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Corrigendum

The affine part of the Picard scheme

(Compositio Math. 145 (2009), 415–422)

Thomas Geisser

Abstract

We give a corrected version of Theorem 3, Lemma 4, and Proposition 9 in the above-mentioned paper, which are incorrect as stated (as was pointed out by O. Gabber).

In [Gei09, Lemma 4] it was incorrectly stated that extensions of abelian group schemes in the category of group schemes and in the category of big étale sheaves agree, but an epimorphism of group schemes may not induce a surjection of étale sheaves. The remaining part of Lemma 4 is correct, and this is sufficient for the rest of the paper.

To restate Theorem 3, we recall the notation of [Gei09]. Let $p : X \to k$ be a proper scheme over a perfect field k of characteristic exponent p with algebraic closure $\bar{k}$, and let Pic_X be its Picard scheme representing the functor $T \mapsto H^0(T_{\text{ét}}, R^1p_*\mathbb{G}_m)$. Its reduced connected component $\mathrm{Pic}_X^{0,\mathrm{red}}$ is an extension of an abelian variety A_X by the direct product of a torus T_X and a unipotent group U_X. We assume that X is reduced, and consider the semi-normalization $X^+ \to X$, that is, the initial object among strongly universal homeomorphisms $X' \to X$. Here a strongly universal homeomorphism is a universal homeomorphism which induces isomorphisms on residue fields. By [Gei09, Corollary 8], U_{X^+} vanishes.

Theorem 3. *Let X be reduced and proper over a perfect field and let $X' \to X$ be a universal homeomorphism of proper k-schemes with X' seminormal.*

(a) *We have a short exact sequence*

$$0 \to K \to \mathrm{Pic}_X^{0,\mathrm{red}} \to \mathrm{Pic}_{X'}^{0,\mathrm{red}} \to 0, \tag{1}$$

and inclusions of unipotent group schemes

$$U_X \subseteq K \subseteq p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$$

with quotients finite unipotent group schemes.

(b) *The group scheme $p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$ represents the functor*

$$T \mapsto \{\mathcal{O}_{X\times T}\text{-line bundles } \mathcal{L} \subseteq \mathcal{O}_{X'\times T} \text{ which are invertible in } \mathcal{O}_{X'\times T}\}.$$

Proof. The proof of (b) is as in [Gei09, Lemma 4]. To prove (a), we can assume that k is algebraically closed because a group scheme is unipotent if and only if it is after a base extension. We need the following lemma, whose easy proof we omit.

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LEMMA. *Any universal homeomorphism* $f : X' \to X$ *induces isomorphisms* $H^i_{\text{ét}}(X,\mathbb{Z}) \cong H^i_{\text{ét}}(X',\mathbb{Z})$ *as well as isomorphisms* $H^i_{\text{ét}}(X,\mu_m) \cong H^i_{\text{ét}}(X',\mu_m)$ *for all* m *prime to* p.

Combining this with [Gei09, Corollary 6],

$$\operatorname{colim} \operatorname{Hom}_k(\mu_m, T_X) \cong H^1_{\text{ét}}(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z},$$
$$\operatorname{colim} \operatorname{Hom}_k(\mu_m, A_X) \cong \operatorname{Div}({}_{\text{tor}}H^2_{\text{ét}}(X,\mathbb{Z})),$$

we obtain that the canonical map $f^* : \operatorname{Pic}^{0,\text{red}}_X \to \operatorname{Pic}^{0,\text{red}}_{X'}$ induces an isomorphism on the torus components, and is an isogeny with kernel a finite unipotent group scheme P on the abelian variety parts by the following lemma. (Gabber showed in a letter to the author of November 2020 that, conversely, any finite unipotent commutative group scheme can appear as P with $X' = X^+$.)

LEMMA. *A* p*-primary finite group scheme* F *over an algebraically closed field* k *with* $\operatorname{Hom}_k(\mu_p, F) = 0$ *is unipotent.*

Proof. It suffices to show that the dual of F is connected. Since F is p-primary, its étale component has connected dual. If the connected-étale component V was non-trivial, then there would be a non-trivial homomorphism $V^D \to \mathbb{Z}/p$, or equivalently there would be a non-trivial homomorphism $\mu_p \to V$, a contradiction. □

We conclude that f^* is surjective with unipotent kernel K, an extension of P by U_X. To prove the remaining statements, we use the following proposition, which replaces [Gei09, Proposition 9]. Its proof is identical to the proof of [Gei09, Proposition 9(a)].

PROPOSITION 9. *Let* $p : X \to k$ *be proper over a perfect field and* $f : X' \to X$ *be a universal homeomorphism. If* X *and* X' *are reduced, then* f *induces an isomorphism* $p_*\mathbb{G}_{m,X} \cong p_*\mathbb{G}_{m,X'}$.

Applying the proposition to the exact sequence of étale sheaves

$$0 \to p_*\mathbb{G}_{m,X} \to p_*\mathbb{G}_{m,X'} \to p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) \to \operatorname{Pic}_X \to \operatorname{Pic}_{X'}$$

on $\operatorname{Spec} k$, we obtain the following commutative diagram with short exact sequences as columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \operatorname{Pic}^{0,\text{red}}_X & \longrightarrow & \operatorname{Pic}^{0,\text{red}}_{X'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow & & \\
0 & \longrightarrow & p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) & \longrightarrow & \operatorname{Pic}_X & \longrightarrow & \operatorname{Pic}_{X'} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{coker} u & \longrightarrow & \operatorname{NS}_X & \longrightarrow & \operatorname{NS}_{X'} & &
\end{array}$$

Since v is injective, so is u. The Néron–Severi group schemes are extensions of finitely generated étale group schemes $\operatorname{NS}^{\text{ét}}_X$ and $\operatorname{NS}^{\text{ét}}_{X'}$ by finite connected group schemes NS^0_X and $\operatorname{NS}^0_{X'}$, respectively (see [SGA6, XIII 5.1]). The isomorphism $H^2_{\text{ét}}(X,\mu_m) \cong H^2_{\text{ét}}(X',\mu_m)$ implies that $\operatorname{Pic}_X(\bar{k})/m \to \operatorname{Pic}_{X'}(\bar{k})/m$ is injective for any $m > 0$ prime to p, and since $\operatorname{Pic}^{0,\text{red}}_X(\bar{k})$ and $\operatorname{Pic}^{0,\text{red}}_{X'}(\bar{k})$ are m-divisible, the same holds for $\operatorname{NS}_X(\bar{k})/m \to \operatorname{NS}_{X'}(\bar{k})/m$. Consequently the kernel of $\operatorname{NS}^{\text{ét}}_X \to \operatorname{NS}^{\text{ét}}_{X'}$, and hence $\operatorname{coker} u$ is a finite p-primary group.

It remains to prove unipotence of $\operatorname{coker} u$. From $H^i_{\text{ét}}(X,\mathbb{Z}) \cong H^i_{\text{ét}}(X',\mathbb{Z})$ we conclude that $H^1_{\text{ét}}(X,\mathbb{Z}/p) \cong H^1_{\text{ét}}(X',\mathbb{Z}/p)$, or equivalently $\operatorname{Hom}_{\bar{k}}(\mu_p, \operatorname{Pic}_X) \cong \operatorname{Hom}_{\bar{k}}(\mu_p, \operatorname{Pic}_{X'})$, and hence we obtain $\operatorname{Hom}_{\bar{k}}(\mu_p, p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})) = 0$. Since there are no non-trivial commutative extensions

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of the group of multiplicative type μ_p by unipotent groups, we see that $\mathrm{Hom}_{\bar{k}}(\mu_p, \operatorname{coker} u) = 0$ and we can conclude by the lemma. □

Remark. In [Gei09, Theorem 3] we mistakenly stated that $\mathrm{Pic}_X \to \mathrm{Pic}_{X^+}$ is surjective. We reproduce a counterexample due to O. Gabber. Let C be a smooth projective irreducible curve of genus greater than zero over the algebraically closed field k with k-rational point p. Let C' be the following push-out.

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_C/m_p^2 & \longrightarrow & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{p} & C' \end{array}$$

Then $C \times C \to C \times C'$ is the semi-normalization, but the line bundle $\mathcal{L} \in \mathrm{Pic}_{C\times C}(k)$ on $C \times C$ defined by the diagonal is not the pull-back of a line bundle on $C \times C'$, that is, not in the image of $\mathrm{Pic}_{C\times C'}(k)$. Indeed, regarding $\mathcal{L}$ as an element of $\mathrm{Pic}_C(C)$ defines a morphism $C \to \mathrm{Pic}_C$, which is a closed embedding and does not factor through C'.

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