq 1$. Our result is related to a theorem of Krushkal and Lee, and depends on the validity of the Novikov conjecture for such fundamental groups.

1 Introduction

The 4-dimensional topological surgery conjecture has been established for a large class of groups which includes the groups of subexponential growth [12, 20]. However, the general case remains open. The proof of the conjecture for "good" fundamental groups is based on the existence of Whitney discs [11], which is not actually known to hold for arbitrary groups. However, in certain cases it may be shown that surgery works even when the disc embedding theorem is not available (see [4, 13, 17]). Here we state another instance when the topological surgery conjecture holds. The result will be proved by use of controlled surgery theory.

Theorem 1 *Let X^4 be a closed connected orientable topological 4-manifold with $\pi_1(X) = A * F(r)$ and $\pi_2(X) = 0$, where $F(r)$, $r \geq 1$, denotes the free group of rank r, and A has one-end. Suppose that the assembly map $\mathcal{A}\colon H_4(A; \mathbb{L}) \to L_4(A)$ is injective. Let $f\colon M \to X$ be a degree one normal map, where M is a closed 4-manifold. Then the vanishing of the Wall obstruction implies that f is normally bordant to a homotopy equivalence $f'\colon M' \to X$. In other words, the surgery sequence*

$$\mathcal{S}(X) \longrightarrow [X, G/\mathrm{TOP}] \longrightarrow L_4(\pi_1(X))$$

is exact.

To explain the assumptions in the theorem, we recall the definition of ends for finitely generated groups (see [15, p. 9], and [18, p. 16]). Such a group G has 0, 1, 2 or infinitely many ends. It has zero ends if and only if it is finite, in which case $H^0(G; \mathbb{Z}[G]) \cong \mathbb{Z}$ and $H^q(G; \mathbb{Z}[G]) = 0$ for $q > 0$. Otherwise, $H^0(G; \mathbb{Z}[G]) = 0$ and $H^1(G; \mathbb{Z}[G])$ is a free abelian group of rank $e(G) - 1$, where $e(G)$ is the number of

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ends of G. The group G has more than one end if and only if it is either a nontrivial generalized free product with amalgamation $G \cong A *_C B$ or an HNN (Higman–Neumann–Neumann) extension $G \cong A *_C \phi$, where C is a finite group. In particular, it has two ends if and only if it is virtually $\mathbb{Z}$ (that is, it has a subgroup of finite index in $\mathbb{Z}$) if and only if it has a (maximal) finite normal subgroup F such that the quotient G/F is either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. Some classes of one-ended groups are described in [16, pp. 9–11].

Examples of 4-manifolds satisfying the homotopy conditions of the theorem are given in Section 4 (see also [6] for the topology of 4-manifolds with vanishing second homotopy). Examples of groups for which the assembly map is injective (that is, for which the integral Novikov conjecture holds) can be found in the survey papers [10, 26] and their references. As a related topic, we refer to [3, 5, 17, 22] for the classification of the homotopy type and the s-cobordism class of closed topological 4-manifolds whose fundamental group is either a nonabelian free group or that of a closed aspherical surface. Further results on surgery theory for compact four-manifolds with special fundamental groups are found in [14, 16, 18, 29, 30].

2 Proof of Theorem 1

First we briefly recall some standard concepts from controlled surgery theory. Let $\mathbb{L}$ denote the 4-periodic simply connected surgery spectrum (see [24] for a geometric definition, and [25] for an algebraic definition). For a topological space B we have the $\mathbb{L}$-*homology*, denoted by $H_*(B;\mathbb{L})$. There is a well-defined *assembly map*

$$\mathcal{A}\colon H_i(B;\mathbb{L}) \to L_i(\pi_1(B)),$$

where $L_i(\pi_1(B))$ denotes the i-th Wall group of obstructions to simple homotopy equivalences [31]. Let $p\colon X \to B$ be a *control map*, where B is a finite-dimensional compact metric ANR. Then p is called a $UV^1(\delta)$-*map*, $\delta > 0$, if every commutative diagram

$$\begin{tikzcd} K_0 \arrow[r, "\alpha_0"] \arrow[d, "j"] & X \arrow[d, "p"] \\ K \arrow[r, "\alpha"] & B \end{tikzcd}$$

where K is a 2-complex, $K_0 \subset K$ is a subcomplex and j is the inclusion map, can be completed by a map $\bar{\alpha}\colon K \to X$ such that $\bar{\alpha} \circ j = \alpha_0$ and $d(p \circ \bar{\alpha}(x), \alpha(x)) < \delta$ for every $x \in K$. Here $d\colon B \times B \to \mathbb{R}$ denotes a metric on B. The map p is called a UV^1-*map* if it is a $UV^1(\delta)$-map for every $\delta > 0$. If $p\colon X \to B$ is a sufficiently small controlled Poincaré complex, then there exists the controlled surgery exact sequence

$$\mathcal{S}_{\epsilon,\delta}(X) \longrightarrow [X, G/\operatorname{TOP}] \longrightarrow H_4(B;\mathbb{L})$$

where $\mathcal{S}_{\epsilon,\delta}$ is the *controlled structure set* (for details see [9, 23]). From the main theorem (assuming its hypothesis) of controlled surgery theory (see the quoted papers) we have a commutative diagram

$$\begin{tikzcd} \mathbb{S}_{\epsilon,\delta}(X) \arrow[r] \arrow[d] & {[X, G/\mathrm{TOP}]} \arrow[r] \arrow[d, equal] & H_4(B;\mathbb{L}) \arrow[d, "\mathcal{A}"] \\ \mathbb{S}(X) \arrow[r] & {[X, G/\mathrm{TOP}]} \arrow[r] & L_4(\pi_1(B)). \end{tikzcd}$$

In order to solve the topological 4-dimensional surgery problem for X, one needs a control map $p\colon X \to B$ such that p is a UV^1-map and the assembly map $\mathcal{A}$ is injective.

Now let X be as in Theorem 1. Let Y be the closed 4-manifold obtained from X by killing the generators of the free part $F(r)$ of $\pi_1(X)$. Then it was proved in [6] that Y is aspherical, *i.e.,* $Y \simeq K(A,1)$, and that X is homotopy equivalent to the connected sum W of Y with r copies of $\mathbb{S}^1 \times \mathbb{S}^3$. An alternative proof of this claim will also be given in Section 3. A homotopy equivalence from X to $W = Y\#r(\mathbb{S}^1 \times \mathbb{S}^3)$ induces an isomorphism of the ordinary short exact surgery sequences. So we can transform a surgery problem for X to a surgery problem for W. We consider the following composite map $p = p_2 \circ p_1\colon W \to B = Y \vee (\vee_r \mathbb{S}^1)$ where

$$p_1\colon W = Y\#r(\mathbb{S}^1 \times \mathbb{S}^3) \to Y \vee (\vee_r \mathbb{S}^1 \times \mathbb{S}^3)$$

is the smash map, and $p_2\colon Y \vee (\vee_r \mathbb{S}^1 \times \mathbb{S}^3) \to B = Y \vee (\vee_r \mathbb{S}^1)$ is induced by the projections $\mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^1$.

By [15, Lemma 3.3] the map $p = p_2 \circ p_1$ is a UV^1-map.

Let us consider the Atiyah–Hirzebruch spectral sequence

$$E^2_{rs} = H_r(B; \pi_s(\mathbb{L})) \Rightarrow H_{r+s}(B;\mathbb{L}),$$

where $\pi_1(B) = A * F(r)$. Following [2], for any finitely generated group Γ, we write

$$L_n(\Gamma) \cong \widetilde{L}_n(\Gamma) \oplus L_n(1),$$

where $\widetilde{L}_n$ denotes the reduced surgery group. In particular, for finitely presented 2-torsion free groups Γ_i we have $\widetilde{L}_4(\Gamma_1 * \Gamma_2) \cong \widetilde{L}_4(\Gamma_1) \oplus \widetilde{L}_4(\Gamma_2)$ (see [2, Theorem 5]). We shall prove in the next section (see Lemma 3) that A is the fundamental group of a well-specified aspherical closed 4-manifold Y. Then A is a PD_4-group (Poincaré duality group; see [18, p. 20] for the definition). So A has finite cohomological dimension and is torsion free. Then, in our case, we obtain

$$\widetilde{L}_4(A * F(r)) \cong \widetilde{L}_4(A) \oplus \widetilde{L}_4(F(r)) \cong \widetilde{L}_4(A),$$

hence $L_4(\pi_1(B)) = L_4(A * F(r)) \cong L_4(A)$. In fact, the isomorphisms

$$\mathbb{Z} \cong L_4(F(r)) \cong \widetilde{L}_4(F(r)) \oplus L_4(1) \cong \widetilde{L}_4(F(r)) \oplus \mathbb{Z}$$

imply that $\widetilde{L}_4(F(r)) \cong 0$. Recall that

$$\pi_s(\mathbb{L}) \cong \begin{cases} 0 & \text{if } s \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } s \equiv 2 \pmod 4, \\ \mathbb{Z} & \text{if } s \equiv 0 \pmod 4. \end{cases}$$

Since $E^2_{rs} = H_r(B; \pi_s(\mathbb{L})) \cong H_r(Y; \pi_s(\mathbb{L})) \cong H_r(A; \pi_s(\mathbb{L}))$ for every $r > 1$, the spectral sequence of B collapses to that of A. This gives $H_4(B; \mathbb{L}) \cong H_4(A; \mathbb{L})$.

In fact, we have

$$\begin{aligned} H_4(B; \mathbb{L}) &\cong H_4(B; \pi_0(\mathbb{L})) \oplus H_3(B; \pi_1(\mathbb{L})) \oplus H_2(B; \pi_2(\mathbb{L})) \\ &\qquad \oplus H_1(B; \pi_3(\mathbb{L})) \oplus H_0(B; \pi_4(\mathbb{L})) \\ &\cong H_4(B; \mathbb{Z}) \oplus H_2(B; \mathbb{Z}_2) \oplus H_0(B; \mathbb{Z}) \cong H_4(Y; \mathbb{Z}) \oplus H_2(Y; \mathbb{Z}_2) \oplus H_0(Y; \mathbb{Z}) \\ &\cong H_4(A; \mathbb{L}). \end{aligned}$$

Since $\mathcal{A}\colon H_4(A; \mathbb{L}) \to L_4(A)$ is injective by hypothesis, $H_4(B; \mathbb{L}) \cong H_4(A; \mathbb{L})$ and $L_4(\pi_1(B)) \cong L_4(A)$, the assembly map $\mathcal{A}\colon H_4(B; \mathbb{L}) \to L_4(\pi_1(B))$ is injective, too. This proves the theorem.

3 Homotopy Type

Let X denote a closed connected orientable topological 4-manifold such that $\pi = A * F(r)$, $r \geq 1$, and $\pi_2 = 0$, where A has one end. Let Y be the closed 4-manifold obtained from X by killing the generators of the free part $F(r)$ of π. This section is devoted to study the homotopy type of X. For a general reference on the homotopy type and obstruction theory see for example [1].

Theorem 2 *With the above notation, the manifold X is homotopy equivalent to $W = Y \# r(\mathbb{S}^1 \times \mathbb{S}^3)$.*

Here we shall prove Theorem 2 under the additional algebraic condition that the second rational homology group of X is not trivial (For the remaining case we refer to [6]). To construct a homotopy equivalence between X and $W = Y \# r(\mathbb{S}^1 \times \mathbb{S}^3)$ we shall use [14, Theorem 1.1(2), Lemma 1.3]. First we need some lemmas.

Lemma 3 *With the above notation, $\pi_2(Y) \cong 0$. Since $\pi_1(Y) \cong A$ has one-end, the manifold Y is aspherical, that is, $Y \simeq K(A, 1)$.*

Proof It suffices to prove the result for $r = 1$ (one can obtain the general case by simple iteration). We have to prove by induction that the condition $\pi_2 = 0$ continues to hold.

(i) First of all, after the surgery the fundamental group of Y is isomorphic to A. This follows from the fact that when we do surgery on a loop γ, we kill the normal closure generated by $[\gamma]$. In the present case $[\gamma]$ is the generator of $\mathbb{Z}$ in $\pi = A * \mathbb{Z}$ and the quotient modulo the normal closure $\langle [\gamma] \rangle \cong \langle \mathbb{Z} \rangle$ is A.

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(ii) From (i) $\pi_1(Y) \cong A$ and the Hurewicz theorem, the second homotopy group of Y is the same as the second homology group $H_2(Y_A)$, where Y_A is the universal covering space of Y, or in other words the regular A-fold covering. In fact, we have $\pi_2(Y) \cong \pi_2(Y_A) \cong H_2(Y_A)$ as Y_A is simply connected.

(iii) Let X_A denote the covering space of X associated to the normal subgroup $\langle[\gamma]\rangle \cong \langle\mathbb{Z}\rangle$ in (i). Then X_A is a regular covering space of X with A as its covering transformation group, hence $\pi_1(X_A) \cong \langle\mathbb{Z}\rangle$. Now Y_A can be obtained from X_A by doing equivariant surgery on the lifting $A\cdot\gamma$ of γ in X_A. Our problem is to understand the effect of these surgeries on X_A.

(iv) Now we have to do a little group theory. From the covering space theory, we know that the fundamental group of X_A is the normal subgroup $\langle[\gamma]\rangle \cong \langle\mathbb{Z}\rangle$. In fact, we claim that this normal subgroup is the free product $\mathbb{Z} * (a\mathbb{Z}a^{-1}) * \cdots$, where $\mathbb{Z}$ is the subgroup $\mathbb{Z}$ in $\pi = A * \mathbb{Z}$, $a\mathbb{Z}a^{-1}$ is the conjugation of $\mathbb{Z}$ by a, and a goes through all the nonzero elements in A. To see this, we can use the model of $K(A * \mathbb{Z}, 1)$ given by the wedge of $K(A, 1)$ and a circle $\mathbb{S}^1$. The regular A-fold covering is the wedge of the contractible space $E(A, 1)$ and a family of circles, one for each element $a \in A$. From Van Kampen's theorem, the above assertion is immediate.

(v) From the free product description of $\pi_1(X_A) \cong \langle\mathbb{Z}\rangle$, it follows that its first homology group $H_1(X_A) \cong \mathbb{Z}[A]$ is the free $\mathbb{Z}[A]$-module of rank one. From surgery theory, the equivariant surgery in (iii) kills this free $\mathbb{Z}[A]$-summand but leaves $H_2(X_A)$ unchanged, that is, $H_2(X_A) \cong H_2(Y_A) \cong \pi_2(Y)$.

(vi) Finally, $H_2(X_A) \cong 0$ because by Hopf's theorem there is an exact sequence

$$\pi_2(X_A) \longrightarrow H_2(X_A) \longrightarrow H_2(\pi_1(X_A)) \longrightarrow 0.$$

Here $\pi_2(X_A) \cong \pi_2(X) \cong 0$ by assumption. Since $\pi_1(X_A)$ is free, it follows that $H_2(\pi_1(X_A)) \cong 0$. Thus we have $H_2(X_A) \cong \pi_2(Y) \cong 0$. Since A has one end, it follows that $\pi_3(\widetilde{Y}) \cong 0$, where $\widetilde{Y}$ is the universal cover of Y. In fact, we have the isomorphisms

$$\pi_3(\widetilde{Y}) \cong H_3(\widetilde{Y}) \cong H_3(Y;\mathbb{Z}[A]) \cong \overline{H}^1(Y;\mathbb{Z}[A]) \cong \overline{H}^1(A;\mathbb{Z}[A]) \cong 0.$$

Furthermore, $H_4(\widetilde{Y}) \cong 0$ because A is infinite. Thus $\widetilde{Y}$ is contractible, and we get $Y \simeq K(A, 1)$. ∎

Lemma 4 *The classifying map $c_X\colon X \to B = Y \vee (\vee_r \mathbb{S}^1)$ induces an isomorphism $H_4(X) \to H_4(B) \cong H_4(Y) \cong H_4(A)$. In particular, we have $(c_X)_*[X] = [Y]$.*

Proof First we examine the case $r = 1$. For a group G, let us define $E(G) = \overline{H}^1(G;\mathbb{Z}[G])$, so $E(\pi) = \overline{H}^1(\pi;\Lambda) \cong H_3(\widetilde{X})$, where $\Lambda = \mathbb{Z}[\pi]$ as usual. We prove that the classifying map $c_X\colon X \to K(\pi, 1) = K(A, 1) \vee \mathbb{S}^1 \simeq Y \vee \mathbb{S}^1$ induces an isomorphism $H_4(X) \to H_4(Y)$. Of course, c_X is 3-connected. Then it induces isomorphisms on H_i and H^i for $i \leq 2$, a monomorphism on H^3, and an epimorphism on H_3. Further, the Betti numbers satisfy relations $\beta_3(X) = \beta_1(X) = \beta_1(Y) + 1 = \beta_3(Y) + 1$. The spectral sequence $H_p(\pi; H_q(\widetilde{X})) \Rightarrow H_{p+q}(X)$ gives

$$\begin{aligned} 0 \longrightarrow H_5(\pi) \longrightarrow H_1(\pi; E(\pi)) \longrightarrow H_4(X) \longrightarrow H_4(\pi) \\ \longrightarrow H_0(\pi; E(\pi)) \longrightarrow H_3(X) \longrightarrow H_3(\pi) \longrightarrow 0. \end{aligned}$$

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Since $K(A,1) \simeq Y$, we have $H_5(\pi) \cong 0$, $H_4(\pi) \cong \mathbb{Z}$ and $H_3(\pi) \cong H^1(Y) \cong H^1(A)$. Further, the epimorphism $H_3(X) \to H_3(\pi) \cong H^1(A)$ has kernel $\mathbb{Z}$ because

$$H_3(X) \cong H^1(X) \cong H^1(Y \vee \mathbb{S}^1) \cong H^1(A) \oplus \mathbb{Z}.$$

So the above sequence becomes

$$0 \longrightarrow H_1(\pi; E(\pi)) \longrightarrow \mathbb{Z} \xrightarrow{\deg(c_X)} \mathbb{Z} \longrightarrow H_0(\pi; E(\pi)) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{3.1}$$

For $\pi = A * \mathbb{Z}$ the Chiswell MV sequence (see [7, Theorem 2, p. 70]) gives

$$0 \longrightarrow \Lambda \longrightarrow E(\pi) \longrightarrow H^1(A;\Lambda) \oplus H^1(\mathbb{Z};\Lambda) \longrightarrow 0$$

or, equivalently,

$$0 \longrightarrow \Lambda \longrightarrow E(\pi) \longrightarrow (E(A) \otimes_{\mathbb{Z}} \mathbb{Z}[A\backslash\pi]) \oplus (E(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}\backslash\pi]) \longrightarrow 0,$$

hence

$$0 \longrightarrow \Lambda \longrightarrow E(\pi) \longrightarrow \mathbb{Z}[\mathbb{Z}\backslash\pi] \cong H^1(\mathbb{Z};\Lambda) \longrightarrow 0 \tag{3.2}$$

as $E(A) \cong 0$ (recall that A has one end) and $E(\mathbb{Z}) \cong \mathbb{Z}$.

Applying the functor $H_*(\pi; -)$ to sequence (3.2) yields

$$\begin{aligned} H_1(\pi;\Lambda) \longrightarrow H_1(\pi;E(\pi)) \longrightarrow H_1(\pi;\mathbb{Z}[\mathbb{Z}\backslash\pi]) \longrightarrow H_0(\pi;\Lambda)& \\ \longrightarrow H_0(\pi;E(\pi)) \longrightarrow H_0(\pi;\mathbb{Z}[\mathbb{Z}\backslash\pi]) \longrightarrow 0.& \end{aligned}$$

Since we have $H_1(\pi;\Lambda) \cong H_1(X;\Lambda) \cong H_1(\widetilde{X}) \cong 0$, $H_0(\pi;\Lambda) \cong H_0(\widetilde{X}) \cong \mathbb{Z}$ and $H_i(\pi;\mathbb{Z}[\mathbb{Z}\backslash\pi]) \cong \mathbb{Z}$ for $i = 0, 1$, the last exact sequence becomes

$$0 \longrightarrow H_1(\pi; E(\pi)) \longrightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \longrightarrow H_0(\pi; E(\pi)) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{3.3}$$

One can see that (3.1) and (3.3) are isomorphic. Thus $\deg(c_X) = \pm 1$ if and only if δ is an isomorphism.

For the connected sum $W = Y\#(\mathbb{S}^1 \times \mathbb{S}^3)$, the spectral sequence

$$H_p(\pi; H_q(\widetilde{W})) \Rightarrow H_{p+q}(W)$$

gives

$$\begin{aligned} 0 \longrightarrow H_1(\pi;E(\pi)) \longrightarrow H_4(W) \longrightarrow H_4(\pi) \longrightarrow H_0(\pi;E(\pi))& \\ \longrightarrow H_3(W) \longrightarrow H_3(\pi) \longrightarrow 0.& \end{aligned}$$

Since the collapsing map from W on Y is of degree one, it induces an isomorphism $H_4(W) \to H_4(Y) \cong H_4(\pi)$. This implies that

$$H_1(\pi; E(\pi)) \cong 0 \quad \text{and} \quad H_0(\pi; E(\pi)) \cong \operatorname{Ker}(H_3(W) \to H_3(\pi)) \cong \mathbb{Z}.$$

Thus δ is an isomorphism in (3.3), and so $\deg(c_X) = \pm 1$. Then the classifying map $c_X \colon X \to Y \vee \mathbb{S}^1$ is of degree one. This proves the statement for $r = 1$.

Let now consider $W = Y \# r(\mathbb{S}^1 \times \mathbb{S}^3)$ for $r > 1$, and the spectral exact sequence

$$0 \to H_1(\pi; E(\pi)) \to H_4(W) \to H_4(\pi) \cong H_4(Y) \to H_0(\pi; E(\pi)) \to H_3(W) \\ \cong H_3(Y) \oplus \mathbb{Z}^r \to H_3(\pi) \cong H_3(Y) \to 0.$$

Of course, the collapsing map $W \to Y$ is of degree one, hence

$$H_4(W) \cong H_4(\pi) \cong H_4(Y).$$

So we get $H_1(\pi; E(\pi)) \cong 0$ and $H_0(\pi; E(\pi)) \cong \operatorname{Ker}(H_3(W) \to H_3(\pi)) \cong \mathbb{Z}^r$.

For a closed 4-manifold X such that $\pi = \pi_1(X) = A * F(r)$, $\pi_2(X) = 0$, and $E(\pi) = \overline{H}^1(\pi; \Lambda) \cong H_3(\widetilde{W}) \cong H_3(\widetilde{X})$, there is a similar exact sequence

$$0 \longrightarrow H_1(\pi; E(\pi)) \longrightarrow H_4(X) \cong \mathbb{Z} \xrightarrow{\deg(c_X)} H_4(\pi) \\ \cong \mathbb{Z} \longrightarrow H_0(\pi; E(\pi)) \longrightarrow \mathbb{Z}^r \longrightarrow 0.$$

By the above, this sequence becomes

$$0 \longrightarrow H_4(X) \cong \mathbb{Z} \xrightarrow{\deg(c_X)} H_4(\pi) \cong \mathbb{Z} \longrightarrow \mathbb{Z}^r \underset{\cong}{\rightarrow} \mathbb{Z}^r \longrightarrow 0,$$

so we obtain again $\deg(c_X) = \pm 1$. Thus the classifying map

$$c_X \colon X \to K(\pi, 1) \simeq Y \vee (\vee_r \mathbb{S}^1)$$

has degree one, as claimed. ∎

Remark. There is an isomorphism $\operatorname{Tor}_1^\Lambda(\Lambda/(z-1)\Lambda, \mathbb{Z}) \to \mathbb{Z}$, where $\Lambda = \mathbb{Z}[\pi]$ and z is a generator of the free factor of π. Applying the functor $\otimes_\Lambda \mathbb{Z}$ to sequence (3.2) yields

$$0 \longrightarrow \operatorname{Tor}_1^\Lambda(E(\pi), \mathbb{Z}) \longrightarrow \operatorname{Tor}_1^\Lambda(H^1(\mathbb{Z}; \Lambda), \mathbb{Z}) \longrightarrow \operatorname{Tor}_1^\Lambda(\Lambda, \mathbb{Z}) \\ \cong \mathbb{Z} \longrightarrow E(\pi) \otimes_\Lambda \mathbb{Z} \longrightarrow H^1(\mathbb{Z}; \Lambda) \otimes_\Lambda \mathbb{Z} \longrightarrow \Lambda \otimes_\Lambda \mathbb{Z} \cong 0.$$

The module $H^1(\mathbb{Z}; \Lambda)$ has the presentation

$$0 \longrightarrow \Lambda \xrightarrow{(z-1)\times} \Lambda \longrightarrow \Lambda/(z-1)\Lambda \cong H^1(\mathbb{Z}; \Lambda) \longrightarrow 0,$$

where $\mathbb{Z} = \langle z \rangle$. Recalling that $E(\pi) \cong H_3(\widetilde{X})$, the above sequence is isomorphic to sequence (3.1) (and also (3.3)). In particular, $\deg(c_X) = \pm 1$ if and only if the map

$$\operatorname{Tor}_1^\Lambda(H^1(\mathbb{Z}; \Lambda), \mathbb{Z}) \cong \operatorname{Tor}_1^\Lambda(\Lambda/(z-1)\Lambda, \mathbb{Z}) \to \mathbb{Z}$$

is an isomorphism. This shows that the Chiswell MV sequence in (3.2) cannot split in general. Otherwise, the last map would be trivial. But, if $\beta_2(A) = \beta_2(Y) > 0$, then $\deg(c_X) = \pm 1$, so that map is an isomorphism.

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To construct a homotopy equivalence between X and $W = Y\#r(\mathbb{S}^1 \times \mathbb{S}^3)$, we now use [14, Theorem 1.1(2), Lemma 1.3] for π infinite and nonzero rational second homology, *i.e.*, $H_2(X;\mathbb{Q}) \neq 0$. The classifying maps

$$X \xrightarrow{c_X} B = Y \vee (\vee_r \mathbb{S}^1) \xleftarrow{c_W} W$$

are 3-equivalences, since we have isomorphisms $\pi_1(X) \cong \pi_1(B) \cong \pi_1(W) \cong A * F(r)$ and $\pi_2(X) \cong \pi_2(B) \cong \pi_2(W) \cong 0$. Furthermore, we have (see Lemma 4)

$$(c_X)_*[X] = (c_W)_*[W].$$

Since π is infinite (and torsion free) and $H_2(X;\mathbb{Q}) \neq 0$, we can apply the following lemma proved in [14] to obtain a degree one map h from X to $W = Y\#r(\mathbb{S}^1 \times \mathbb{S}^3)$. Then Theorem 2 follows from the Whitehead theorem.

Lemma 5 *Let X_1 and X_2 be connected oriented 4-dimensional Poincaré spaces with the same 2-stage Postnikov system B and with 3-equivalences $f_i\colon X_i \to B$, for $i = 1, 2$. Let us assume that $H_2(X_i;\mathbb{Q}) \neq 0$ for $i = 1, 2$, and that $\pi \cong \pi_1(X_i)$ is infinite. Then there exists an orientation preserving homotopy equivalence $h\colon X_1 \to X_2$ such that the diagram*

$$\begin{tikzcd} X_1 \arrow[r, "h"] \arrow[d, "f_1"] & X_2 \arrow[d, "f_2"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

commutes, up to homotopy, if and only if $f_{1*}[X_1] = f_{2*}[X_2]$. Here $[X_i] \in H_4(X_i;\mathbb{Z})$ denotes the fundamental class of X_i, as usual.

4 Examples

In this section we give examples of closed orientable 4-manifolds X which satisfy the homotopy conditions of our theorem. First, note that connected sums of closed 4-manifolds with $\pi_2 = 0$ again have $\pi_2 = 0$. So it suffices to consider closed orientable 4-manifolds M with $\pi = \pi_1(M)$ one-ended and $\pi_2(M) = 0$. Since $\pi_3(M) \cong H^1(\pi;\mathbb{Z}[\pi]) \cong 0$ and π is infinite, such manifolds are aspherical. Then π is a PD_4-group and so it is torsion free. Now taking connected sums of these aspherical 4-manifolds with copies of $\mathbb{S}^1 \times \mathbb{S}^3$ gives examples of manifolds X as requested above.

Knot manifolds A 2-*knot* is a locally flat embedding $K\colon \mathbb{S}^2 \to \mathbb{S}^4$. Let $M(K)$ denote the closed 4-manifold obtained from $\mathbb{S}^4$ by surgery on K. Then $M(K)$ is orientable, has Euler characteristic zero, and $\pi K = \pi_1(M(K))$ has weight 1 (*i.e.*, it is the normal closure of a single element) and infinite cyclic abelianization. Suppose that πK is an elementary amenable group, that is, a group which belongs to the class of groups generated from the class of finite groups and $\mathbb{Z}$ by the operations of extension and increasing union. Then $M(K)$ is aspherical if and only if πK has one end and

$H^2(\pi K; \mathbb{Z}[\pi K]) = 0$ (see [16, p. 142]). If $M(K)$ admits a geometry, then it fibres over $\mathbb{S}^1$ (see [18, p. 334]).

Bundles $\mathbb{S}^1$-bundles over closed aspherical 3-manifolds and surface-bundles over aspherical surfaces with aspherical fibers are examples of closed aspherical 4-manifolds. A lot of information about such manifolds can be found in [18, Ch. 4–5]. Further examples are given by 4-manifolds which are finitely covered by a manifold simple homotopy equivalent to a surface-bundle over the torus (see [18, p. 95].

Mapping tori Let M be a closed orientable 4-manifold whose fundamental group $\pi = \pi_1(M)$ is an extension of $\mathbb{Z}$ by a finitely generated normal subgroup ν. By [18, Theorem 4.1, p. 70], if $\chi(M) = 0$, then M is aspherical if and only if ν is infinite and $H^2(\pi; \mathbb{Z}[\pi]) = 0$. In particular, π is one-ended and torsion free. As an example, let M be the product of an aspherical orientable closed 3-manifold N with $\mathbb{S}^1$. Then $\pi = \nu \times \mathbb{Z}$, where $\nu = \pi_1(N)$ is one-ended and torsion free. It is known that the Whitehead group of ν vanishes. By [2, Corollary 15], (see also [31, p. 60] we have isomorphisms $L_4(\pi) = L_4(\nu \times \mathbb{Z}) \cong L_4(\nu) \oplus L_3(\nu)$. Furthermore, $H_4(\pi; \mathbb{L}) = H_4(N \times \mathbb{S}^1; \mathbb{L}) \cong H_4(N; \mathbb{L}) \oplus H_3(N; \mathbb{L}) = H_4(\nu; \mathbb{L}) \oplus H_3(\nu; \mathbb{L})$. It was proved in [21, Theorem 2, Corollary 3] (see also [28, Corollary 1.3, p. 110]) that the integral Novikov conjecture is true for aspherical 3-manifolds. Thus the assembly map from $H_n(\nu; \mathbb{L})$ to $L_n(\nu)$ is a monomorphism, and $\mathcal{A}\colon H_4(\pi; \mathbb{L}) \to L_4(\pi)$ is injective, too. Then the manifolds $X = M\#r(\mathbb{S}^1 \times \mathbb{S}^3)$ satisfy all the conditions of our theorem and are not covered by previously surgery results.

Fibred manifolds A closed connected 4-manifold is *Seifert fibred* if it is the total space of an orbifold-bundle with general fibre a torus or Klein bottle over a 2-orbifold. If the base orbifold is hyperbolic, then such 4-manifolds are aspherical (see [18, Theorem 9.2, Corollary 9.2.1, pp. 181–182]). Many classes of aspherical Seifert fibred 4-manifolds are described in [18, Ch. 9]. See also [18, §13.4, pp. 257–260] for several classes of *complex surfaces* (*i.e.,* compact connected nonsingular complex analytic manifolds of complex dimension 2) which are aspherical Seifert fibred 4-manifolds.

Geometric manifolds The *Davis hyperbolic manifold* [8, 27] is the simplest and most well-known example of a closed connected orientable hyperbolic 4-manifold. It is the orbit space of the unique torsion free normal subgroup of index 14400 of the $(5, 3, 3, 5)$ Coxeter simplex reflection group acting on the hyperbolic 4-space $\mathbb{H}^4$. By [10, 26], fundamental groups of closed hyperbolic orientable manifolds satisfy the integral Novikov conjecture. So taking the connected sums of such manifolds with copies of $\mathbb{S}^1 \times \mathbb{S}^3$ gives examples of X satisfying the conditions of our theorem and which are not covered by previously known surgery results. Several results on aspherical geometric 4-manifolds can be found in [18, Part II]. In particular, a closed 4-manifold which admits a finite decomposition into geometric pieces is (essentially) either geometric or aspherical (see [18, Theorem 7.1, p. 138]).

References

[1] H. J. Baues, *Homotopy Type and Homology.* Oxford Science Publ., Clarendon Press, Oxford, 1996.

[2] S. E. Cappell, *Mayer-Vietoris sequences in Hermitian K-theory.* In: Algebraic K-Theory, III: Hermitian K-Theory and Geometric Applications. Lectures Notes in Math. 343, Springer-Verlag, Berlin, 1973, pp. 478–512.

[3] A. Cavicchioli and F. Hegenbarth, *On* 4*-manifolds with free fundamental group.* Forum Math. **6**(1994), 415–429.

[4] A. Cavicchioli, F. Hegenbarth, and D. Repovš, *On the stable classification of certain* 4*-manifolds.* Bull. Austral. Math. Soc. **52**(1995), 385–398.

[5] ———, *Four-manifolds with surface fundamental groups.* Trans. Amer. Math. Soc. **349**(1997), no. 10, 4007–4019.

[6] A. Cavicchioli and F. Spaggiari, *Topology of four-manifolds with special homotopy groups.* Bull. Austral. Math. Soc. **74**(2006), 321–335.

[7] I. M. Chiswell, *Exact sequences associated with a graph of groups.* J. Pure and Applied Algebra **8**(1976), no. 1, 63–74.

[8] M. W. Davis, *A hyperbolic 4-manifold.* Proc. Amer. Math. Soc. **93**(1985), no. 2, 325–328.

[9] S. C. Ferry and E. K. Pedersen, *Epsilon Surgery Theory.* In: Novikov Conjectures, Index Theorems and Rigidity. London Math. Soc. Lecture Note Ser. 227, Cambridge University Press, Cambridge, 1995, pp. 167–226.

[10] S. C. Ferry, A. Ranicki, and J. Rosenberg, *A History and Survey of the Novikov Conjecture.* In: Novikov Conjectures, Index Theorems and Rigidity. London Math. Soc. Lecture. Note Ser. 226, Cambridge University Press, Cambridge, 1995, pp. 7–66.

[11] M. H. Freedman and F. Quinn, *Topology of* 4*-Manifolds.* Princeton Mathematical Series 39, Princeton University Press, Princeton, NJ, 1990.

[12] M. H. Freedman and P. Teichner, 4*-Manifold topology I: Subexponential groups.* Invent. Math. **122**(1995), no. 3, 509–529.

[13] ———, 4*-Manifold topology II: Dwyer' s filtration and surgery kernels.* Invent. Math. **122**(1995), no. 3, 531–557.

[14] I. Hambleton and M. Kreck, *On the classification of topological* 4*-manifolds with finite fundamental group.* Math. Ann. **280**(1988), no. 1, 85–104.

[15] F. Hegenbarth and D. Repovš, *Solving four-dimensional surgery problems using controlled surgery theory.*(Russian) Fundam. Prikl. Mat. **11**(2005), no. 4, 221–236; translation in J. Math. Sci. (N. Y.) **144**(2007), no. 5, 4516–4526.

[16] J. A. Hillman, *The Algebraic Characterization of Geometric* 4*-Manifolds.* London Math. Soc. Lecture Note Series 198, Cambridge University Press, Cambridge, 1994.

[17] ———, *Free products and* 4*-dimensional connected sums.* Bull. London Math. Soc. **27**(1995), no. 4, 387–391.

[18] J. A. Hillman, *Four-manifolds, Geometries and Knots.* Geometry and Topology Monographs 5, Geometry & Topology Publications, Coventry, (2002),

[19] V. S. Krushkal and R. Lee, *Surgery on closed* 4*-manifolds with free fundamental group.* Math. Proc. Cambridge Philos. Soc. **133**(2002), no. 2, 305–310.

[20] V. S. Krushkal and F. Quinn, *Subexponential groups in* 4*-manifold topology.* Geom. Topol. **4**(2000), 407–430.

[21] B. Jahren and S. Kwasik, *Three-dimensional surgery theory, UNil-groups and the Borel conjecture.* Topology **42**(2003), no. 6, 1353–1369.

[22] T. Matumoto and A. Katanaga, *On* 4*-dimensional closed manifolds with free fundamental groups.* Hiroshima Math. J. **25**(1995), no. 2, 367-370.

[23] E. K. Pedersen, F. Quinn, and A. Ranicki, *Controlled surgery with trivial local fundamental groups.* In: High-dimensional Manifold Topology, World Science Press, River Edge, NJ, 2003, pp. 421–426.

[24] F. Quinn, *A geometric formulation of surgery.* In: Topology of Manifolds. Markham, Chicago, 1970, pp. 500–511.

[25] A. Ranicki, *Algebraic L-Theory and Topological Manifolds.* Cambridge Tracts in Mathematics 102, Cambridge University Press, Cambridge, 1992.

[26] ———, *On the Novikov conjecture.* In: Novikov Conjectures, Index Theorems and Rigidity. London Math. Soc. Lecture Note Series 226, Cambridge University Press, Cambridge, 1995, pp. 272–337

[27] J. G. Ratcliffe and S. T. Tschantz, *On the Davis hyperbolic 4-manifold.* Topology Appl. **111**(2001), no. 3, 327–342.

[28] S. K. Roushon, *L-theory of 3-manifolds with nonvanishing first Betti number.* Internat. Math. Res. Notices **2000**, no. 3, 107–113.

[29] F. Spaggiari, *Four-manifolds with π_1-free second homotopy.* Manuscripta Math. **111**(2003), no. 3, 303–320.
[30] F. Spaggiari, *On the stable classification of Spin four-manifolds.* Osaka J. Math. **40**(2003), no. 4, 835–843.
[31] C. T. C. Wall, *Surgery on Compact Manifolds.* London Mathematical Society Monographs 1, Academic Press, London, 1970.

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