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ON ODA'S PROBLEM AND SPECIAL LOCI

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Abstract Oda's problem, which deals with the fixed field of the universal monodromy representation of moduli spaces of curves and its independence with respect to the topological data, is a central question of anabelian arithmetic geometry. This paper emphasizes the stack nature of this problem by establishing the independence of monodromy fields with respect to finer special loci data of curves with symmetries, which we show provides a new proof of Oda's prediction.

1. Introduction

Let $\mathcal{M}_{g,m}$ be the moduli stack of smooth projective curves of genus g with m (disjoint ordered) sections satisfying the hyperbolicity condition $2g-2+m \geq 1$, which is a smooth geometrically connected Deligne-Mumford stack over $\mathbb{Q}$, and is endowed with a universal punctured curve $\mathcal{C}_{g,m} \to \mathcal{M}_{g,m}$. For X a punctured curve over $\mathbb{Q}$ of topological type (g,m), associated to a morphism $x\colon \operatorname{Spec}\mathbb{Q} \to \mathcal{M}_{g,m}$, one obtains two short exact sequences of étale fundamental groups

$$1 \to \pi_1^{et}(X \otimes \overline{\mathbb{Q}}) \to \pi_1^{et}(\mathcal{C}_{g,m}) \to \pi_1^{et}(\mathcal{M}_{g,m}) \to 1 \text{ and}$$
$$1 \to \pi_1^{et}(\mathcal{M}_{g,m} \otimes \overline{\mathbb{Q}}) \to \pi_1^{et}(\mathcal{M}_{g,m}) \xrightarrow{p} G_{\mathbb{Q}} \to 1, \qquad (1.1)$$

where the fundamental groups are given with respect to a choice of compatible base points that we omit. Denoting $X \otimes \overline{\mathbb{Q}}$ by $X_{\overline{\mathbb{Q}}}$, the left-hand one gives rise to the *universal ℓ-monodromy representation*

$$\Phi_{g,m}^{\ell}\colon \pi_1^{et}(\mathcal{M}_{g,m}) \to \operatorname{Out}\pi_1^{et}(X_{\overline{\mathbb{Q}}}) \to \operatorname{Out}\pi_1^{\ell}(X_{\overline{\mathbb{Q}}}),$$

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where the right-hand side morphism comes, for ℓ a fixed prime, from the surjective map $\pi_1^{et}(X_{\overline{\mathbb{Q}}}) \to \pi_1^{\ell}(X_{\overline{\mathbb{Q}}})$ to the pro-ℓ geometric fundamental group of X (also the maximal pro-ℓ quotient of the geometric one). Composing with the section induced by x between Galois and étale fundamental groups, one furthermore recovers the ℓ-adic representation associated to X

$$\varphi_X^{\ell}\colon G_{\mathbb{Q}} \to \operatorname{Out}\pi_1^{\ell}(X_{\overline{\mathbb{Q}}}),$$

which, contrary to the classical profinite geometric Galois action, has a nontrivial kernel whose corresponding fixed field contains $\mathbb{Q}_{g,m}^{\ell} = \overline{\mathbb{Q}}^{p(\operatorname{Ker}\Phi_{g,m}^{\ell})}$.

The following prediction, as formulated in [17] §1.4, stems from Takayuki Oda's original conjecture formulated in [27].

Oda's prediction. *For $g, m \in \mathbb{N}$ such that $2g-2+m>0$, the ℓ-monodromy fixed field $\mathbb{Q}_{g,m}^{\ell}$ associated to $\Phi_{g,m}^{\ell}$ is independent of (g,m).*

As noted in [27], the group $\operatorname{Out}\pi_1^{\ell}(X_{\overline{\mathbb{Q}}})$ is 'almost intractable', which motivates Oda to formulate his conjecture in terms of a seemingly more reachable but stronger weight-filtration version of the above prediction, and for fixed $g \geq 0$, see ibid. *§2. Theorem and conjectures*. Oda's prediction is finally settled[1] for every (g,m) by Takao in [31] following successive advances on the independence in g or m in terms of arithmetic-geometry (see Ihara and Nakamura in [17]), of group theoretic and Lie algebra computations (see Nakamura-Takao-Ueno [25] and Matsumoto [20]) and by the use of the (divisorial) Knudsen-Mumford stratification of $\mathcal{M}_{g,m}$ (see [23]). An independant proof was later given in terms of combinatorial anabelian geometry by Hoshi and Mochizuki in [13]. We also refer to [32] for a recent panorama.

Oda's problem – that is, to which extent canonical arithmetic and geometric data such as g and m, produce independent ℓ-monodromy fixed fields – is a central question of anabelian arithmetic geometry: it allows the study of the Deligne-Ihara Lie algebra [15] related to motivic multiple zeta values, since for $(g,m)=(0,3)$, the morphism $\Phi_{0,3}^{\ell}$ is one of Ihara's 山 = 天 questions on $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$ [14], which in turn, is related to the Rasmussen-Tamagawa conjecture [28]. It also has application in low-dimensional topology via the Johnson homomorphism and the Morita obstructions [22]. This conjecture has since motivated the anabelian notion of monodromic fullness [12].

We remark that, as presented in [20] Remark 3.3, while Oda's problem is essentially of stack-theoretic nature – by $\mathcal{M}_{g,m}$ as a solution to a fine moduli problem and the very existence of the universal punctured curve $\mathcal{C}_{g,m}$ – the field $\mathbb{Q}_{g,m}^{\ell}$ was expressed and dealt with in a scheme-theoretic way. This paper develops a setup and techniques that allow to exploit the stack-theoretic aspects of Oda's problem.

Oda's problem for G-special loci

Let $\mathcal{M}_{g,[m]}$ denote the moduli stack of curves of genus g with m (unordered) marked points (in particular, $\mathcal{M}_{g,[m]}$ is not represented by a scheme), which is naturally endowed

[1] Publication of the proof, established in 1995, was indeed postponed to 2012 for unfortunate non-mathematical ground.

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with a stack inertia stratification (i.e., by the automorphism group of objects). Each strata corresponds to a *G-special locus* $\mathcal{M}_{g,[m]}(G)$ of curves whose automorphism group contains a given finite group G. It is shown that the geometric irreducible components $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ for G cyclic automorphism group, that are among the biggest nontrivial strata, are $\mathbb{Q}$-rational and can be described by combinatorial Hurwitz data $\underline{kr}$; see [4].

This context also provides an ℓ-universal G-monodromy representation; see Theorem 2.2.

There exists a universal monodromy representation

$$\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}}\colon \pi_1^{et}(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow \operatorname{Out}\pi_1^{\ell}(X)$$

for X a smooth curve with compactification $\overline{X}$ represented by a $\overline{\mathbb{Q}}$-point on $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ and where $\overline{X}\setminus X$ is a divisor of degree m on $\overline{X}$.

In particular, this setup provides an ℓ-monodromy fixed field $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}} = \overline{\mathbb{Q}}^{p(\operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}})}$ where p denotes the usual projection to $G_{\mathbb{Q}}$. In this paper, we deal with the following $\mathbb{Z}/\ell^n\mathbb{Z}$-special loci version of Oda's problem.

Oda's problem for cyclic special loci. *For g, $m\in\mathbb{N}$ such that $2g-2+m>0$ and G cyclic group of order ℓ^n is the ℓ-monodromy fixed field $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}$ independent of all the special loci data (g,m), n and $\underline{kr}$?*

While a positive answer to this problem may at first seems 'unreasonable' – Oda's problem for cyclic special loci is finer and implies Oda's prediction – it is supported by a series of indirect results that exhibit *similar arithmetic properties of the stack inertia stratification to the classical divisorial one*: the Galois actions have the same type [5], and the related Grothendieck-Teichmüller groups are isomorphic [3]. More concretely, one notices that the curves used in [20] §4 to establish Oda's prediction for $2g=0 \mod (\ell-1)$ live in $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ with $G=\mathbb{Z}/\ell\mathbb{Z}$, quotient genus $g'=0$ and some $\underline{kr}$ data with $k=(1,\dots,1,j,-(1+1\cdots+1+j))$ for $j=1$ or 2; see Section 2.1.1 for notations.

Indeed, the main results of this paper can be summarized as follows; see Section 2.2 for the compatibility of the various ℓ-universal monodromy fields and morphisms and Theorem 5.3.

Theorem. *Let ℓ be a fixed prime. Let $g,m\in\mathbb{N}$ be such that $2g-2+m>0$ and $\underline{kr}$ an associated abstract Hurwitz data such that $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is nonempty. The map $\Phi^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is compatible with the map $\Phi^{\ell}_{g,m}$ and the ℓ-monodromy fixed field $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is constant equal to $\mathbb{Q}^{\ell}_{0,3}$.*

As a corollary, see Corollary 5.4, we recover the containment $\mathbb{Q}^{\ell}_{g,m}\subset\mathbb{Q}^{\ell}_{0,3}$ and thus the classical version of Oda's prediction; that is, for all g, $m\in\mathbb{N}$ such that $2g-2+m>0$, we have $\mathbb{Q}^{\ell}_{g,m}=\mathbb{Q}^{\ell}_{0,3}$ (see also [17] Theorem 3 B). Both proofs of Oda's problem for special loci and classical settings still rely on the previously established $\mathbb{Q}^{\ell}_{0,3}\subset\mathbb{Q}^{\ell}_{g,m}$; see [23, 20, 31].

The organization of the paper is as follows. In Section 2, we recall the $\underline{kr}$ combinatorial description of irreducible components of cyclic special loci of [4] and introduce the

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ℓ-universal G-monodromy representation, whose fixed field we relate within a lattice of other ℓ-monodromy fixed fields, which, in particular, includes the more traceable Hurwitz spaces $\mathcal{M}_{g,[m]}[G]_{\underline{kr}}$

$$\begin{array}{ccccc} \mathbb{Q}^{\ell}_{g,m} & \hookrightarrow & \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})_{\underline{kr}} & \longrightarrow & \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} \\ \uparrow & & & & \uparrow \\ \mathbb{Q}^{\ell}_{0,3} & \hookleftarrow\dashrightarrow & \mathbb{Q}^{\ell}_{g',m'} & \hookrightarrow & \mathbb{Q}^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}, \end{array}$$

where (g',m'), resp.$\mathbb{Q}^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}$, denotes the topological data, resp. a certain monodromy fixed field, obtained by G-quotient. At this stage, establishing the G-special version of Oda's prediction relies on showing that $\mathbb{Q}^{\ell}_{g,m}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{0,3}$; our proof adapts Ihara-Nakamura's [17]. Section 3 deals with the construction of tangential base points, or one-parameter families, on the G-stable compactification of Hurwitz spaces in terms of formal patching of certain Matsumoto-Seyama $\mathbb{Z}/\ell\mathbb{Z}$-stable curves, whose Galois action properties are established in Section 4 via Grothendieck-Murre theory and by comparison with Deligne's original tangential base point. This results in the inclusion of the ℓ-monodromy fixed field of the generic fiber of the constructed one-parameter families into $\mathbb{Q}^{\ell}_{0,3}$. We conclude with a general Theorem 4.12 that can be applied to multiple geometric situations. Section 5 ties everything together for $\mathbb{Z}/\ell\mathbb{Z}$, starting with the case of proper loci for which the deformation method does not apply. In the diagram above, Oda's classical prediction then follows the bottom row reading.

Notations and conventions. For G a finite group, we write $\mathcal{M}_{g,[m]}[G]$ for the Hurwitz space of G-covers and $\mathcal{M}_{g,[m]}(G)^{\nu}$ for the quotient $\mathcal{M}_{g,[m]}[G]/\operatorname{Aut}G$. We denote by $\overline{\mathcal{M}}_{g,[m]}(G)$ the G-stable compactification of the G-special locus $\mathcal{M}_{g,[m]}(G)$, and by $\overline{\mathcal{M}}_{g,[m]}(G)^{\nu}$ the stable compactification of $\mathcal{M}_{g,[m]}(G)^{\nu}$. The topological data (g,m) of a curve are said to be of hyperbolic type if they satisfy $2g-2+m>0$.

2. Oda's conjecture for G-special loci

After some brief reminders on the description of irreducible components $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ of cyclic G-special loci in terms of combinatorial Hurwitz data $\underline{kr}$, we define the ℓ-universal G-monodromy representation $\Phi^{\ell}_{g,m}(G)_{\underline{kr}}\colon \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow \operatorname{Out}\pi^{\ell}_1(X)$ – for G any finite group – where X is a hyperbolic curve of type (g,m). Relying on the forgetful functor and the quotient functor

$$\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}} \to \mathcal{M}_{g,[m]}(G)_{\underline{kr}} \to \mathcal{M}_{g,[m]}, \text{ and } \mathcal{M}_{g,[m]}[G]_{\underline{kr}} \xrightarrow{\delta} \mathcal{M}_{g',[m']}$$

and some properties of the stack inertia $\mathcal{I}_{\mathcal{M}}$, we build step-by-step *a lattice of relations between the various ℓ-monodromy fixed fields arising from this context* – that is, between $\mathbb{Q}^{\ell}_{g,m}$, $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})_{\underline{kr}}$, $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}}$, and $\mathbb{Q}^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}}$.

2.1. Universal monodromy representations and Oda's fields for G-special loci

2.1.1. Hurwitz data of special loci. Let $\mathcal{M}_{g,[m]}[G]$ be the moduli stack of curves of genus g with m marked points endowed with a faithful G-action, or *Hurwitz stack*,

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whose S-sections for a $\mathbb{Q}$-scheme S are defined as follows: the objects of $\mathcal{M}_{g,[m]}[G](S)$ are triplets (C,D,ι) where

$$\begin{cases} C \text{ is a smooth projective curve of genus } g \text{ over } S, \\ \iota\colon G \to \operatorname{Aut}_S C \text{ an injective homomorphism}, \\ D \text{ an tale Cartier divisor of degree } m \\ \quad \text{stabilized by the } G\text{-action induced by } \iota; \end{cases}$$

see [4] §2.1 as well as for the rest of this section. The *G-special locus $\mathcal{M}_{g,[m]}(G)$ of $\mathcal{M}_{g,[m]}$* is obtained as the image of $\mathcal{M}_{g,[m]}[G]$ in $\mathcal{M}_{g,[m]}$ under the forgetful functor defined by

$$\begin{array}{ccc} \mathcal{M}_{g,[m]}[G](S) & \longrightarrow & \mathcal{M}_{g,[m]}(S) \\ (C,D,\iota) & \longmapsto & (C,D). \end{array}$$

In particular, the S-sections of $\mathcal{M}_{g,[m]}(G)$ are curves over S whose geometric fibers admits a faithful G-action. The stack $\mathcal{M}_{g,[m]}[G]$ having a canonical right-action of $\operatorname{Aut} G$ via ι, we can form the quotient stack $\mathcal{M}_{g,[m]}[G]/\operatorname{Aut} G$ that we denote by $\mathcal{M}_{g,[m]}(G)^\nu$ since, apart from a few exceptional cases[2] (see [19] Theorem 5.1 and Section 4 for an account with $g \geq 2$ and also Remark 2.1 2.1.1), it identifies with the normalization of $\mathcal{M}_{g,[m]}(G)$ by the proof of [29] Proposition 3.4.1. All the stacks $\mathcal{M}_{g,[m]}[G]$, $\mathcal{M}_{g,[m]}(G)$ and $\mathcal{M}_{g,[m]}(G)^\nu$ are Deligne-Mumford stacks over $\operatorname{Spec}\mathbb{Q}$ – with $\mathcal{M}_{g,[m]}[G]$ and $\mathcal{M}_{g,[m]}(G)^\nu$ moreover smooth over $\operatorname{Spec}\mathbb{Q}$.

From now on, *we assume that $G \simeq \mathbb{Z}/n\mathbb{Z}$ is cyclic*, so that following [4], we can investigate the subloci $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ of $\mathcal{M}_{g,[m]}(G)$ of S-curves whose G-action ramification data correspond to certain Hurwitz data $\underline{kr} = (k,r)$ modulo the diagonal $(\mathbb{Z}/n\mathbb{Z})^\times$-action, which are abstractly defined as follows:

- The part k corresponds to an N-tuple in $(\mathbb{Z}/n\mathbb{Z})^N$, where N is the degree of the branch divisor, whose terms sum to 0, and which is taken up to permutation. Each component of k corresponds to a generator of one of the G-isotropy groups.
- The second part r is an element of $\mathbb{N}^n$, whose i-th component, in the case of a quotient map $\psi\colon C \to C/G$, corresponds to

  $$r(i) = \operatorname{Card}\{y \in D/G \mid br(y) = i \bmod n\},$$

  where $br(y)$ is the branching order at y – that is, the ramification index of any point in the fiber $\psi^{-1}(y)$.

Note that the $(\mathbb{Z}/n\mathbb{Z})^\times$-quotient in $\underline{kr}$ should be seen as the $(\operatorname{Aut} G)$-quotient previously introduced. We refer to ibid. Definitions 3.5 and 3.9, and Example 3.11 for further details.

The construction of abstract Hurwitz data from G-curves defines a map

$$\underline{kr}\colon \mathcal{M}_{g,[m]}[G]_N \longrightarrow ((\mathbb{Z}/n\mathbb{Z})^N/\mathfrak{S}_N \times \mathbb{N}^n)/(\mathbb{Z}/n\mathbb{Z})^\times,$$

where $\mathcal{M}_{g,[m]}[G]_N$ denotes the substack of $\mathcal{M}_{g,[m]}[G]$ of curves whose branch divisor is of degree N, which is locally constant – see [4] Lemma 3.13. For a fixed value of $\underline{kr}$, one thus

[2] Erratum: Proposition 2.4 and Corollaire 2.5 of [4] are subject to the same exceptions.

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obtains a substack $\mathcal{M}_{g,[m]}[G]_{\underline{kr}}$ of $\mathcal{M}_{g,[m]}[G]$ of G-curves with abstract Hurwitz data $\underline{kr}$ so that one can define the following:

Definition 2.1. For G cyclic and given abstract Hurwitz data $\underline{kr}$, the special sublocus $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ is the image of $\mathcal{M}_{g,[m]}[G]_{\underline{kr}}$ under the forgetful functor $\mathcal{M}_{g,[m]}[G] \to \mathcal{M}_{g,[m]}(G)$.

Also, since the action of $\operatorname{Aut} G$ stabilizes $\mathcal{M}_{g,[m]}[G]_{\underline{kr}}$ by definition, we have substacks $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ of $\mathcal{M}_{g,[m]}(G)^{\nu}$. The stacks $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ and $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ are defined over $\mathbb{Q}$ by construction and are geometrically irreducible by Proposition 3.12 and Theorem 4.3 of [4].

One particular case of interest is when the ramification divisor is contained in the marked divisor D. In this case, we can recover r by the data of D and k. Indeed, we have

$$\begin{cases} r(i) = \operatorname{Card}\{j \mid k(j) = i\} / \gcd(i,n) \text{ for } i \neq 0 \\ r(0) = \deg D - \sum_{i \in \mathbb{Z}/n\mathbb{Z}\setminus\{0\}} \operatorname{Card}\{j \mid k(j) = i\}. \end{cases}$$

Similarly to the moduli stacks of curves, the stacks $\mathcal{M}_{g,[m]}(G)$, resp. $\mathcal{M}_{g,[m]}(G)^{\nu}$, are not necessarily proper. We denote by $\overline{\mathcal{M}}_{g,[m]}(G)_{\underline{kr}}$ the G-stable compactification of the G-special locus $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$, and by $\overline{\mathcal{M}}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ the G-stable compactification of $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$. These are obtained from the original stacks by adding stable curves endowed with a stable G-action. We refer to [6] and [2] §4 and 6 for details.

Remark 2.1.

1. The correspondence between the abstract Hurwitz data $\underline{kr}$ and the Hurwitz data ξ of [2] §2.2 in terms of equivalence classes $[H_i, \chi_i]$ of characters χ_i at G-inertia group H_i is straightforward by considering generators of the G-isotropy groups.
2. The difference between $\mathcal{M}_{g,[m]}(G)$ and $\mathcal{M}_{g,[m]}(G)^{\nu}$ comes from the potential existence of a curve whose geometric fiber has an automorphism group that contains 2 topologically but not holomorphically conjugate subgroups. We refer to [7] for examples.

2.1.2. The universal *G*-monodromy representation. We now consider $\mathcal{C}_{g,[m]}(G)_{\underline{kr}}$ the universal G-curve of genus g with m punctures and abstract Hurwitz data $\underline{kr}$. We denote by $\mathcal{M}_{g,[m]+1}$ the stack of smooth projective curves with a degree m divisor and an additional marked point. We have an identification $\mathcal{C}_{g,[m]}(G)_{\underline{kr}} \simeq \mathcal{M}_{g,[m]}(G)_{\underline{kr}} \times_{\mathcal{M}_{g,[m]}} \mathcal{M}_{g,[m]+1}$. The S-sections of $\mathcal{C}_{g,[m]}(G)_{\underline{kr}}$ are the elements of $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}(S)$ with the additional data of a section outside the marked points D; similarly, the universal punctured curve over $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ is given by the stack $\mathcal{C}_{g,[m]}(G)^{\nu}_{\underline{kr}} = \mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}} \times_{\mathcal{M}_{g,[m]}} \mathcal{M}_{g,[m]+1}$.

One obtains the ℓ-universal G-monodromy representation of étale fundamental groups.

Theorem 2.2. *Let g, $m \in \mathbb{N}$ such that $2g-2+m>0$, G a finite cyclic group and $\underline{kr}$ a Hurwitz data with respect to g, m and G. Then there is an exact sequence*

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(\mathcal{C}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow 1, \qquad (2.1)$$

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where X denotes a geometric curve of type (g,m). The ℓ-universal G-monodromy representation is the induced monodromy map

$$\Phi^{\ell}_{g,m}(G)_{\underline{kr}} \colon \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow \operatorname{Out}\pi_1^{\ell}(X), \tag{2.2}$$

which is universal in the following sense: for any curve C over a connected $\mathbb{Q}$-scheme S in $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}(S)$ and $\overline{\mathbb{Q}}$-point $\overline{s}$ of S, the natural representation $\pi_1(S) \to \operatorname{Out}\pi_1^{\ell}(C_{\overline{s}})$ factors through $\Phi^{\ell}_{g,m}(G)_{\underline{kr}}$. A similar result holds for $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$.

Note that the exact sequence above is independent of the choice of the geometric curve X of type (g,m). In the case $m \geq 1$, one identifies $\pi_1^{\ell}(X) \simeq \widehat{F}^{\ell}_{2g+m-1}$ with the pro-ℓ completion of a free group; for general hyperbolic type (g,m), the group $\pi_1^{\ell}(X)$ is the pro-ℓ completion of a surface group.

Proof. The exactness of Equation (2.1) follows a classical argument on the geometric parts of the étale fundamental groups. Considering $\mathcal{C}_{g,[m]}(G)_{\underline{kr}}$ as the stack of m-marked curves with G-symmetries and an additional marked point, then erasing this additional point, provide a Birman point-erasing exact sequence between orbifold fundamental groups:

$$1 \longrightarrow \pi_1^{top}(X) \longrightarrow \pi_1^{top}(\mathcal{C}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow \pi_1^{top}(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \longrightarrow 1$$

whose exactness is preserved after profinite completion: The profinite completion is always right-exact; the left-exactness follows from [1] Proposition 4 and above by residual finitness of $\pi^{top}(X)$ and the centerfreeness of the profinite completion of surface groups; see ibid. Proposition 8 and Proposition 18.

A similar argument provides the result for $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ with *ad hoc* substitutions. □

For a curve C over S as in Theorem 2.2, the *ℓ-monodromy representation of C*

$$\varphi^{\ell}_{C} \colon \pi_1(S) \to \operatorname{Out}\pi_1^{\ell}(C_{\overline{s}})$$

is obtained from the relative homotopy exact sequence as usual. Notice that the $\mathbb{Q}$-scheme S also sits in a classical arithmetic-geometric homotopy exact sequence, so that $\pi_1(S)$ is naturally equipped with a projection map $p_S \colon \pi_1(S) \to G_{\mathbb{Q}}$. We recall that, similarly, we have a canonical homomorphism $p \colon \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \to G_{\mathbb{Q}}$.

Definition 2.2. The field $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}$, resp. $\mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}}$, is the fixed field of $p(\operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}})$, resp. of $p(\operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}})$. For a curve C over a connected $\mathbb{Q}$-scheme S, the field $\mathbb{Q}^{\ell}_{C}$ is the fixed field of $p_S(\operatorname{Ker}\varphi^{\ell}_{C})$.

Lemma 2.3. *For C a curve over a connected $\mathbb{Q}$-scheme S represented by an S-point on $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$, resp. on $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$, we have the inclusion*

$$\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{C}, \ \textit{resp.}\ \mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{C}.$$

The ℓ-monodromy fixed field $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}$ is furthermore obtained as the intersection of all the $\mathbb{Q}^{\ell}_{C}$ for such C/S where S varies in the category of connected $\mathbb{Q}$-schemes.

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The field $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}$ can also be obtained as $\mathbb{Q}^{\ell}_{C_0}$ where $C_0 = \mathcal{C}_{g,[m]}(G)_{\underline{kr}} \times_{\mathcal{M}_{g,[m]}(G)_{\underline{kr}}} \mathcal{M}_{g,[m]+m'}(G)_{\underline{kr}}$ is a curve over $S = \mathcal{M}_{g,[m]+m'}(G)_{\underline{kr}}$ with m' large enough for S to be a scheme.

Proof. By the universality of the map $\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}}$, we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(S) & \longrightarrow & \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) & \longrightarrow & \operatorname{Out}\pi_1^{\ell}(C) \\ {\scriptstyle p_S}\downarrow & & \downarrow{\scriptstyle p} & & \\ G_{\mathbb{Q}} & = & G_{\mathbb{Q}}, & & \end{array}$$

where C denotes a hyperbolic curve of type (g,m), and where φ^{ℓ}_C appears as the composition $\pi_1(S) \to \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \to \operatorname{Out}\pi_1^{\ell}(C)$. The compatibility with the projections to $G_{\mathbb{Q}}$ ensures that we have $p_S(\operatorname{Ker}\varphi^{\ell}_C) \subset p(\operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}})$ and thus the inclusion. To prove the last point, by commutativity of the diagram, it suffices to show the existence of a curve C in $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}(S)$ such that the induced map $\pi_1(S) \to \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}})$ is surjective. This is done by taking $C_0 = \mathcal{C}_{g,[m]}(G)_{\underline{kr}} \times_{\mathcal{M}_{g,[m]}(G)_{\underline{kr}}} \mathcal{M}_{g,[m]+m'}(G)_{\underline{kr}}$ over $S = \mathcal{M}_{g,[m]+m'}(G)_{\underline{kr}}$ with m' large enough for S to be a scheme.

The case of $\mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ is similar after replacing $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ by $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$. □

2.1.3. The case of Hurwitz data of étale type. Let us now relate the general situation to the one where the divisor of marked points D contains the ramification divisor R of the G-action, a property that we recall can be seen directly on the abstract Hurwitz data.

By base change to an algebraically closed field and reading of the $\underline{kr}$ data, one notices that the divisor $R \cup D$ is finite étale over S for a curve C/S as before.

Lemma 2.4. *Let (C,D) be a curve represented by an S-point on $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ as before. Then the degree of the ramification divisor R of C and of the divisor $R \cup D$ are determined by the abstract Hurwitz data $\underline{kr}$.*

Proof. As everything is locally constant on the base, it is enough to treat the case where S is the spectrum of an algebraically closed field. By definition of $\underline{kr}$, the degree $\deg R = N$ of the ramification divisor is the length of k. Furthermore, since the degree of $R \cap D$ is given by $\sum_{i=1}^{n-1} \gcd(i,n) \cdot \underline{r}(i)$, we have the formula

$$\deg R \cup D = m + N - \sum_{i=1}^{n-1} \gcd(i,n) \cdot r(i),$$

which is entirely determined by m, $\underline{kr}$ and $G = \mathbb{Z}/n\mathbb{Z}$. □

For an abstract Hurwitz data $\underline{kr}$, we introduce $\underline{kr}^{et}$ as the minimal associated Hurwitz data such that the ramified points are contained in the marked divisor – thata is, minimal in the sense that the new marked divisor is the smallest one containing D and R – and

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which is thus defined by

$$\begin{cases} r^{et}(0) = r(0) \\ r^{et}(i) = \mathrm{Card}\{j \in \{1,\ldots,N\} \mid k(j) = i\}, \quad i \geq 1. \end{cases}$$

Proposition 2.5. *There is a natural map of stacks*

$$\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}} \longrightarrow \mathcal{M}_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{et}},$$

where $\underline{r}^{et}$ *and* $s = \deg R - \deg R \cap D$ *can be explicitly determined as above.*

Proof. By the previous lemma, we have that if (C,D) is in $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}(S)$, then $(C, R \cup D)$ is an element of $\mathcal{M}_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{et}}(S)$. This association defines a map of groupoids as any isomorphism preserving the G-action must also preserve the ramification divisor. □

Theorem 2.6. *We have the following inclusion of* ℓ*-monodromy fixed fields:*

$$\mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{et}}. \tag{2.3}$$

Proof. Let $\sigma \in p(\mathrm{Ker}\,\Phi^{\ell}_{g,[m+s]}(G)_{\underline{kr}^{et}}) \subset G_{\mathbb{Q}}$. By Lemma 2.3, there is a connected $\mathbb{Q}$-scheme S and a curve (C,D) over S represented by an S-point on $\mathcal{M}_{g,[m+s]}(G)_{\underline{kr}^{et}}$ such that σ has a lift τ in the kernel of the map

$$\pi_1(S) \xrightarrow{s_C} \pi_1(\mathcal{M}_{g,[m+s]}(G)_{\underline{kr}^{et}}) \xrightarrow{\Phi^{\ell}_{g,[m+s]}(G)_{\underline{kr}^{et}}} \mathrm{Out}\,\pi_1^{\ell}(X),$$

where X denotes a hyperbolic curve of type (g,m). The divisor D admits a decomposition $D = D^{un} \cup D^{ram}$ where D^{un} is given by the unramified marked points and D^{ram} by the ramified marked points. By definition of the component $\underline{r}^{et}$ of $\underline{kr}^{et}$, the divisor D^{ram} corresponds to all the ramified points. The divisor D^{ram} splits into a disjoint union of geometrically irreducible divisors over a finite étale extension $S' = S_K$ of S where K is defined by the property that G_K stabilizes each geometric component of D^{ram}. In particular, $\pi_1(S')$ contains the subgroup $\{\alpha \in \pi_1(S) \mid p_S(\alpha) \in G_K\}$, which contains τ by construction. We can thus assume that $S = S'$.

By removing some chosen orbits of ramified points in D^{ram} according to the data given by $\underline{r}$, we can form a divisor $D' = D^{un} \cup D^{ram'}$ such that (C, D') gives an S-point of $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$. Hence, it is sufficient to show that σ is the image of an element of $\pi_1(S)$ that acts trivially on the pro-ℓ-fundamental group of a geometric fiber $C_{\overline{s}} \setminus D'_{\overline{s}}$ of $C \setminus D'$. This now comes from the fact that the outer actions of $\pi_1(S)$ on $\pi_1^{\ell}(C_{\overline{s}} \setminus D_{\overline{s}})$ and $\pi_1^{\ell}(C_{\overline{s}} \setminus D'_{\overline{s}})$ are compatible with the canonical surjection $\pi_1^{\ell}(C_{\overline{s}} \setminus D_{\overline{s}}) \to \pi_1^{\ell}(C_{\overline{s}} \setminus D'_{\overline{s}})$. □

2.2. From the classical to the special loci settings

In order to relate the ℓ-monodromy fixed fields $\mathbb{Q}^{\ell}_{g,m}$ and $\mathbb{Q}^{\ell}_{g,m}(G)_{\underline{kr}}$, let us start by showing that we can move from $\mathcal{M}_{g,m}$ to $\mathcal{M}_{g,[m]}$ without harm. Let $\mathbb{Q}^{\ell}_{g,[m]}$ be the fixed field of $p(\mathrm{Ker}\,\Phi^{\ell}_{g,[m]})$ where $p\colon \mathcal{M}_{g,[m]} \to \mathrm{Spec}\,\mathbb{Q}$ is the structure map and $\Phi^{\ell}_{g,[m]}\colon \pi_1(\mathcal{M}_{g,[m]}) \to \mathrm{Out}\,\pi_1^{\ell}(C)$ the outer Galois action coming from the exact sequence

$$1 \longrightarrow \pi_1(C) \longrightarrow \pi_1(\mathcal{M}_{g,[m]+1}) \longrightarrow \pi_1(\mathcal{M}_{g,[m]}) \longrightarrow 1,$$

where C is a geometric fiber of $\mathcal{M}_{g,[m]+1} \to \mathcal{M}_{g,[m]}$. The following can also be seen as a special case of [11] Lemma 1.4 (ii).

Lemma 2.7. *We have* $\mathbb{Q}^{\ell}_{g,m} = \mathbb{Q}^{\ell}_{g,[m]}$.

Proof. It suffices to see that the equality $\operatorname{Ker}\Phi^{\ell}_{g,m} = \operatorname{Ker}\Phi^{\ell}_{g,[m]}$ holds in $\pi_1(\mathcal{M}_{g,[m]})$ as we have $\pi_1(\mathcal{M}_{g,m}) \subset \pi_1(\mathcal{M}_{g,[m]})$ with cokernel $\mathfrak{S}_m$. For a presentation of $\pi_1^{\ell}(C)$ given by

$$\langle y_1,\dots,y_{2g},x_1,\dots,x_m \mid [y_1,y_2]\cdots[y_{2g-1},y_{2g}]x_1\cdots x_m = 1\rangle,$$

it is clear that an element $\tau \in \pi_1(\mathcal{M}_{g,[m]})$ has image $\sigma \in \mathfrak{S}_m$ if and only if the permutation induced by τ on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1^{\ell}(C_{\overline{\mathbb{Q}}})$, which is in bijection with the set $\{x_1,\dots,x_m\}$, is the one given by σ. Such an element τ thus has trivial outer action on $\pi_1^{\ell}(C)$ only if it has trivial image in $\mathfrak{S}_m$ and thus belongs to $\pi_1(\mathcal{M}_{g,m})$. □

2.2.1. First monodromy fixed fields comparisons. The comparison via the forgetful functor $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}} \to \mathcal{M}_{g,[m]}(G)_{\underline{kr}} \to \mathcal{M}_{g,[m]}$ is now straightforward.

Proposition 2.8. *For all* (g,m) *of hyperbolic type and compatible Hurwitz data* $\underline{kr}$*, we have* $\mathbb{Q}^{\ell}_{g,m} \subset \mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}}$.

Proof. Let C be a curve over $\overline{\mathbb{Q}}$ represented on $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$. First, see that the sequence of maps

$$\pi_1(\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}) \to \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) \to \pi_1(\mathcal{M}_{g,[m]}) \to \operatorname{Out}\pi_1^{\ell}(C)$$

induces a sequence

$$\operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}} \to \operatorname{Ker}\Phi^{\ell}_{g,[m]}(G)_{\underline{kr}} \to \operatorname{Ker}\Phi^{\ell}_{g,[m]},$$

where the second map is obtained by considering the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C) & \longrightarrow & \pi_1(\mathcal{C}_{g,[m]}(G)_{\underline{kr}}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[m]}(G)_{\underline{kr}}) & \longrightarrow & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C) & \longrightarrow & \pi_1(\mathcal{M}_{g,[m]+1}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[m]}) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn}\pi_1^{\ell}(C) & \longrightarrow & \operatorname{Aut}\pi_1^{\ell}(C) & \longrightarrow & \operatorname{Out}\pi_1^{\ell}(C) & \longrightarrow & 1,
\end{array}$$

and the first map is obtained in a similar way.

By applying the canonical projections to $G_{\mathbb{Q}}$, and Lemma 2.7 for $\mathbb{Q}^{\ell}_{g,m} = \mathbb{Q}^{\ell}_{g,[m]}$, one obtains the desired sequence of inclusions. □

Corollary 2.9. *With the notations of Theorem 2.6 we have*

$$\mathbb{Q}^{\ell}_{0,3} \subset \mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}} \subset \mathbb{Q}^{\ell}_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{et}}.$$

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Proof. The inclusion $\mathbb{Q}^\ell_{0,3} \subset \mathbb{Q}^\ell_{g,m}$ for all hyperbolic (g,m) is essentially Theorem 3.6 of [31]. The rest of the inclusions follow from Proposition 2.8 and Theorem 2.6. □

Remark 2.10. In Proposition 2.8, there is no difficulty to move to the weight version of Oda's conjecture, and we get, for all (g,m) of hyperbolic type, any compatible Hurwitz data $\underline{kr}$, and all weight w

$$\mathbb{Q}^\ell_{g,m}(w) \subset \mathbb{Q}^\ell_{g,[m]}(G)_{\underline{kr}}(w) \subset \mathbb{Q}^\ell_{g,[m]}(G)^\nu_{\underline{kr}}(w).$$

In contrast, see Remark 2.13 2.

2.2.2. Quotient vs G-special loci fixed fields.

The quotient map $\delta\colon \mathcal{M}_{g,[m]}[G]_{\underline{kr}} \to \mathcal{M}_{g',[m']}$ defined by $(C,D,\iota) \mapsto (C/\iota(G), D/\iota(G))$ allows the comparison of ℓ-monodromy fixed fields. We first remark that the map δ is well defined at the level of the stack $\mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}}$, since δ is equivariant under the action of $\operatorname{Aut} G$.

Therefore, we have a map $\delta\colon \mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}} \to \mathcal{M}_{g',[m']}$ that fits in a commutative square

$$\begin{array}{ccc} \mathcal{C}_{g,[m]}(G)^\nu_{\underline{kr}} & \longrightarrow & \mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}} \\ \downarrow & & \downarrow \delta \\ \mathcal{M}_{g',[m']+1} & \longrightarrow & \mathcal{M}_{g',[m']}, \end{array}$$

where the map on the left is induced by the quotient in the same way. For a curve $\overline{X}$ over $\overline{\mathbb{Q}}$ represented on $\mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}}$, let us denote $\overline{Y}$ the quotient proper curve, and X, Y their open counterparts. This leads to a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(\mathcal{C}_{g,[m]}(G)^\nu_{\underline{kr}}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(\mathcal{M}_{g',[m']+1}) & \longrightarrow & \pi_1(\mathcal{M}_{g',[m']}) & \longrightarrow & 1, \end{array}$$

which in turn provides an ℓ-monodromy representation

$$\Phi^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}}\colon \pi_1(\mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}}) \to \operatorname{Out}\pi^\ell_1(Y)$$

in the quotient curve, so that one obtains

$$p(\operatorname{Ker}\Phi^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}}) \subset p(\operatorname{Ker}\Phi^\ell_{g',[m']}) \text{ or equivalently } \mathbb{Q}^\ell_{g',[m']} \subset \mathbb{Q}^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}} \quad (2.4)$$

where $\mathbb{Q}^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}}$ denotes the fixed field of the subgroup $p(\operatorname{Ker}\Phi^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}})$ as usual. Lemma 2.7 then gives $\mathbb{Q}^\ell_{g',m'} \subset \mathbb{Q}^\ell_{g',m'}(\delta G)^\nu_{\underline{kr}}$.

In the rest of this section, we finally establish that $\mathbb{Q}^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}} = \mathbb{Q}^\ell_{g,m}(G)^\nu_{\underline{kr}}$ in the case where $X \to Y$ is a finite étale[3] geometric cover and where $G \simeq \mathbb{Z}/\ell^n\mathbb{Z}$. The finite

[3] That is, $\underline{kr}$ is of étale type (i.e., $\underline{kr}^{et} = \underline{kr}$). See Section 2.1.3 for definition.

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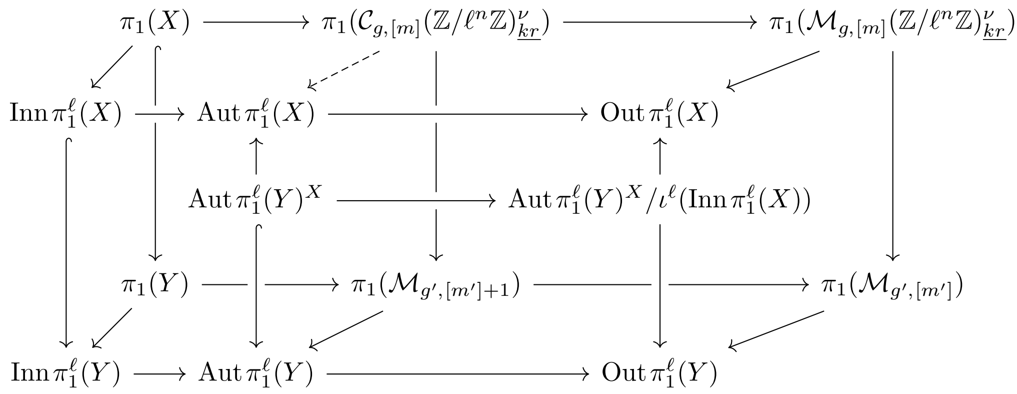
étale condition guarantees that the inclusion $\iota\colon \pi_1(X) \to \pi_1(Y)$ induces an inclusion at the pro-ℓ completion level $\iota^\ell\colon \pi_1^\ell(X) \to \pi_1^\ell(Y)$.

Denoting by $\operatorname{Aut}\pi_1^\ell(Y)^X$ the subgroup of the automorphisms of $\pi_1^\ell(Y)$ that stabilizes $\pi_1^\ell(X)$, we thus obtain a big commutative diagram



By tracking the conjugation action of $\pi_1(\mathcal{C}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}})$ on $\pi_1^\ell(X)$ on the first square of the back face, we see that the dashed arrow $\pi_1(\mathcal{C}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}) \to \operatorname{Aut}\pi_1^\ell(X)$ factors by $\operatorname{Aut}\pi_1^\ell(Y)^X$ through its conjugation action on $\pi_1^\ell(Y)$ and the restriction map.

Theorem 2.11. *For (g,m) of hyperbolic type, and $\underline{kr}$ an abstract Hurwitz data of étale type associated to $\mathbb{Z}/\ell^n\mathbb{Z}$ with quotient topological data (g',m'), we have the following inclusions of ℓ-monodromy fixed fields:*

$$\mathbb{Q}^\ell_{g',[m']} \subset \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}.$$

Proof. Since $\mathbb{Q}^\ell_{g',[m']} \subset \mathbb{Q}^\ell_{g',[m']}(\delta G)^\nu_{\underline{kr}}$, by Equation (2.4), it suffices to show the equality $\mathbb{Q}^\ell_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}} = \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}$. We do so by introducing some intermediate fields as can be seen in Diagram (2.5).

We first have a map

$$\Psi\colon \pi_1^\ell(\mathcal{C}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}) \longrightarrow \operatorname{Aut}\pi_1^\ell(X) \times \operatorname{Aut}\pi_1^\ell(Y)^X$$

such that $\Phi^\ell_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}$ and $\Phi^\ell_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}$ are obtained by composing Ψ with the projections and quotients by the inner automorphisms. One checks directly that $\operatorname{Inn}\iota^\ell\pi_1^\ell(X)$ is a normal subgroup of $\operatorname{Aut}\pi_1^\ell(Y)^X$. We thus have a quotient map

$$\operatorname{Aut}\pi_1^\ell(X) \times \operatorname{Aut}\pi_1^\ell(Y)^X \longrightarrow \operatorname{Out}\pi_1^\ell(X) \times \operatorname{Aut}\pi_1^\ell(Y)^X/\operatorname{Inn}\iota^\ell(\pi_1^\ell(X)),$$

which by composition with Ψ results in a map

$$S^\ell\colon \pi_1(\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^\nu_{\underline{kr}}) \longrightarrow \operatorname{Out}\pi_1^\ell(X) \times \operatorname{Aut}\pi_1^\ell(Y)^X/\operatorname{Inn}\iota^\ell(\pi_1^\ell(X)).$$

Considering the quotient map $p_Y\colon \operatorname{Aut}\pi_1^\ell(Y)^X/\operatorname{Inn}\iota^\ell(\pi_1^\ell(X)) \to \operatorname{Out}\pi_1^\ell(Y)$ and the canonical projections p_i, $i = 1,2$, of the product $\operatorname{Out}\pi_1^\ell(X) \times \operatorname{Aut}\pi_1^\ell(Y)^X/\operatorname{Inn}\iota^\ell(\pi_1^\ell(X))$,

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one observes that by construction,

$$\Phi^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} = p_1 \circ S^{\ell} \text{ and } \Phi^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} = p_Y \circ p_2 \circ S^{\ell}.$$

By setting $\mathbb{Q}^{\ell}_S$ to be the fixed field of $p(\operatorname{Ker} S^{\ell})$ and $\mathbb{Q}^{\ell}_{S_Y}$ to be the fixed field of $p(\operatorname{Ker} p_2 \circ S^{\ell})$, we obtain the following diagram of inclusions of ℓ-monodromy fixed fields:

$$\begin{array}{ccccc} \mathbb{Q}^{\ell}_{g,m} & \hookrightarrow & \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} & & \\ & & & \searrow & \\ & & & & \mathbb{Q}^{\ell}_S \\ & & & \nearrow & \\ \mathbb{Q}^{\ell}_{g',[m']} & \hookrightarrow & \mathbb{Q}^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} \hookrightarrow \mathbb{Q}^{\ell}_{S_Y} & & \end{array} \tag{2.5}$$

It remains to show some equalities. First, $\mathbb{Q}^{\ell}_{S_Y} = \mathbb{Q}^{\ell}_S = \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}}$ since by the inclusion ι^{ℓ}, we have that p_2 restricted to the image of S^{ℓ} is injective, and by slimness of $\pi_1^{\ell}(X)$ (see Section 3.1.2 for a definition), we have that p_1 restricted to the image of S^{ℓ} is also injective.

For the remaining equality $\mathbb{Q}^{\ell}_{S_Y} = \mathbb{Q}^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}}$, we consider the stack inertia injection $G \subset \mathcal{I}_{\mathcal{M},\overline{x}} \hookrightarrow \pi_1(\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}})$ as in [26], where $\overline{x} \in \mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}(\overline{K})$ corresponds to the curve $\overline{X}$, and where the injectivity follows from ibid. Theorem 6.2 with the arguments of Remark 4.4 of [5]. The injection $G \hookrightarrow \pi_1(\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}})$ can be shown to be independent of the choice of point $\overline{x}$ and maps, through our construction, $G = \mathbb{Z}/\ell^n\mathbb{Z}$ isomorphically to the quotient $\operatorname{Inn}\pi_1^{\ell}(Y)/\operatorname{Inn}\iota^{\ell}(\pi_1^{\ell}(X)$. Let $\sigma \in p(\operatorname{Ker} p_Y \circ p_2 \circ S^{\ell})$, which lifts to $\tau \in \pi_1(\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}})$ by definition which in turn maps to $h \in \operatorname{Inn}\pi_1^{\ell}(Y)/\operatorname{Inn}\iota^{\ell}(\pi_1^{\ell}(X)) \simeq G$. The element $h^{-1}\tau \in \pi_1(\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}})$ is in $\operatorname{Ker} p_2 \circ S^{\ell}$ and verifies $p(h^{-1}\tau) = p(\tau)$. Thus, we have proven that $p(\operatorname{Ker}\Phi^{\ell}_{g',[m']}(\delta\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}) \subset p(\operatorname{Ker} S^{\ell}_Y)$, and the reverse inclusion is given by Diagram 2.5. □

By Theorem A of [23], Theorem 4.3 of [20] and Theorem 3.6 [31], there is an inclusion $\mathbb{Q}^{\ell}_{0,3} \subset \mathbb{Q}^{\ell}_{g,m}$ for all (g,m) of hyperbolic type. Thus, we can complete the diagram Diagram (2.5) of field inclusions as follows.

Corollary 2.12. *For (g,m) of hyperbolic type and $\underline{kr}$ compatible Hurwitz data, we have a diagram of inclusions of fields*

$$\begin{array}{ccccc} \mathbb{Q}^{\ell}_{g,m} & \hookrightarrow & \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})_{\underline{kr}} & \hookrightarrow & \mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell^n\mathbb{Z})^{\nu}_{\underline{kr}} \\ \uparrow & & & \nearrow & \\ \mathbb{Q}^{\ell}_{0,3} & \dashrightarrow & \mathbb{Q}^{\ell}_{g',m'} & & \end{array}$$

Remark 2.13.

1. While for some well-chosen Hurwitz data $\underline{kr}$, we have $(g',m') = (0,3)$ in the diagram above, the above references [23, 20, 31] are still required for the final comparison of monodromy fields.

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2. In *the setting of Oda's weight conjecture*, where the pro-ℓ-fundamental groups are replaced by quotients $\pi_1^\ell(-)[w]$ with respect to a certain weight filtration $\pi_1^\ell(-)(w)$, the map $\pi_1^\ell(X)[w] \to \pi_1^\ell(Y)[w]$ fails to be injective. Thus, the end of the proof of Theorem 2.11 does not adapt well, since we cannot recover the equality $\mathbb{Q}_{S_Y}^\ell(w) = \mathbb{Q}_S^\ell(w)$, where $\mathbb{Q}_{S_Y}^\ell(w)$ and $\mathbb{Q}_S^\ell(w)$ are defined in the obvious manner. See also Remark 2.10

Establishing the G-special loci Oda's conjecture in the case of $G = \mathbb{Z}/\ell\mathbb{Z}$ – that is, that $\mathbb{Q}_{g,m}^\ell(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is independent of the topological and Hurwitz data and indeed equal to $\mathbb{Q}_{0,3}^\ell$ – is thus reduced to establishing the last inclusion $\mathbb{Q}_{g,[m]}^\ell(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}^\nu \subset \mathbb{Q}_{0,3}^\ell$. We proceed to do so in the rest of this paper by developing for G-special loci a refinement of Ihara-Nakamura's degeneration method used in their original proof of the containment $\mathbb{Q}_{g,m}^\ell \subset \mathbb{Q}_{0,3}^\ell$ in [17].

3. Maximal degeneration families for G-stable compactification

After some brief reminder on Deligne's tangential base point on $\mathcal{M}_{0,4}$, we construct, following [17] for generic curves, some tangential base points on $\mathcal{M}_{g,[m]}(G)$ as 1-parameter deformation families $X/\operatorname{Spf} K[[q]]$ of some maximally degenerated G-stable curves in some well-chosen strata of $\overline{\mathcal{M}}_{g,[m]}(G)_{\underline{kr}}^\nu$. These curves are defined as certain $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagrams X^0 that are obtained, via Grothendieck's formal patching technique, from well-chosen arrangements of so-called Matsumoto-Seyama curves C_r. In particular, the associated $\mathbb{Z}/\ell\mathbb{Z}$-quotient curves and their deformation will be the $\mathbb{P}^1 \setminus \{0,1,\infty\}$-diagrams and their canonical 1-dimensional deformation constructed by Ihara and Nakamura in [17] 2.1.3.

We enunciate, under the anabelian slimness hypothesis, some immediate results for the kernel of universal monodromy representations, and for $\mathbb{Q}_{C_r'}^\ell$. Consequences for the ℓ-monodromy fixed fields $\mathbb{Q}_{g,m}^\ell$, $\mathbb{Q}_{g,[m]}^\ell(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ and $\mathbb{Q}_{0,3}^\ell$, and for Oda's conjecture are exploited in Section 5.

3.1. Tangential Galois actions and universal monodromy properties

3.1.1. Tangential base points on curves. We follow the elementary definition of tangential base point of the survey [24] Section I; that is, for X connected smooth curve over a field K, *a K-tangential base point v on X* is a morphism $v\colon \operatorname{Spec} K((t)) \to X$ (see ibid. Definition 1.1).

The key feature of such a choice of a K-rational tangential base point is, via the field of Puiseux series $\overline{K}\{\{t\}\}$, to provide at once a geometric base point for the étale fundamental group of X and a section to the related homotopy exact sequence:

$$1 \longrightarrow \pi_1(X_{\overline{K}}, \vec{v}) \longrightarrow \pi_1(X, \vec{v}) \underset{s_v}{\curvearrowleft}\!\!\longrightarrow G_K \longrightarrow 1. \tag{3.1}$$

In other words, one obtains a specific G_K-action $\varphi_{\vec{v}}$ on $\pi_1(X_{\overline{K}}, v)$ given by conjugation which lifts the canonical outer Galois action

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$$\begin{array}{ccc} G_K & \xrightarrow{\varphi_v} & \operatorname{Aut}\pi_1(X_{\overline{K}},v) \\ & \searrow^{\varphi_X} & \downarrow \\ & & \operatorname{Out}\pi_1(X_{\overline{K}},v) \end{array} \tag{3.2}$$

and can be chosen to reflect some good arithmetic properties of X. More explicitly, the G_K action φ_v is given, via the function fields of X, by the action on the coefficients of the formal series in $\overline{K}\{\{t\}\}$; see also Equation (3.3) below.

Remark 3.1.

1. By the valuative criterion of properness this is equivalent to giving a map $v\colon \operatorname{Spec} K[[t]] \to \overline{X}$ where $\overline{X}$ is the compactification of X (i.e., $X \subset \overline{X}$ is a Zariski open given by finitely many punctures of the proper curve $\overline{X}$).
2. By Grothendieck-Murre theory, the category $\operatorname{Rev}^D(\overline{X})$ of finite étale coverings of $\overline{X}$ tamely ramified along the divisor $D = \overline{X} \setminus X$ is equivalent to the category of finite étale coverings of X. The choice of a tangential base point gives a fiber functor of this Galois category in the following way. Let $Y \in \operatorname{Rev}^D(\overline{X})$ and B the $K[[t]]$-algebra obtained by the pullback of Y along our tangential base point v. With this formalism, the fiber functor $\vec{v}$ is defined by
$$\begin{array}{cccc} \vec{v}\colon & \operatorname{Rev}^D(\overline{X}) & \longrightarrow & \text{Set} \\ & Y & \longmapsto & \operatorname{Hom}_{K[[T]]}(B,\overline{K}\{\{t\}\}). \end{array} \tag{3.3}$$
3. The above formalism provides a fundamental group $\pi_1^D(\overline{X},v)$ which is canonically isomorphic to $\pi_1(X,v)$ and carries the same tangential Galois action.

For $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$, let us denote the set of fiber functor associated, as in 2 above, to Deligne-Ihara's original $\mathbb{Q}$-tangential base points by

$$\mathbb{B} = \{\vec{01},\vec{0\infty},\vec{10},\vec{1\infty},\vec{\infty 1},\vec{\infty 0}\},$$

where, for example, $\vec{01}\colon \operatorname{Spec}\mathbb{Q}((t)) \to \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$ and $\vec{0\infty}\colon \operatorname{Spec}\mathbb{Q}((-t)) \to \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$, and refer to the Appendix of [16] for further details on the associated $G_{\mathbb{Q}}$-action. For our study, the main property of these tangential base points is that

$$\operatorname{Ker}\varphi^{\ell}_{\vec{ij}} = \operatorname{Ker}\varphi^{\ell}_{\mathbb{P}^1\setminus\{0,1,\infty\}} \quad \text{for every } \vec{ij} \in \mathbb{B}. \tag{3.4}$$

While even the simplest rational scaling of the parameter (see, for example, $\vec{01}$ vs $\vec{0\infty}$ above or [33] Section 1.5), changes the tangential Galois action, we have the following Galois invariance property.

Lemma 3.2. *The G_K-action induced by a K-rational tangential base point $v\colon \operatorname{Spec} K((t)) \to X$ depends only on the closed point $x \in \overline{X}(K)$ in the closure of the image of v and the class of the image of t in the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

Proof. Let $x \in \overline{X}$ be a closed K-rational point. It suffices to show that if t and t' are both uniformizers at x (i.e., we have $\widehat{\mathcal{O}}_{\overline{X},x} \simeq K[[t]] \simeq K[[t']]$ and $t' = t(1+tF)$ in $K[[t]]$ with

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$F \in K[[t]]$), then the isomorphism $\delta_{t',t}\colon \overline{K}\{\{t'\}\} \to \overline{K}\{\{t\}\}$ is G_K-equivariant. But as $\delta_{t',t}$ is defined by $t'^{\frac{1}{N}} \mapsto t^{\frac{1}{N}}(1+tF)^{\frac{1}{N}}$ for $N \geq 0$, this comes from the fact that $(1+tF)^{\frac{1}{N}} = G_N$ with $G_N \in K[[t]]$ by the series expansion of $(1+tF)^{\frac{1}{N}}$.

Indeed, let v_t (resp. v'_t) be the tangential base points given by t (resp. t') and denote by φ_{v_t} (resp. $\varphi_{v'_t}$) the associated tangential G_K-action. Let $\sigma \in G_K$ and consider a meromorphic function $f = \sum_k a_k t'^{\frac{k}{N}} \in \mathcal{M}_{v_{t'}} \subset K\{\{t'\}\}$. Then we have

$$\begin{aligned}\sigma_{v_t}^{-1} \circ \delta_{t',t} \circ \sigma_{v_{t'}}(f) &= \sigma_{v_t}^{-1} \circ \delta_{t',t}\Big(\sum_k \sigma(a_k) t'^{\frac{k}{N}}\Big)\\ &= \sigma_{v_t}^{-1}\Big(\sum_k \sigma(a_k) t^{\frac{k}{N}} G_N\Big)\\ &= \sum_k a_k t^{\frac{k}{N}} G_N;\end{aligned}$$

that is,

$$\sigma_{v_t}^{-1} \circ \delta_{t',t} \circ \sigma_{v_{t'}}(f) = \delta_{t',t}(f),$$

which shows that $\sigma_{v_t}^{-1} \circ \delta_{t',t} \circ \sigma_{v_{t'}} = \delta_{t',t}$ and thus $\varphi_{v'_t} = \varphi_{v_t} \circ \delta_{t',t}$ as intended. □

3.1.2. Monodromy fixed fields and tangential base points. We recall that *a profinite group is said to be slim* if any of its open subgroup has trivial centralizer. Examples of slim groups include the absolute Galois group of rational numbers and the pro-ℓ fundamental group of hyperbolic curves; see [21] Proposition 1.4.

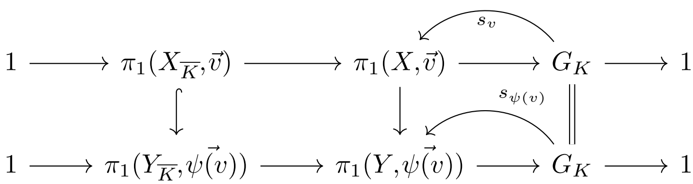
We record the following inclusions between the ℓ-monodromy fixed fields of the various tangential and non-tangential Galois actions in the case of étale coverings.

Lemma 3.3. *Let $\psi\colon X \to Y$ be a finite étale covering of geometrically irreducible curves over a field K of degree a power of ℓ. Let $v\colon \operatorname{Spec} K((t)) \to X$ be a tangential base point on X and $\psi(v)$ the induced tangential base point on Y. We have the following inclusions of subgroups of G_K:*

1. $\operatorname{Ker}\varphi^{\ell}_{\psi(\vec{v})} \subset \operatorname{Ker}\varphi^{\ell}_{\vec{v}}$
2. $\operatorname{Ker}\varphi^{\ell}_{\vec{v}} \subset \operatorname{Ker}\varphi^{\ell}_{X}$ *and* $\operatorname{Ker}\varphi^{\ell}_{\psi(\vec{v})} \subset \operatorname{Ker}\varphi^{\ell}_{Y}$.

Furthermore, when $\pi_1^{\ell}(Y,\psi(\vec{v}))$ is slim, we have $\operatorname{Ker}\varphi_{\vec{v}} = \operatorname{Ker}\varphi_{\psi(\vec{v})}$ and $\operatorname{Ker}\varphi^{\ell}_{X} \subset \operatorname{Ker}\varphi^{\ell}_{Y}$.

Proof. The homotopy exact sequence for X and Y and the covering map ψ gives the diagram



that is commutative by definition of $\psi(v)$ and the étaleness of ψ. One thus recovers, via $\pi_1(Y,\psi(\vec{v})) \to \operatorname{Aut}\pi_1^{\ell}(Y_{\overline{K}},\psi(\vec{v}))$ whose image stabilizes $\pi_1^{\ell}(X_{\overline{K}},\vec{v})$, the monodromy action

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$\varphi^{\ell}_{\vec{v}}$ as the composition

$$G_K \to \operatorname{Aut}\pi_1^{\ell}(Y_{\overline{K}}, \vec{\psi(v)})^{X_{\overline{K}}} \to \operatorname{Aut}\pi_1^{\ell}(X_{\overline{K}}, \vec{v}),$$

which leads to the inclusion given in 1. In the case of slimness, the right restriction map is injective, which yields the equality.

The remaining inclusions are obtained by adding the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Aut}\pi_1^{\ell}(X_{\overline{K}}, \vec{v}) & \longleftarrow & \operatorname{Aut}\pi_1^{\ell}(Y_{\overline{K}}, \vec{\psi(v)})^{X_{\overline{K}}} \\
\downarrow & & \downarrow{\scriptstyle d_X} \\
\operatorname{Out}\pi_1^{\ell}(X_{\overline{K}}, \vec{v}) & \longleftarrow & \operatorname{Aut}\pi_1^{\ell}(Y_{\overline{K}}, \vec{\psi(v)})^{X_{\overline{K}}}/\operatorname{Inn}\pi_1^{\ell}(X_{\overline{K}}, \vec{v}) \\
 & & \downarrow{\scriptstyle d_Y} \\
 & & \operatorname{Out}\pi_1^{\ell}(Y_{\overline{K}}, \vec{\psi(v)}).
\end{array}$$

The inclusions of 2 are thus direct by the diagram and the definitions of the maps involved. To see the remaining inclusion, we remark that by slimness $\operatorname{Ker}\varphi^{\ell}_X = \operatorname{Ker} d_X \circ \varphi^{\ell}_{\psi(\vec{v})}$, and the inclusion follows as $\varphi^{\ell}_Y = d_Y \circ d_X \circ \varphi^{\ell}_{\psi(\vec{v})}$. □

3.2. The Matsumoto-Seyama curves

We now introduce the Matsumoto-Seyama curves C_r, for $r \in \{0, 1, \dots, \ell-2\}$, that live in certain special loci $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ and that have $\mathbb{P}^1_K$ as $\mathbb{Z}/\ell\mathbb{Z}$-quotient, where K denotes $\mathbb{Q}(\mu_\ell)$.

For $r \in \{1, \dots, \ell-2\}$, the curves C_r are those of [30] – that is, some smooth projective curves of genus $g = \ell - 1/2$ that are birationally equivalent to the affine curve

$$y^r(y-1) = x^{\ell} \text{ with } \mathbb{Z}/\ell\mathbb{Z}\text{-action } \begin{cases} \text{given by } x \mapsto \zeta_\ell x \\ \text{ramified at } P_{r,0}, P_{r,1}, P_{r,\infty} \text{ over } 0, 1, \infty. \end{cases} \tag{3.5}$$

The quotient $\psi\colon C_r \setminus \{P_{r,0}, P_{r,1}, P_{r,\infty}\} \to \mathbb{P}^1_K \setminus \{0, 1, \infty\}$ is finite étale and Galois of group $\mathbb{Z}/\ell\mathbb{Z}$. The abstract Hurwitz data of C_r is $\underline{k} = (r, 1, -(r+1))$ which, when r varies, is seen to represent every possible abstract Hurwitz data of a $\mathbb{Z}/\ell\mathbb{Z}$-curve with three ramified points.

3.2.1. Tangential base points and Galois actions comparisons. The set of curves $\{C_r \mid r = 1, \dots, \ell-2\}$ admits an $\mathfrak{S}_3$-action that is compatible with the $\mathbb{Z}/\ell\mathbb{Z}$-action and, in particular, with the $\mathfrak{S}_3$-action on $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$ through the quotient map; see [30] Corollary 2.5. This allows us to define, for every r, the tangential base points on $C'_r = C_r \setminus \{P_{r,0}, P_{r,1}, P_{r,\infty}\}$ at the punctures by doing so at $P_{r,1}$.

Indeed, for $\sigma \in \mathfrak{S}_3$, we have $\sigma(P_{r,1}) = P_{\sigma(r),\sigma(1)}$ so that for every $r \in \{1, \dots, \ell-1\}$ and every $P \in \{P_{r,0}, P_{r,1}, P_{r,\infty}\}$, there is an element $\sigma \in \mathfrak{S}_3$ such that P is the image of $P_{r,1}$ for some r. Now, the smooth affine open $U = C_r \setminus \{P_{r,0}, P_{r,\infty}\}$ is given by $U = \operatorname{Spec} K[x, y, \frac{1}{y}]$ where $x^{\ell} = y^r(y-1)$. Looking at the equation, we see that x is a uniformizer at $P_{r,1}$, and we have $\psi(x) = x^{\ell} = y^r(y-1)$ where ψ is the quotient map to $\mathbb{P}^1_K$.

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Lemma 3.4. *The tangential base point* $T_{10}^r\colon \operatorname{Spec} K((t)) \to C_r'$ *defined by* $t \mapsto \zeta_{2\ell}x$ *induces a tangential base point* $\psi(T_{10}^r)$ *on* $\mathbb{P}^1_K \setminus \{0,1,\infty\}$ *that defines the same* G_K*-action on* $\pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \vec{10})$ *as* $\vec{10}$.

Proof. By Lemma 3.2, it suffices to check that $\psi(T_{10}^r)$ and $\vec{10}$ have, after taking the closure, the same closed points in $\mathbb{P}^1_K$ and the same class in $\mathfrak{m}_1/\mathfrak{m}_1^2$. The first part is obvious. For the second one, by definition, we have that $\psi(T_{10}^r)$ is $-y^r(y-1) \in K[[y-1]] \simeq \widehat{\mathcal{O}}_{\mathbb{P}^1_K,1}$ so that its class modulo $\mathfrak{m}_1^2$ is equal to -1 as required. $\square$

As stated before, by using the $\mathfrak{S}_3$-action on the previous subset of Matsumoto-Seyama curves, we obtain tangential base points T_{ij}^r for $i,j \in \{0,1,\infty\}$, whose set of associated fiber functors on the categories of finite étale covers $\operatorname{Et}(C_r')$ we denote by

$$\mathbb{B}^r = \{\overrightarrow{T_{ij}^r} \mid i,j \in \{0,1,\infty\}\}, \text{ for } r \in \{1,\dots,\ell-2\}.$$

These tangential base points induce the same tangential G_K-actions on the fundamental group of $\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}$ given by Deligne-Ihara as in Lemma 3.4.

Theorem 3.5. *The* G_K*-action defined by the* $\overrightarrow{T_{ij}^r}$*s on the groupoid* $\Pi_1(C'_{r,\overline{K}}, \mathbb{B}^r)$ *induces a* G_K*-action on the groupoid* $\Pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \mathbb{B})$ *that is compatible with the Deligne-Ihara one. Furthermore, an element of* G_K *acts trivially on* $\Pi_1^\ell(C'_{r,\overline{K}}, \mathbb{B}^r)$ *if and only if it acts trivially on* $\Pi_1^\ell(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \mathbb{B})$.

Proof. The first part of the statement is the result of the previous paragraph. For the second part, let $\sigma \in G_K$. As the tangential base points of $\mathbb{B}^r$ are K-rational, the action of σ on $\Pi_1(C'_{r\overline{K}}, \mathbb{B}^r)$ stabilizes each fundamental group or set of étale paths. Now as this action is compatible with the one on $\Pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \mathbb{B})$ and each of the inclusions maps between $\Pi_1(C'_{r,\overline{K}}, \overrightarrow{T_{ij}}, \overrightarrow{T_{jk}})$ and $\Pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \vec{ij}, \vec{jk})$ remains injective after passing to the pro-ℓ-completion for all $i,j,k \in \{0,1,\infty\}$, it follows that the reverse implication holds. By Lemma 3.3, it also holds that $\operatorname{Ker}\varphi^\ell_{\overrightarrow{T_{ij}}} = \operatorname{Ker}\varphi^\ell_{\vec{ij}}$ for all $i,j \in \{0,1,\infty\}$. Thus, if σ acts trivially on $\Pi_1(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, \mathbb{B})$, it acts trivially on each of the fundamental groups appearing in $\Pi_1(C'_{r\overline{K}}, \mathbb{B}^r)$, and thus on the whole groupoid. $\square$

In what follows, r will be omitted from notations when clear from context.

3.2.2. The fixed field of the Matsumoto-Seyama curves. For $r=0$, we consider the covering of $\mathbb{P}^1_K$ given by

$$C_0\colon x = y^\ell \text{ with usual } \mathbb{Z}/\ell\mathbb{Z}\text{-action having two ramified points } 0 \text{ and } \infty$$

with abstract Hurwitz data $\underline{k} = (1,-1)$. The $\ell+2$-marking is given by the two ramified points and by the unramified points $P_1,\dots,P_\ell$ of the fiber at 1.

We further set

$$\mathbb{B}^0 = \{\overrightarrow{T^0_{0\infty}}, \overrightarrow{T^0_{\infty 0}}\} \text{ and } C_0' = C_0 \setminus \{0,\infty,P_1,\dots,P_\ell\},$$

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where the fiber functors $\overrightarrow{T^0_{0\infty}}$ and $\overrightarrow{T^0_{\infty 0}}$ are induced by the tangential base points associated to the parameter x and $\frac{-1}{x}$, and which are direct lifts of the Deligne tangential base points $\overrightarrow{0\infty}$ and $\overrightarrow{\infty 0}$.

Proposition 3.6. *The action of G_K on $\Pi_1(C'_{0\overline{K}},\mathbb{B}^0)$ is compatible with its action on $\Pi_1(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\},\mathbb{B})$. Furthermore, an element of G_K acts trivially on $\Pi_1^\ell(C'_{0\overline{K}},\mathbb{B}^0)$ if and only if it acts trivially on $\Pi_1^\ell(\mathbb{P}^1_{\overline{K}}\setminus\{0,1,\infty\},\mathbb{B})$.*

Proof. The only part of the statement that is not already proven is a direct consequence of [17] Corollary 4.1.4 (ii). □

We finish this section by showing that the ℓ-monodromy fixed field of the Matsumoto-Seyama curves is $\mathbb{Q}^\ell_{0,3}$.

Corollary 3.7. *We have $\mathbb{Q}^\ell_{C'_r}=\mathbb{Q}^\ell_{0,3}$ for all $r\in\{0,\dots,\ell-2\}$.*

Proof. For $r\in\{0,\dots,\ell-2\}$ given, it follows from Lemma 3.3 that $\operatorname{Ker}\varphi^\ell_{\overrightarrow{T_{0\infty}}}=\operatorname{Ker}\varphi^\ell_{\vec{01}}$ as $\pi_1(\mathbb{P}^1_K\setminus\{0,1,\infty\},\vec{0\infty})$ is slim. From the same lemma, we also get the inclusions

$$\operatorname{Ker}\varphi^\ell_{\overrightarrow{T_{0\infty}}}\subset\operatorname{Ker}\varphi^\ell_{C'_r}\subset\operatorname{Ker}\varphi^\ell_{\mathbb{P}^1\setminus\{0,1,\infty\}}.$$

Since the two outmost terms are equal as in Equation (3.4), it follows that $\operatorname{Ker}\varphi^\ell_{C'_r}=\operatorname{Ker}\varphi^\ell_{\mathbb{P}^1\setminus\{0,1,\infty\}}$, thus the desired equality. □

Remark 3.8.

1. At this stage, one can already obtain, by following Matsumoto's approach as in [20], that $\mathbb{Q}^\ell_{g,m}=\mathbb{Q}^\ell_{0,3}$ for the specific values of $(g,m)=((\ell-1)/2),3)$ and $(g,m)=(0,\ell+2)$.
2. The curves introduced in this section are chosen so that the corresponding stacks $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ have only one geometric point.

3.3. Diagrams in the $\mathbb{Z}/\ell\mathbb{Z}$-stable compactification

Similarly to the $\mathbb{P}^1_K\setminus\{0,1,\infty\}$-diagrams construction of [17] 1.2, we construct some $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram X^0 over a field K, here as gluing the previously defined Matsumoto-Seyama $\mathbb{Z}/\ell\mathbb{Z}$-curves.

3.3.1. Gluing curves with G-actions. While the gluing, or clutching, of marked points for stable curves can be found in details in [18], the similar gluing for curves with G-action requires an additional constraint as follows.

Consider two curves C_r and $C_{r'}$ with $r,r'\in\{1,\dots,\ell-2\}$. The gluing of both curves at the points $P_{r,1}$ and $P_{r',1}$ can be constructed as the union

$$C^{1,1}_{r,r'}=C_r\times\{P_{r',1}\}\cup C_{r'}\times\{P_{r,1}\}\text{ in the fiber product }C_r\times_{\operatorname{Spec}K}C_{r'}.$$

The result of the gluing is a curve X^0 of genus $\ell-1$ with 2 irreducible components and 4 marked points given by $\{P_{r,0},P_{r,\infty},P_{r',0},P_{r',\infty}\}$, that is equipped with a $\mathbb{Z}/\ell\mathbb{Z}$-action by pullback of the action on the product.

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For X^0 to be a G-stable curve, the G-actions must be chosen such that Hurwitz data at the points $P_{r,1}$ and $P_{r',1}$ have opposite characters (see [2] Section 4.1), which is easily done by choosing that $G=\mathbb{Z}/\ell\mathbb{Z}$ acts by $x\mapsto\overline{\zeta_\ell}x$ on $C_{r'}$ and by $x\mapsto\zeta_\ell x$ on C_r. The same construction can be made by gluing together any two ramified points $P_{r,i}$ and $P_{r',j}$ into a curve $C^{i,j}_{r,r'}$, where $i,j\in\{0,1,\infty\}$ denotes which points are glued.

Note that the $\mathfrak{S}_3$-action on the curves $(C_r)_{r\in\{1,\dots,\ell-2\}}$ extends naturally to a $\mathfrak{S}_3\times\mathfrak{S}_3$-action on the fiber products $(C_r\times_{\operatorname{Spec}K}C_{r'})_{r,r'\in\{1,\dots,\ell-2\}}$ of such curves. One checks that this action stabilizes the closed subsets $(C^{i,j}_{r,r'})_{r,r'\in\{1,\dots\ell-2\},\ i,j\in\{0,1,\infty\}}$ globally; that is, for $\sigma,\tau\in\mathfrak{S}_3\times\mathfrak{S}_3$, we have $(\sigma,\tau)\cdot C^{i,j}_{r,r'}=C^{\sigma(i),\tau(j)}_{\sigma(r),\tau(r')}$. It results that the affine neighborhood of $C^{i,j}_{r,r'}$ with the 4 marked points removed is, for some r, always isomorphic to

$$C^{1,1}_{r,r'}\setminus\{P_{r,0},P_{r,\infty},P_{r',0},P_{r',\infty}\}=\operatorname{Spec}K[x,y,x',y'][\frac{1}{y},\frac{1}{y'}]/(xx'),$$

which serves as a model for the construction of the U_μs as in Section 3.4.1.

3.3.2. Diagrams of Matsumoto-Seyama curves.

We will build our $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagrams from the two types of Matsumoto-Seyama curves C_r of Section 3.2. Recall that the genus 0 curves have two distinguished rational sections given by the ramified points, and that the genus $(\ell-1)/2$ ones have three.

Definition 3.1. A *$\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram* is a connected curve X^0 over K that is defined by the following data:

1. *A finite collection of curves* X^0_λ $(\lambda\in\Lambda\sqcup\Lambda')$ where X^0_λ is either isomorphic to C_r with $r\geq 1$ if $\lambda\in\Lambda$ or to C_0 if $\lambda\in\Lambda'$.
2. *A finite collection of pairs of distinguished section* P^0_μ $(\mu\in M)$ of the X^0_λ, $\lambda\in\Lambda\sqcup\Lambda'$. The pairs P^0_μ are such that the Hurwitz data at those sections are opposite and such that two distinct pairs P^0_μ and $P^0_{\mu'}$ $(\mu\neq\mu')$ have no common element. Let $\mu\in M$ and set $\lambda(\mu)=(\lambda,\lambda')$ where the sections of P^0_μ land in X^0_λ and $X^0_{\lambda'}$.

The curve X^0 is obtained from the disjoint union $\bigsqcup_{\lambda\in\Lambda}X^0_\lambda$ by identifying the pair of points given by the P^0_μ. Given a $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram X^0, we shall denote by Q^0_v, $v\in N$, the distinguished sections of X^0 coming from the X^0_λ that do not appear in the pairs P^0_μ, $\mu\in M$.

The isomorphisms of 1 come with choices of variables x_λ,y_λ and choices of tangential base points $T^\lambda_{ij}\colon\operatorname{Spec}K((t))\to X^0_\lambda$ with the properties of the ones defined in Section 3.2. The corresponding set of fiber functors will be denoted by $\mathbb{B}^r_\lambda$. We will omit λ and r from the notations when it is clear from context.

The following three kinds of $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagrams will be used as basic building blocks for the special fiber of our 1-parameter deformation families.

- **Seyama curve** (Figure 1a): a curve of genus $g=(\ell-1)/2$ with $\nu=3$ ramified points and $\underline{k}$ free;
- **A $\mathbb{Z}/\ell\mathbb{Z}$-curve of genus** 0 (Figure 1b): a curve of genus $g=0$ with $\nu=2$ ramified points, ℓ unramified points and $\underline{k}=(1,-1)$;

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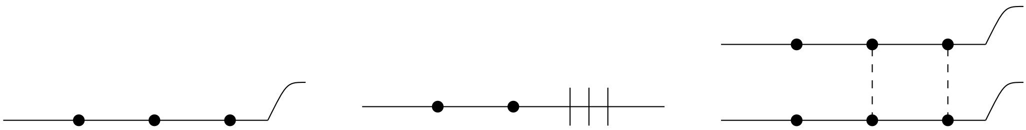

(a) Seyama curve (b) G-curve of genus 0 (c) 2-Seyama curve

Figure 1. Elementary building blocks of $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagrams.

- **A 2-Seyama curve** (Figure 1c): a curve of genus $g = \ell$ with $\nu = 2$ ramified points and $\underline{k} = (1, -1)$. These curves are obtained by gluing two Seyama curves twice. The correspond data of the $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram is $\Lambda = \{1,2\}$ where X_1^0, X_2^0 are Matsumoto-Seyama curves isomorphic to C_r with $r \in \{1,\ldots,\ell-2\}$ and Hurwitz data $\underline{k} = (1,a,-(1+a))$ with ramified points $\{\nu_1,\nu_2,\nu_3\}$ and $\underline{k'} = (-1,-a,1+a)$ with ramified points $\{\nu_1',\nu_2',\nu_3'\}$. We have $M = \{1,2\}$, and the pairs P_1^0, P_2^0 are, respectively, (ν_1,ν_1') and (ν_2,ν_2'). The remaining set of distinguished sections is $\{\nu_3,\nu_3'\} = \{Q_v^0\}_{v\in N}$.

In Figure 1 above, the bold points represent ramified points under the G-action, the lined markings represent the unramified points ($\ell = 3$ here), and the dashed lines represent the glued points. The hook at the end of the genus $g = (\ell-1)/2$ curves is to differentiate them from the genus 0 ones, which are represented by straight lines.

Recall that we denote by $\overline{\mathcal{M}}_{g,[m]}(G)^\nu$ the stable compactification of $\mathcal{M}_{g,[m]}(G)^\nu$, and accordingly, $\overline{\mathcal{M}}_{g,[m]}(G)_{\underline{kr}}$ the closure of $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ in $\overline{\mathcal{M}}_{g,[m]}(G)^\nu$.

Proposition 3.9. *Let g,m and $\underline{kr}$ be given as below, such that $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}^\nu$ is nonempty. Then there exists a $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram in the boundary of $\overline{\mathcal{M}}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}^\nu$.*

Let us first recall that, by [4] Proposition 3.7, the locus $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is nonempty as soon as g can be obtained by the Hurwitz formula

$$g = (N-2)\frac{\ell-1}{2} + g'\ell \text{ with } g' \geq 0 \text{ and } N \geq 0, N \neq 1,$$

where N is the number of ramified points in the cover, and $N \neq 1$ by the nullity assumption modulo $\mathbb{Z}/\ell\mathbb{Z}$ as given in ibid. Définition 3.5. The m marked points are distributed freely in G-orbits. This is a particular instance of the Frobenius coin problem, and it is thus known that all $g \geq (\frac{\ell-1}{2})(\frac{\ell-3}{2})$ are attainable with $N-2 \geq 0$, as well as one element of each pair $(k, \ell\frac{\ell-1}{2} - \ell - \frac{\ell-1}{2} - k)$ for $k \in \{0,\ldots,\frac{\ell-1}{2}\frac{\ell-3}{2} - 1\}$. When $g \in \{0,\ldots,\frac{\ell-1}{2}\frac{\ell-3}{2} - 1\}$ is attainable only by the choice $N = 0$, we say that *g is an unramified case*. For example, this is the case for $g = 1$ by considering the translation action by a choice of order ℓ point on an elliptic curve.

Proof. First, suppose g is not unramified. Then by gluing along the dotted lines as in Figure 2a, we obtain the desired $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram X^0 as follows. The first part is made by gluing p copies of $\mathbb{Z}/\ell\mathbb{Z}$-curves of genus 0, which contributes to the $p\ell$ unramified

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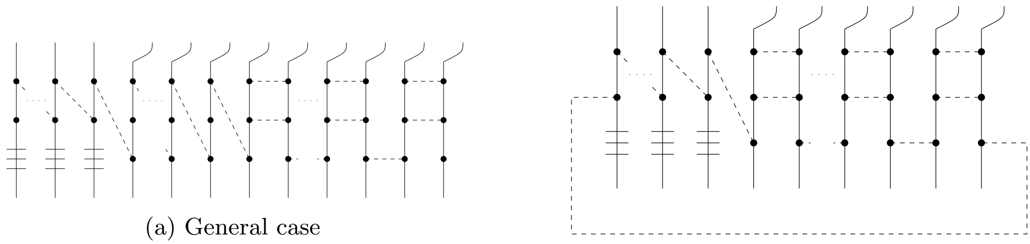

(a) General case

(b) Unramified case

Figure 2. The $\mathbb{Z}/\ell\mathbb{Z}$-stable curve X^0.

marked points that are permuted by $\mathbb{Z}/\ell\mathbb{Z}$ to 1 ramified point with Hurwitz data $\underline{k} = (1)$ and does not contribute to the genus. The second portion is composed of $N-2$ Seyama curves of genus $(\ell-1)/2$ glued in a chain, which contributes to $(N-2)(\ell-1)/2$ to the genus and to $N-2$ to the ramified points with free Hurwitz data. The last part is made by gluing g' copies of 2-Seyama curves. It contributes to $g'\ell$ to the genus and to 1 ramified point with imposed Hurwitz data.

To achieve the unramified g, we remove the middle section made of Seyama curves in the previous construction and glue the remaining parts on the added dotted line as in Figure 2b. One can easily check in the same way that it gives a desired curve. □

Remark 3.10. It is readily seen that the G-quotient of the G-stable diagrams that we constructed is a $\mathbb{P}^1 \setminus \{0,1,\infty\}$-diagram as in [17] 2.1.3.

3.4. The deformation family of $\mathbb{Z}/\ell\mathbb{Z}$-stable diagrams

We now start with a $\mathbb{Z}/\ell\mathbb{Z}$-stable C_r-diagram X^0 with $\operatorname{Card}\Lambda \sqcup \Lambda' \geq 2$ which is in the boundary of $\overline{\mathcal{M}}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}}$ and build, by patching local formal schemes $\mathcal{W}_\bullet$, $\mathcal{V}_\bullet$ and $\mathcal{U}_\bullet$ into a $\mathcal{S}$-scheme $\mathfrak{X}$ over an affine cover of X^0, a family of deformations $X/\operatorname{Spf} K[[q]]$ of X^0.

3.4.1. The three families. Consider the following kind of families W^0_λ, U^0_μ, and V^0_v of affine open of X^0.

1. *The family* $(W^0_\lambda)_{\lambda\in\Lambda}$, resp. $(W^0_\lambda)_{\lambda'\in\Lambda'}$, given for each $\lambda \in \Lambda$, resp. $\lambda' \in \Lambda'$, by the open complement in X^0_λ of the three ramified points, resp. of the two ramified points, and represented as below:

$W^0_\lambda = \operatorname{Spec} K[y_\lambda, x_\lambda, \frac{1}{y_\lambda}, \frac{1}{1-y_\lambda}]$ $\qquad$ $W^0_{\lambda'} = \operatorname{Spec} K[y_{\lambda'}, \frac{1}{y_{\lambda'}}]$

2. *The family* $(U^0_\mu)_{\mu\in M}$, that we will specify as three subfamilies $U^0_{\mu,0,0}$, $U^0_{\mu,0,1}$, and $U^0_{\mu,1,1}$, which for $\mu \in M$ are defined such that P^0_μ consists of a pair of distinguished

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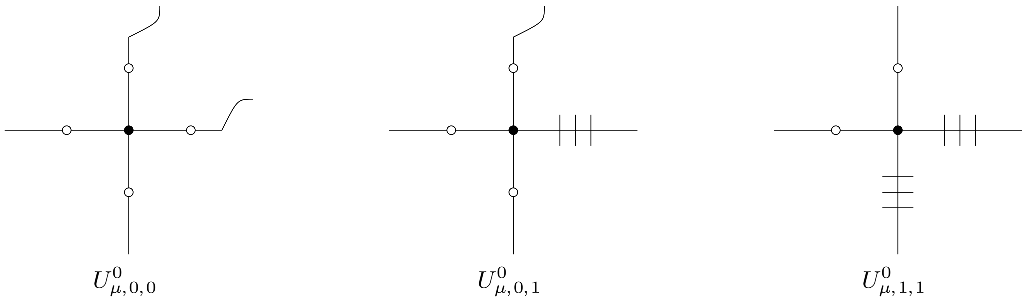

Figure 3. The three subfamilies of U^0_μ.

sections over X^0_λ and $X^0_{\lambda'}$ with $\lambda,\lambda'\in\Lambda\sqcup\Lambda'$, and are, respectively, given as below (see also Figure 3):

$$U^0_{\mu,0,0}=\operatorname{Spec} K[y_\lambda,x_\lambda,y_{\lambda'},x_{\lambda'}][\frac{1}{y_\lambda},\frac{1}{y_{\lambda'}}]/(T^\lambda_{ij}T^{\lambda'}_{kl}),$$
$$U^0_{\mu,0,1}=\operatorname{Spec} K[x_\lambda,y_\lambda,y_{\lambda'},\frac{1}{y_\lambda}]/(T^\lambda_{ij}T^{\lambda'}_{kl}),$$
$$U^0_{\mu,1,1}=\operatorname{Spec} K[y_\lambda,y_{\lambda'}]/(T^\lambda_{ij}T^{\lambda'}_{kl}),$$

where T^λ_{ij} denotes the image of $T^\lambda_{ij}\colon \operatorname{Spec} K((t))\to X^0_\lambda$ and the couple (λ,λ') is related to $\mu\in M$ as in Definition 3.1 and below.

3. *The family* $(V^0_v)_{v\in N}$ given for each $v\in N$ by taking the component X^0_λ that supports the section Q^0_v and removing all the other distinguished sections, to obtain

$$V^0_v=\operatorname{Spec} K[y_\lambda,x_\lambda,\frac{1}{1-y_\lambda}],\ \text{resp. } V^0_v=\operatorname{Spec} K[y_\lambda]. \tag{3.6}$$

for $\lambda\in\Lambda$, resp. $\lambda\in\Lambda'$.

We thus obtain an affine cover of the $\mathbb{Z}/\ell\mathbb{Z}$-stable curve X^0

$$X^0=\bigcup_{\lambda\in\Lambda\sqcup\Lambda'}W^0_\lambda\bigcup_{\mu\in M}U^0_\mu\bigcup_{v\in N}V^0_v,$$

where each open is $\mathbb{Z}/\ell\mathbb{Z}$-stable by construction, and such that:

1. For $\mu\in M$ such that P^0_μ contains a distinguished section of X^0_λ and $X^0_{\lambda'}$ we have W^0_λ and $W^0_{\lambda'}$ as open subsets of U^0_μ and $W^0_\lambda\cap W^0_{\lambda'}=\varnothing$.
2. For $v\in N$ such that Q^0_v is a distinguished section of X^0_λ, we have W^0_λ as an open subset of V^0_v.
3. The intersection of U^0_μ or V^0_v with any other member of the affine cover is either empty, W^0_λ or $W^0_\lambda\sqcup W^0_{\lambda'}$.

These properties ensure, in the next section, the possibility of patching local formal schemes over the affine cover that we just defined.

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3.4.2. Patching formal schemes in algebraic family. Consider the affine formal scheme $\mathcal{S} = \operatorname{Spf} K[[q]]$ with ideal of definition $\mathfrak{q} = (q)$ equipped with the G-action $q \mapsto \zeta_\ell q$ by our choice of isomorphism $G \simeq \mu_\ell$. In order to construct a formal scheme $\mathfrak{X}$ with base X^0 over $\mathcal{S}$ with a compatible G-action, we shall define affine formal $\mathcal{S}$-schemes $\mathcal{W}_\lambda$, $\mathcal{U}_\mu$ and $\mathcal{V}_v$ with bases W_λ^0, U_μ^0 and V_v^0 with ideal of definition the pullbacks of $\mathfrak{q}$ denoted $\mathfrak{q}$ again.

For λ, v, we set

$$\mathcal{W}_\lambda = \operatorname{Spf}\Gamma(W_\lambda^0, \mathcal{O}_{X^0})[[q]] \text{ and } \mathcal{V}_v = \operatorname{Spf}\Gamma(V_v^0, \mathcal{O}_{X^0})[[q]],$$

where the K-algebras of sections $\Gamma(-,\mathcal{O}_{X^0})$ are given by one of the explicit K-algebra of the affine schemes of Section 3.4.1 items 1. and 3. above.

Whenever λ and v are such that W_λ^0 is an open subset of V_v^0, the open immersion $j_{v/\lambda}\colon \mathcal{W}_\lambda \to \mathcal{V}_v$ over it is obtained without effort. For instance, let us assume $W_\lambda^0 = \operatorname{Spec} K[y_\lambda, x_\lambda, \frac{1}{y_\lambda}, \frac{1}{1-y_\lambda}]$ and $V_v^0 = \operatorname{Spec} K[y_\lambda, x_\lambda, \frac{1}{1-y_\lambda}]$. By [8] Proposition 10.1.4, it suffices to check that the map $K[y_\lambda, x_\lambda, \frac{1}{1-y_\lambda}][\frac{1}{y_\lambda}][[q]]^\wedge \to \Gamma(\mathcal{W}_\lambda)$, where $\wedge$ denotes the $\mathfrak{q}$-adic completion, is an isomorphism. But this is clear by construction. Note that $j_{v/\lambda}$ is an $\mathcal{S}$-morphism.

Let us consider the case of $\mathcal{U}_\mu$, whose base U_μ^0 is obtained via 3 subfamilies $U_{\mu,0,0}^0$, $U_{\mu,0,1}^0$, and $U_{\mu,1,1}^0$ as in Section 3.4.1 2.

Proposition 3.11. *For $\mu \in M$ such that U_μ^0 is of the form $U_{\mu,0,0}^0$. Let us define*

$$\mathcal{U}_{\mu,0,0} = \operatorname{Spf} K[T,T',X,X'][\tfrac{1}{1-T}, \tfrac{1}{1-T'}][[q]]/(T_r T'_{r'} - q)$$
$$\text{with } \begin{cases} X^\ell = T^r(1-T), & T_r = \zeta_{2\ell} X \\ X'^\ell = T'^{r'}(1-T'), & T'_{r'} = \zeta_{2\ell} X'. \end{cases}$$

Then we can identify $\mathcal{U}_{\mu,0,0} \bmod \mathfrak{q}$ with $U_{\mu,0,0}^0$ by $(T,T') \mapsto (y_\lambda, y_{\lambda'})$ with the choices $(X,X') \mapsto (x_\lambda, x_{\lambda'})$. Furthermore, for $\lambda \in \Lambda$ such that $W_\lambda^0 = \operatorname{Spec} K[y_\lambda, x_\lambda, \frac{1}{y_\lambda}, \frac{1}{1-y_\lambda}]$, the scheme W_λ^0 is an open subset of $U_{\mu,0,0}^0$ given by inverting y_λ, so that $\Gamma(\mathcal{U}_\mu)[\frac{1}{T}]^\wedge \to \Gamma(\mathcal{W}_\lambda)$, given by $T \to y_\lambda$, is an isomorphism, which induces an open immersion $j_{\mu/\lambda}\colon \mathcal{W}_\lambda \to \mathcal{U}_{\mu,0,0}$.

Proof. By assumption, $T_r T'_{r'} = q$ so that for $N \geq 1$, we have

$$\Gamma(\mathcal{U}_\mu)[\frac{1}{T}]/\mathfrak{q}^N = K[T,T',X,X'][\frac{1}{1-T}, \frac{1}{1-T'}, \frac{1}{T'}]/(T_r T'_r - q)^N.$$

As T and $1-T$ are invertible, so is $X^\ell = T^r(T-1)$, and thus so is X and $T_r = \zeta_{2\ell} X$. It follows that $(T_r T'_{r'})^N = 0$ if and only if ${T'_{r'}}^N = 0$. Now as ${T'_{r'}}^{\ell N} = -{T'^{r'}}^N (1-T')^N$, we have $(1-T')^N = 0$, which gives that T'^{-1} can be written as $\sum_{k=0}^{N-1}(1-T')^k$. To recover T' and X' first as ${T'}^\ell_{r'} = -T'^{r'}(T'-1)$, we have ${T'}^\ell_{r'} = P(T')$ with P invertible for the composition in $K[[T'-1]]$. So there is $F \in K[[T'-1]]$ such that $F({T'}^\ell_{r'}) = T'$. As ${T'}^\ell_{r'}$ is nilpotent of order N, we can truncate F to get a polynomial $\widetilde{F}$ that verifies the equality $\widetilde{F}({T'}^\ell_{r'}) = T'$ in $\Gamma(\mathcal{U}_\mu)[\frac{1}{T}]/\mathfrak{q}^N$. Thus, as $T'_{r'} = \frac{q}{\zeta_{2\ell} T}$, we have

$$\Gamma(\mathcal{U}_\mu)[\frac{1}{T}]/\mathfrak{q}^N = K[X,T,\frac{1}{T},\frac{1}{1-T}][q]/(q^N)$$

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and the desired isomorphism by passing to the $\mathfrak{q}$-adic completion. It is clear that this isomorphism is compatible with the $\mathbb{Z}/\ell\mathbb{Z}$-action on both sides. $\square$

The other open immersions are proven in the same way.

One thus obtains a proper formal regular $\mathcal{S}$-scheme $\mathfrak{X}$ with a collection of sections $(\mathcal{Q}_v)_{v\in N}$ with base space X^0 by gluing along the affine formal schemes $\mathcal{U}_\mu,\mathcal{V}_v$ and $\mathcal{W}_\lambda$.

The formal scheme $\mathfrak{X}$ has the property that, for each μ, λ or v, we have $\mathcal{S}$-isomorphisms

$$\varphi_\mu\colon \mathfrak{X}_{|U^0_\mu}\to\mathcal{U}_\mu,\ \varphi_\lambda\colon \mathfrak{X}_{|W^0_\lambda}\to\mathcal{W}_\lambda,\varphi_v\colon \mathfrak{X}_{|V^0_v}\to\mathcal{V}_v$$

extending the identity maps of U^0_μ, W^0_λ and V^0_v, respectively, such that

1. for each $v\in N$, $\mathcal{Q}_v$ is induced from the canonical section $\mathcal{S}\to\mathfrak{X}_{|V^0_v}$ that lift the section Q_v of V^0_v,
2. the isomorphisms φ_λ, φ_λ and φ_v are compatible with the open immersions $j_{v/\lambda}$ and $j_{\mu/\lambda}$.

A direct application of Grothendieck's existence theorem [9] 5.4 as in [17] 2.4.1 and 3.1 provides the algebraization of the formal scheme $\mathfrak{X}$ into a scheme X over $\operatorname{Spec} K[[q]]$, whose generic fiber X_η is a smooth geometrically irreducible genus g curve with m marked points and a $\mathbb{Z}/\ell\mathbb{Z}$-action, coming by pullback of the one on X, with Hurwitz data $\underline{kr}$, and whose special fiber is X^0.

The sets of divisors $D=((X^0_\lambda)_{\lambda\in\Lambda\sqcup\Lambda'},(Q_v)_{v\in N})$ and $\mathcal{D}=((X^0_\lambda)_{\lambda\in\Lambda\sqcup\Lambda'},(\mathcal{Q}_v)_{v\in N})$ are regular with normal crossings on X and $\mathfrak{X}$, respectively, in the sense of [10] Section 1.8.3; see [17] 3.2 for details.

Remark 3.12. The generic fiber X_η of the scheme X should be interpreted as a tangential base point $\eta\colon \operatorname{Spec} K((q))\to\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^\nu_{\underline{kr}}$ in the moduli space.

3.4.3. Local-global tracking tangential base points. Another important output of our construction, that will be of interest in the next section, is that we can explicitly track our tangential base points in the different formal completions of $\mathfrak{X}$ along chosen closed subsets of the special fiber.

Consider the completion $\mathfrak{X}_\mu$ of $\mathfrak{X}$ along P^0_μ. By construction, $\mathfrak{X}_{|U^0_\mu}=\operatorname{Spf} A/(T_{r,s}T'_{r',s'}-q)$ for a ring A given in the construction of U^0_μ and P^0_μ corresponds to the ideal $(T_{r,s},T'_{r',s'})$, so that

$$\mathfrak{X}_\mu=\operatorname{Spf} K[[T_{ij},T'_{kl}]] \text{ with } T_{ij}T'_{kl}=q \text{ as usual.}$$

Let T_1,T_2 be two indeterminacies. We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spf} K[[T_1,T_2]] & \xrightarrow{\ \mu\ } & \mathfrak{X}_\mu \\ {\scriptstyle q\mapsto T_1T_2}\downarrow & & \downarrow \\ \operatorname{Spf} K[[q]] & \xrightarrow{\ s\ } & \mathcal{S}, \end{array}$$

where $K[[T_1,T_2]]$ has ideal of definition (T_1T_2), and where the top horizontal map is an isomorphism. The formal scheme $\mathfrak{X}_\mu$ comes with a divisor $\mathcal{D}_\mu$ given by the pullback of $\mathcal{D}$

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which has two components corresponding to X^0_λ and $X^0_{\lambda'}$ where $\lambda(\mu) = (\lambda,\lambda')$. They are defined by $T_1 = 0$ and $T_2 = 0$, respectively, so that $\mathcal{D}_\mu$ is a set of divisors with regular normal crossing on $\mathfrak{X}_\mu$.

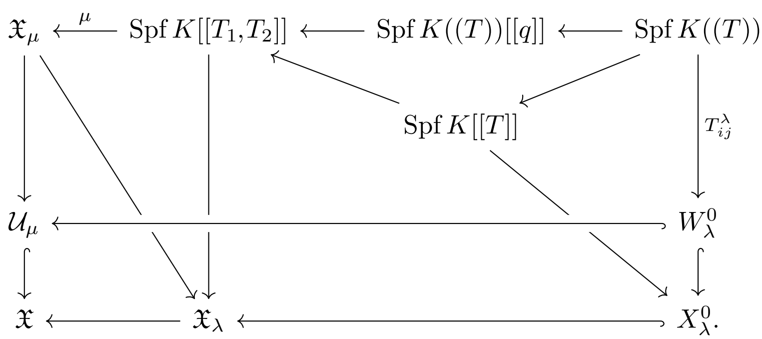
We shall also consider the completion $\mathfrak{X}_\lambda$ of $\mathfrak{X}$ along X^0_λ. It is also equipped with a divisor $\mathcal{D}_\lambda$ as the pullback of $\mathcal{D}$ to $\mathfrak{X}_\lambda$ which consists of the union of two divisors:

1. $\mathcal{D}^0_\lambda$ given by X^0_λ
2. $\mathcal{D}'_\lambda$ given by the distinguished sections of X^0_λ.

It is again a set of divisors with regular normal crossings.

By arguing as in the proof of Proposition 3.11, one further obtain the following compatibility result between tangential base points and formal completions.

Proposition 3.13. *Let $\mu \in M$ and $\lambda \in \lambda(\mu)$. Then we have the following commutative diagram in the category of formal schemes:*



One remarks that the map $\iota_\lambda\colon \operatorname{Spec} K[[T]] \to \operatorname{Spf} K[[T_1,T_2]]$ which is given by the quotient by T_2 factors through the restriction to the special fiber $\operatorname{Spec} K[[T_1,T_2]]/(T_1T_2)$.

4. Galois actions by Grothendieck-Murre theory

Starting with a G-stable diagram X^0 with $\operatorname{Card}\Lambda \sqcup \Lambda' \geq 2$, the end result of the previous section gives us a smooth curve X_η represented by a $K((q))$-point on $\mathcal{M}_{g,[m]}(G)^\nu_{\underline{kr}}$ which comes with a model X over $\mathcal{S}$ with special fiber X^0.

We will now relate the Galois action on the fundamental groupoid $\Pi_1(X_\eta \setminus \{(Q_v)_{v\in N}\},(\vec{\mu})_{\mu\in M})$ of X_η based at the punctures coming from the double points $(P_\mu)_{\mu\in M}$ of X to the ones on the curves $(C_r)_{r\in\{0,\dots,\ell-2\}}$ obtained by the tangential base points T^r_{ij} that we defined in Section 3.2. To do so, we follow some equivalence between categories of covers as in [17]: the category $\operatorname{Rev}^D(X)$ of finite étale covers of X tamely ramified along the divisor D, made of the union $X^0 \cup \{(Q_v)_{v\in N}\}$, is canonically equivalent both to $\operatorname{Rev}^{\mathcal{D}}(\mathfrak{X})$ and $\operatorname{Rev}(X_\eta \setminus \{(Q_v)_{v\in N}\})$. For $\mu \in M$, we then define some fiber functors $\vec{\mu}$, so that, by the previous canonical equivalences of categories, we have the isomorphism

$$\pi_1^{\mathcal{D}}(\mathfrak{X},\vec{\mu}) \simeq \pi_1^D(X,\vec{\mu}) \simeq \pi_1(X_\eta \setminus \{(Q_v)_{v\in N}\},\vec{\mu}).$$

Those equivalences are Galois equivariant, so in order to determine whenever an element of G_K acts trivially on the geometric part of $\pi_1(X_\eta \setminus \{(Q_v)_{v\in N}\},\vec{\mu})$, it is enough to do so on the left-hand side.

4.1. Tamely ramified fundamental groups and fiber functors

4.1.1. Tangential base points: Fiber functors and Galois actions. We start by defining fiber functors on $\mathrm{Rev}^{\mathcal{D}}(\mathfrak{X})$ locally by fixing $\mu \in M$ and considering $\mathfrak{X}_\mu$. Recall that we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spf} K[[T_1,T_2]] & \xrightarrow{\ \mu\ } & \mathfrak{X}_\mu \\ \downarrow & & \downarrow \\ \operatorname{Spf} K[[q]] & \xrightarrow[\ s\ ]{} & \mathcal{S} \end{array}$$

given by the map $q \mapsto T_1T_2$. Both maps s and μ define fiber functors, $\vec{\mu}$ for $\mathrm{Rev}^{\mathcal{D}_\mu}(\mathfrak{X}_\mu)$ and $\vec{s}$ for $\mathrm{Rev}^{S^0}(\mathcal{S})$; see [17] 3.3.1 and 3.3.2.

To be explicit, consider a compatible choice of indeterminates $\{T_1^{\frac{1}{N}},T_2^{\frac{1}{N}}\}_{N\in\mathbb{N}}$ and $\{q^{\frac{1}{N}}\}_{N\in\mathbb{N}}$ to form the fields $K\{\{T_1,T_2\}\}$ and $K\{\{q\}\}$. Then for $\mathfrak{B} = \operatorname{Spf}\mathcal{B} \in \mathrm{Rev}^{\mathcal{D}}(\mathfrak{X}_\mu)$, resp. $\mathfrak{A} = \operatorname{Spf}\mathcal{A} \in \mathrm{Rev}^{S^0}(\mathcal{S})$, the values of the fiber functors are given by

$$\vec{\mu}(\mathfrak{B}) = \mathrm{Hom}_{K[[T_1,T_2]]}(\mathcal{B},\overline{K}\{\{T_1,T_2\}\}),\ \text{resp.}\ \vec{s}(\mathfrak{A}) = \mathrm{Hom}_{K[[q]]}(\mathcal{A},\overline{K}\{\{q\}\}).$$

By choosing geometric points such that $q^{\frac{1}{N}} \mapsto (T_1T_2)^{\frac{1}{N}}$, one obtains two compatible homotopy exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\mathbb{Z}}(1)\times\widehat{\mathbb{Z}}(1) & \xrightarrow{\ j_\mu\ } & \pi_1^{\mathcal{D}_\mu}(\mathfrak{X}_\mu,\vec{\mu}) & \xrightarrow{\ p_\mu\ } & G_K & \longrightarrow & 1 \\ & & {\scriptstyle j_\lambda}\uparrow\ \downarrow\ \uparrow{\scriptstyle j_{\lambda'}} & & {\scriptstyle p_{\mu/S}}\downarrow & {\scriptstyle s_\mu}\nwarrow\ \ {\scriptstyle s_s}\swarrow & \| & & \\ 1 & \longrightarrow & \widehat{\mathbb{Z}}(1) & \xrightarrow{\ j_S\ } & \pi_1^{S^0}(\mathcal{S},\vec{s}) & \xrightarrow{\ p_S\ } & G_K & \longrightarrow & 1, \end{array} \tag{4.1}$$

where the geometric parts $\widehat{\mathbb{Z}}(1)\times\widehat{\mathbb{Z}}(1)$ and $\widehat{\mathbb{Z}}(1)$ are equipped with the Galois actions coming from the sections defined by the choices of tangential base points μ and s. We refer to [17] 3.3.1–3.3.4 for details.

We will now track explicitly the fiber functors defined by $\vec{\mu}$ on $\mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$ and $\mathrm{Rev}^{\mathcal{D}_{\lambda'}}(\mathfrak{X}_{\lambda'})$ for $(\lambda,\lambda') = \lambda(P_\mu^0)$ and compare them to the one given by the tangential base points T_{ij}^λ of Definition 3.1. First of all, remark that the map $\mathfrak{X}_\mu \to \mathfrak{X}_\lambda$ pulls back the divisor $\mathcal{D}_\lambda$ to $\mathcal{D}_\mu$ so that it induces a base change functor

$$\mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda) \longrightarrow \mathrm{Rev}^{\mathcal{D}_\mu}(\mathfrak{X}_\mu),$$

and thus, we have a fiber functor on $\mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$ that is given by composition with $\vec{\mu}$, which we also denote by $\vec{\mu}$. In particular, this comes with a map on the étale fundamental groups

$$p_{\mu/\lambda}\colon \pi_1^{\mathcal{D}_\mu}(\mathfrak{X}_\mu,\vec{\mu}) \longrightarrow \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}).$$

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In the same way, the morphism $f_\lambda\colon \mathfrak{X}_\lambda \to \mathcal{S}$ defines a map

$$p_{\lambda/\mathcal{S}}\colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow \pi_1^{S^0}(\mathcal{S},\vec{s})$$

by the fact that the pullback of S^0 is the divisor $\mathcal{D}_\lambda^0 \cup \mathcal{D}_\lambda''$ where $\mathcal{D}_\lambda''$ is given by $\mathcal{D}_\lambda'$ restricted to X_λ^0. As the map $\mathfrak{X}_\mu \to \mathfrak{X}_\lambda$ is a map of $\mathcal{S}$-schemes, we have the commutativity condition

$$p_{\lambda/\mathcal{S}} \circ p_{\mu/\lambda} = p_{\mu/\mathcal{S}}$$

and compatibility with the previous homotopy exact sequences of Equation (4.1).

4.1.2. Étale fundamental group comparisons and Galois-compatible actions. By Theorem 4.3.2 of [10] the restriction map to X_λ^0 gives a categorical equivalence

$$\mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda) \simeq \mathrm{Rev}^{D_\lambda}(X_\lambda^0),$$

and the last one is canonically equivalent to $\mathrm{Rev}(W_0^\lambda)$.

Proposition 4.1. *The isomorphisms* $\mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda) \simeq \mathrm{Rev}^{\mathcal{D}_\lambda}(X_\lambda^0) \simeq \mathrm{Rev}(W_\lambda^0)$ *transform the fiber functor* $\vec{\mu}$ *in* $\overrightarrow{T_{ij}}$ *and thus yield a Galois compatible isomorphism*

$$\pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \simeq \pi_1(W_\lambda^0,\vec{T}_{ij}).$$

Proof. By Proposition 3.13, the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Spf}\,K[[T_1,T_2]] & & \xleftarrow{\quad\iota_\lambda\quad} & & \mathrm{Spf}\,K[[T]] \\
\downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle T_{ij}} \\
\mathfrak{X}_\mu & \longrightarrow & \mathfrak{X}_\lambda & \longleftarrow & X_\lambda^0,
\end{array}$$

where we recall the map $\iota_\lambda\colon \mathrm{Spec}\,K[[T]] \to \mathrm{Spf}\,K[[T_1,T_2]]$ is given by the quotient by T_2.

It thus suffices to check that the fiber functors on $\mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda)$ given by μ and $\mu \circ \iota_\lambda$ are canonically equivalent and that they are also equivalent to the one given by composition of the pullback to the special fiber and $\vec{T}_{ij}$.

Let $\mathfrak{B} \in \mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda)$ and consider $A \in \mathrm{Rev}^{D_\lambda}(X_\lambda^0)$ obtained from $\mathfrak{B}$ by base change to the special fiber. The pullback of $\mathfrak{B}$ to $\mathfrak{X}_\mu$ is $\mathrm{Spf}\,\mathcal{B} \in \mathrm{Rev}^{(T_1=0)}(\mathfrak{X}_\mu)$ with $\mathcal{B}$ a direct sum of subalgebras of $\overline{K}[[T_1^{\frac{1}{N}},T_2]]$ for some $N \geq 1$. Then we have

$$\begin{aligned}
\vec{\mu}(\mathfrak{B}) &= \mathrm{Hom}_{K[[T_1,T_2]]}(\mathcal{B},\overline{K}\{\{T_1,T_2\}\}) \\
&= \mathrm{Hom}_{K[[T_1,T_2]]}(\mathcal{B},\overline{K}\{\{T_1\}\}[[T_2]]) \\
&= \mathrm{Hom}_{K[[T]]}(\mathcal{B}/T_2,\overline{K}\{\{T\}\}) \\
\vec{\mu}(\mathfrak{B}) &= \overrightarrow{\mu \circ \iota_\lambda}(\mathfrak{B}) = \overrightarrow{T_{ij}}(A).
\end{aligned}$$

□

Remark 4.2. The map $\mathrm{Spf}\,K[[T]] \to \mathrm{Spf}\,K[[T_1,T_2]]$ does not define a base change $\mathrm{Rev}^{\mathcal{D}_\mu}(\mathfrak{X}_\mu) \to \mathrm{Rev}^{(T=0)}(\mathrm{Spf}\,K[[T]])$ as the pullback of the divisor $\mathcal{D}_\mu$ is $\mathrm{Spf}\,K[[T]]$ and not $(T=0)$. Thus, we cannot define a fiber functor for the first category in this way.

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We can now compare $\pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda,\vec{\mu})$ and $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu})$ by Grothendieck-Murre theory since $\mathcal{D}_\lambda$ and $\mathcal{D}'_\lambda$, as defined in Section 3.4.3, are two divisors that differ by the special fiber; see [10] Corollary 5.1.11.

Proposition 4.3. *We have an exact sequence*

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \xrightarrow{\ \alpha\ } \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow \pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow 1,$$

where $\alpha = p_{\mu/\lambda}\circ j_\mu \circ j_\lambda$ and where β comes from the canonical projection induced by the inclusion $\mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda)\subset \mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$.

Proof. By [10] Theorem 7.3.1, we have the exactness of the sequence

$$\widehat{\mathbb{Z}}(1) \xrightarrow{\ \alpha\ } \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow \pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow 1.$$

The injectivity of α can be deduced from the injectivity of $p_{\lambda/\mathcal{S}}\circ\alpha = j_{\mathcal{S}}$. □

Remark 4.4. With the equality $p_{\lambda/\mathcal{S}}\circ p_{\mu/\lambda}\circ s_\mu = s_s$, we also have the surjectivity of $p_{\lambda/\mathcal{S}}$.

4.2. Geometric Galois actions and groupoids

For the fundamental group of a curve X over K, the geometric part is defined to be the fundamental group of $X_{\overline{K}}$ and coincides with the kernel of the projection to G_K given by the arithmetic geometric fundamental homotopy exact sequence Equation (3.1).

4.2.1. Galois actions and inertia.

Following [17] 3.4.7, we define geometric parts of the fundamental groups $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu})$ as the kernels of such projections to G_K.

Definition 4.1. The geometric part $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu})$ is the kernel of $p_\lambda = p_{\mathcal{S}}\circ p_{\lambda/\mathcal{S}}$.

Proposition 4.5. *We have the following results on the structure of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu})$.*

1. *We have an exact sequence*

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \xrightarrow{\ \overline{\alpha}\ } \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu}) \longrightarrow \pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}}) \longrightarrow 1$$

and an isomorphism $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu}) \simeq \widehat{\mathbb{Z}}(1)\times \pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$.

2. *The exact sequence*

$$1 \longrightarrow \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu}) \longrightarrow \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \longrightarrow G_K \longrightarrow 1$$

admits a splitting, and we have an isomorphism $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda,\vec{\mu}) \simeq \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})\rtimes G_K$.

Furthermore, the action of G_K on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ preserves the direct product decomposition of 1 and induces the Galois action on $\pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$ given by the tangential base point T^λ_{ij}.

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Proof.

1. We deduce the exact sequence from the one of Proposition 4.3, where we replaced the last term via the equivalence of categories $\mathrm{Rev}^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda) \simeq \mathrm{Rev}^{D_\lambda}(X^0_\lambda)$; see §4.1.2.

   We know that $\widehat{\mathbb{Z}}(1)$ is the kernel of β so that its image lands in the geometric part is a given. The short exact sequence follows.

   The projection $p_{\lambda/\mathcal{S}}$ induces a geometric counterpart

$$\overline{p_{\lambda/\mathcal{S}}}\colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu}) \longrightarrow \widehat{\mathbb{Z}}(1)$$

   which verifies $\overline{p_{\lambda/\mathcal{S}}} \circ \overline{\alpha} = \mathrm{id}_{\widehat{\mathbb{Z}}(1)}$. It follows that $\mathrm{Ker}\overline{p_{\lambda/\mathcal{S}}} \cap \overline{\alpha}(\widehat{\mathbb{Z}}(1)) = \{1\}$ so that $\mathrm{Ker}\overline{p_{\lambda/\mathcal{S}}}$ is isomorphic to $\pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$ and we have the direct product decomposition.
2. The splitting is given by $s_\mu \circ p_{\mu/\lambda}$. The fact that the resulting G_K-action preserves the direct product decomposition and induces the G_K-action on $\pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$ given by the tangential base point T_{ij} follows directly from the compatibility of the fiber functors $\vec{\mu}$, $\vec{s}$ and $\overrightarrow{T_{ij}}$. □

4.2.2. A trivial Galois action condition. We can now state the basic result that determines when an element of G_K acts trivially on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$.

Proposition 4.6. *An element of G_K acts trivially on $\pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ if and only if it acts trivially on $\pi_1^{\ell,D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$.*

Proof. The decomposition of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ given by 4.2.1 of the previous result passes to the pro-ℓ-completion, which gives

$$\pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu}) \simeq \widehat{\mathbb{Z}_\ell}(1) \times \pi_1^{\ell,D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}}).$$

As the G_K-action preserves the product, the implication is straightforward. For the reciprocal, let $\sigma \in G_K$ that acts trivially on $\pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$. Let us choose a representation $(y_1,\dots,y_{2g},x_1,\dots x_n \mid \prod_i [y_i,y_{i+1}]x_1\cdots x_n)$ of $\pi_1^{\ell,D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$ in the usual way, where x_1 denotes the loop around the closed point image of T_{ij} in X^0_λ. We have $\sigma(x_1) = x_1^{\chi_\ell(\sigma)} = x_1$ by assumption. But σ also acts by $\chi_\ell(\sigma)$ on the first factor $\widehat{\mathbb{Z}_\ell}(1)$, so the action of σ on $\pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ is trivial. □

Remark 4.7. More generally, the result also holds in the case of any almost full class of finite groups $\mathcal{C}$ and the maximal pro-$\mathcal{C}$-quotients of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}},\vec{\mu})$ and $\pi_1^{D_\lambda}(X^0_{\lambda\overline{K}},\overrightarrow{T_{ij}})$; see [17] Proposition 3.4.8.

4.2.3. From fundamental groups to groupoids. In order to conclude, we first we need to explain how to move from fundamental groups to fundamental groupoids. This is essentially formal and comes down to the fact that the set of étale paths are principal homogeneous spaces under the translation actions of the fundamental groups. As such, the technical details will mostly be avoided.

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Let $M_\lambda = \{\mu \in M \mid \lambda \in \lambda(\mu)\}$ and fix $\lambda \in \Lambda \sqcup \Lambda'$. Let $\mu_1, \mu_2 \in M_\lambda$. The set of étale paths between the fiber functors $\vec{\mu_1}$ and $\vec{\mu_2}$ of the category $\mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$ is the profinite set $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$ of isomorphisms between these two functors. The fundamental groups $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1})$ and $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_2})$ act by left and right translation canonically on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$, and these actions are simply transitive. By construction, $\vec{\mu_1}$ and $\vec{\mu_2}$ are turned into the fiber functor $\vec{s}$ of $\mathrm{Rev}^{S^0}(\mathcal{S})$ through the base change by the map $f_\lambda \colon \mathfrak{X}_\lambda \to \mathcal{S}$ so that we have a map

$$p_{\lambda/\mathcal{S}} \colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2}) \longrightarrow \pi_1^{S^0}(\mathcal{S}, \vec{s}).$$

By composition, we get a canonical map $p_\lambda = p_{\mathcal{S}} \circ p_{\lambda/\mathcal{S}} \colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2}) \to G_K$.

Definition 4.2. The geometric part $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1}, \vec{\mu_2})$ of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$ is the set $p_\lambda^{-1}(\{1\})$.

The maps p_λ (for varying $\mu \in M_\lambda$) induce a groupoid homomorphism from $\Pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, (\vec{\mu})_{\mu \in M_\lambda})$ to G_K. This groupoid compatibility ensures that the canonical actions of the groups $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1})$ and $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_2})$ on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$ induce by restriction simply transitive actions from their geometric part to the geometric part of the latter.

This construction can be made when considering $\vec{\mu_1}$ and $\vec{\mu_2}$ as fiber functors with respect to the category of étale covers of $\mathfrak{X}_\lambda$ tamely ramified over $\mathcal{D}'_\lambda$ instead of of $\mathcal{D}_\lambda$. As in Proposition 4.3, we have a natural map

$$\beta_{\mu_1,\mu_2} \colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2}) \longrightarrow \pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$$

which is compatible with the canonical actions on both sides with regard to the maps β_{μ_1} and β_{μ_2}. In particular, the map β_{μ_1,μ_2} is surjective and also induces a bijection from $\overline{p_{\lambda/\mathcal{S}}}^{-1}(\{1\})$ to $\pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1}, \vec{\mu_2})$ as in Proposition 4.5. Moreover, the base change functor to the special fiber induces again a canonical bijection

$$\pi_1^{\mathcal{D}'_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2}) \simeq \pi_1^{D_\lambda}(X^0_\lambda, \overrightarrow{T_{ij}}, \vec{T_{kl}}).$$

Definition 4.3. We define an action of G_K on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$ in the following way. For $\gamma \in \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1}, \vec{\mu_2})$ and $\sigma \in G_K$, let

$$\sigma \cdot \gamma = s_{\lambda/\mu_1}(\sigma) \circ \gamma \circ s_{\lambda/\mu_2}(\sigma)^{-1},$$

where $s_{\lambda/\mu} = p_{\mu/\lambda} \circ s_\mu$ for $\mu \in M_\lambda$.

By the compatibility with p_λ, this action induces an action of G_K on the geometric part of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$. This action is compatible with the bijection $p_{\lambda/\mathcal{S}}^{-1}(\{1\}) \simeq \pi_1^{D_\lambda}(X^0_{\lambda\overline{K}}, \overrightarrow{T_{ij}}, \vec{T_{kl}})$, and we recover the G_K-action induced by our choice of tangential base points on the right-hand side.

We can now state the groupoid analog of Proposition 4.6 and establish the main result of this section.

Proposition 4.8. *Let $\mu_1, \mu_2 \in M$. An element of G_K acts trivially on $\pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1}, \vec{\mu_2})$ if and only if it acts trivially on $\pi_1^{\ell,D_\lambda}(X^0_{\lambda\overline{K}}, \overrightarrow{T_{ij}}, \vec{T_{kl}})$.*

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Proof. As the bijection $p_{\lambda/\mathcal{S}}^{-1}(\{1\}) \simeq \pi_1^{D_\lambda}(X_\lambda^0, \overrightarrow{T_{ij}}, \vec{T_{kl}})$ is a G_K-isomorphism, the implication is straightforward again.

For the converse, let $\sigma \in G_K$. We first remark that by the simple transitiveness of the action of $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1})$ on $\pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2})$ and its compatibility with the map $p_{\lambda/\mathcal{S}}$, we have that for every $\gamma \in \pi_1^{\ell,\mathcal{D}_\lambda}(\mathfrak{X}_{\lambda\overline{K}}, \vec{\mu_1}, \vec{\mu_2})$, there exists $\alpha \in \widehat{\mathbb{Z}}_\ell(1)$ such that $\alpha \cdot \gamma \in p_{\lambda/\mathcal{S}}^{-1}(\{1\})$.

Now, by assumption, we have $\sigma(\alpha \cdot \gamma) = \alpha \cdot \gamma$ so that $\sigma(\gamma) = \sigma(\alpha)^{-1} \cdot (\alpha \cdot \gamma)$, and thus, it is enough to see that σ acts trivially on $\widehat{\mathbb{Z}}_\ell(1)$. This follows as in the proof of Proposition 4.6, since σ acting trivially on $\pi_1^{D_\lambda}(X_{\lambda\overline{K}}^0, \overrightarrow{T_{ij}}, \vec{T_{kl}})$ implies it acts trivially on $\pi_1^{D_\lambda}(X_{\lambda\overline{K}}^0, \overrightarrow{T_{ij}})$, again by simple transitiveness and Galois compatibility. □

Remark 4.9. The result holds in more generality by using an almost full class of finite groups instead of the pro-ℓ completion.

Consider the formal scheme $\mathfrak{X}$. The maps $\mathfrak{X}_\lambda \to \mathfrak{X}$ for $\lambda \in \Lambda \sqcup \Lambda'$, which send $\mathcal{D}$ to $\mathcal{D}_\lambda$ by pullback, induce base change functors $\mathrm{Rev}^{\mathcal{D}}(\mathfrak{X}) \to \mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$. Hence, for $\mu \in M$, we have fiber functors $\vec{\mu}$ for $\mathrm{Rev}^{\mathcal{D}}(\mathfrak{X})$ and a fundamental groupoid $\Pi_1^{\mathcal{D}}(\mathfrak{X}, (\vec{\mu})_{\mu\in M})$ which comes with a geometric part $\Pi_1^{\mathcal{D}}(\mathfrak{X}_{\overline{K}}, (\vec{\mu})_{\mu\in M})$ equipped with a Galois action. For every $\lambda \in \Lambda \sqcup \Lambda'$ and $\mu_1, \mu_2 \in M_\lambda$, the induced canonical maps

$$p_{\lambda/\mathfrak{X},\mu_1,\mu_2} \colon \pi_1^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda, \vec{\mu_1}, \vec{\mu_2}) \longrightarrow \pi_1^{\mathcal{D}}(\mathfrak{X}, \vec{\mu_1}, \vec{\mu_2})$$

are compatible with taking geometric parts and Galois actions on both sides.

Theorem 4.10. *If an element of G_K acts trivially on the groupoids $\Pi_1^{\ell,D_\lambda}(X_{\lambda\overline{K}}^0, \mathbb{B}_\lambda^r)$ for every $\lambda \in \Lambda \sqcup \Lambda'$, then it acts trivially on the groupoid $\Pi_1^{\ell,\mathcal{D}}(\mathfrak{X}_{\overline{K}}, (\vec{\mu})_{\mu\in M})$.*

Proof. The main result of [10] paragraph 8.2.6 gives an equivalence of categories between $\mathrm{Rev}^{\mathcal{D}}(\mathfrak{X})$ and a system of certain subcategories of the $\mathrm{Rev}^{\mathcal{D}_\lambda}(\mathfrak{X}_\lambda)$ which yields that the fundamental groupoid $\Pi_1^{\ell,\mathcal{D}}(\mathfrak{X}, (\vec{\mu})_{\mu\in M})$ is generated by the images of the $p_{\lambda/\mathfrak{X},\mu_1,\mu_2}$ for all $\lambda \in \Lambda \sqcup \Lambda'$ and $\mu_1, \mu_2 \in M_\lambda$. This generation statement carries to the geometric parts by [17] Section 3.6.

The statement of the theorem now follows from Proposition 4.8. □

By Theorem 4.3.2 of [10], there is a canonical isomorphism

$$\Pi_1^{\ell,\mathcal{D}}(\mathfrak{X}, (\vec{\mu})_{\mu\in M}) \simeq \Pi_1^{\ell,D}(X, (\vec{\mu})_{\mu\in M}),$$

where the right-hand side is isomorphic to $\Pi_1^{\ell}(X_\eta \setminus \{(Q_v)_{v\in N}\}, (\vec{\mu})_{\mu\in M})$, and the choice of $\vec{\mu}$ defines compatible G_K-actions.

Corollary 4.11. *We have the inclusion of ℓ-monodromy fixed fields $K_{X_\eta}^{\ell} \subset \mathbb{Q}_{0,3}^{\ell}$.*

Proof. For any $\mu \in M$ and $\vec{\mu}$, coming from a tangential base point of X_η, and seen as a fiber functor on $\mathrm{Rev}(X_\eta \setminus \{(Q_v)_{v\in N}\}$, we have the usual inclusion $K_{X_\eta}^{\ell} \subset K_{\vec{\mu}}^{\ell}$. The inclusion $K_{\vec{\mu}}^{\ell} \subset \mathbb{Q}_{0,3}^{\ell}$ follows by [17] Corollary 4.1.4 (ii). Indeed, by Theorem 3.5, an element of G_K acts trivially on the groupoids $\Pi_1^{\ell,D_\lambda}(X_{\lambda\overline{K}}^0, \mathbb{B}_\lambda^r)$, $\lambda \in \Lambda \sqcup \Lambda'$ if and only

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if it acts trivially on the groupoid $\Pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}}\setminus\{0,1,\infty\},\mathbb{B})$. If so, it also acts trivially on $\Pi_1^{\ell,\mathcal{D}}(\mathfrak{X}_{\overline{K}},(\vec{\mu})_{\mu\in M})$ and thus on $\Pi_1^{\ell}(X_\eta\setminus\{(Q_v)_{v\in N}\},(\vec{\mu})_{\mu\in M})$ by Theorem 4.10. □

For future use, let us summarize the results of Section 4 in a statement that can be applied for various well-chosen geometric constructions as in Section 3 of this paper.

Theorem 4.12. *Let X/S be a stable curve with S the spectrum of a discrete valuation ring with residue field K of characteristic 0. Let $D\subset X$ be a normal crossing divisor containing X^0 the special fiber of X. Let us denote by X_η the generic fiber of X such that X_η, equipped with D_η, is a proper smooth marked curve. Let $(X_\lambda)_{\lambda\in\Lambda}$ be the irreducible components of X^0, which are equipped with a divisor D_λ by pullback from D, and M the set of double points of X^0. Suppose given for each $\mu\in M$ a morphism*

$$\mu\colon \operatorname{Spf} K[[T_1,T_2]]\simeq\mathfrak{X}_\mu\to\mathfrak{X}.$$

If $\sigma\in G_K$ acts trivially on $\Pi_1^{D_\lambda}(X_\lambda,\{\vec{\mu}_\lambda\}_{\{\mu|\mu\ni\lambda\}})$ for every $\lambda\in\Lambda$, then it acts trivially on $\Pi_1^{D_\eta}(X_\eta,\{\vec{\mu}\}_{\mu\in M})$, where $\{\vec{\mu}_\lambda\}_{\{\mu|\mu\ni\lambda\}}$ are the associated fiber functors of $\operatorname{Rev}^{D_\lambda}X_\lambda$.

5. Oda's problem for $\mathbb{Z}/\ell\mathbb{Z}$-special loci

In the rest of this section, we fix a prime ℓ and specialize the previous study of this paper to the case $G=\mathbb{Z}/\ell\mathbb{Z}$ to establish Oda's prediction for $\mathbb{Z}/\ell\mathbb{Z}$-special loci – that is, the ℓ-monodromy fixed field $\mathbb{Q}^{\ell}_{g,m}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is constant independent of the topological g, m and Hurwitz $\underline{kr}$ data and equal to $\mathbb{Q}^{\ell}_{0,3}$ – which provides a new proof of Oda's original prediction (i.e., that is $\mathbb{Q}^{\ell}_{g,m}=\mathbb{Q}^{\ell}_{0,3}$).

We proceed by considering two types of irreducible components $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$, whose associated monodromy fixed fields $\mathbb{Q}^{\ell}_{g,m}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ are compared to those of other components by the G-quotient of Section 2.1 and the G-deformation of Section 3.4.

5.1. The case of proper special loci

Let us consider the case where $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ is such that the quotient loci is $\mathcal{M}_{0,3}$, – that is, when the quotient loci is proper. As the quotient map is itself quasi-finite and proper, the stack $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ is proper if and only if it is the case of the stack of the quotient curves. In this case, both stacks $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ and $\mathcal{M}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ are geometrically given by a single point and are equal.

The following lemma enumerates the possible values of g, m and $\underline{kr}$ that make this possible for a $\mathbb{Z}/\ell\mathbb{Z}$-special loci in the étale quotient case.

Lemma 5.1. *Assuming the ramified points are marked, the moduli space $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is proper in the following cases:*

1. *$g=0$, $m=2+\ell$, $\underline{k}=(1,-1)$;*
2. *$g=\frac{\ell-1}{2}$, $m=3$, and the abstract Hurwitz data $\underline{k}$ is free.*

Proof. In the case of a quotient by $\mathbb{Z}/\ell\mathbb{Z}$, the Hurwitz formula is

$$2g-2=(2g'-2)\ell+N(\ell-1),$$

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where N is the number of ramified points, and setting $g'=0$ yields

$$g=(N-2)(\frac{\ell-1}{2}).$$

Since the ramified points are assumed to be marked, we have $N\in\{2,3\}$, since the cases $N=0$ or 1 are not possible.

For $N=2$, we have $g=0$ and $\underline{k}=(1,-1)$. The $m=2+\ell$ marked points are given by two ramified points and ℓ points permuted under the action of $\mathbb{Z}/\ell\mathbb{Z}$.

For $N=3$, we have $g=(\ell-1)/2$, and the marked points are the ramified points. In this case, there is no condition on the abstract Hurwitz data. □

Let us remark that the case $N=3$ (resp. $N=2$) is given by the Seyama curves (resp. the G-curves of genus 0) discussed in Section 3.2.

Theorem 5.2. *For $g,m\in\mathbb{N}$ and compatible abstract Hurwitz data $\underline{kr}$ such that the stack $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is proper and nonempty, we have the equality*

$$\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}=\mathbb{Q}^{\ell}_{0,3}.$$

Note that following our assumptions one as also $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}=\mathbb{Q}^{\ell}_{g,[m]}(G)^{\nu}_{\underline{kr}}$.

Proof. Corollary 2.12 (see diagram below) gives the inclusions $\mathbb{Q}^{\ell}_{0,3}\subset\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}\subset\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}}$. Let us consider s and the abstract Hurwitz data $\underline{kr}^{et}$, as defined in Proposition 2.5, and the map $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}\to\mathcal{M}_{g,[m+s]}(G)_{\underline{kr}^{et}}$ which is finite. Thus, $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ is proper if and only if $\mathcal{M}_{g,[m+s]}(G)_{\underline{kr}^{et}}$ is, and it is sufficient to establish the reverse inclusion $\mathbb{Q}^{\ell}_{0,3}\supset\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ in the étale quotient case, since $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}\subset\mathbb{Q}^{\ell}_{g,[m+s]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}^{et}}$ by Theorem 2.6. In this case, it follows from Lemma 5.1 that there is a K-point in the special loci that represents a curve C isomorphic to either a Seyama curve or a G-curve of genus 0.

The result then follows from the inclusion $\mathbb{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}}\subset\mathbb{Q}^{\ell}_{C}=\mathbb{Q}^{\ell}_{0,3}$ obtained from Lemma 2.3 and Corollary 3.7. □

5.2. General conclusion

We can now establish the main result of this paper for prime cyclic special loci, which also recovers Oda's weak classical conjecture.

Theorem 5.3. *For $g,m\in\mathbb{N}$ such that $2g-2+m>0$ and compatible abstract Hurwitz data $\underline{kr}$ such that $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is nonempty, we have $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}=\mathbb{Q}^{\ell}_{0,3}$.*

Proof. By Corollary 2.9, we can assume that the marked points contain the ramified points of the G-action. Since Theorem 5.2 gives the equalities $\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}}=\mathbb{Q}^{\ell}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}=\mathbb{Q}^{\ell}_{0,3}$ in the case where $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}}$ is proper, let us assume otherwise.

In this case, let us consider the G-stable diagram X^0 over K, with $\operatorname{Card}\Lambda\sqcup\Lambda'\geq 2$, in the boundary of $\overline{\mathcal{M}}_{g,[m]}(G)^{\nu}_{\underline{kr}}$ such as provided by Proposition 3.9. The stable curve X^0 admits a formal deformation $\mathfrak{X}$ which is algebraizable into a scheme X with generic fiber $X_\eta \in \mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}}(K((T)))$ as given by Section 3.4.2. The groupoid $\Pi_1^\ell(X_\eta \setminus \{(Q_v)_{v\in N}\}, (\vec{\mu})_{\mu\in M})$ is equipped with the tangential Galois action of G_K constructed in Section 4 coming from the choices of the fiber functors $(\vec{\mu})_{\mu\in M}$. It results from Corollary 4.11 that $K_{X_\eta} \subset \mathbb{Q}^\ell_{0,3}$.

It follows that $\mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}} \subset \mathbb{Q}^\ell_{0,3}$, since $\mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}} \subset K_{X_\eta}$ by Lemma 2.3, which concludes the first statement by the diagram below Corollary 2.12. In short, we obtained

$$\mathbb{Q}^\ell_{0,3} \hookrightarrow \mathbb{Q}^\ell_{g,m} \hookrightarrow \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}} \hookrightarrow \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}} \hookrightarrow K_{X_\eta} \hookrightarrow \mathbb{Q}^\ell_{0,3}.$$

□

Recovering Oda's weak conjecture relies on previous work of Nakamura and the consideration of certain étale type loci in $\mathcal{M}_{g,[m+s]}(G)$.

Corollary 5.4. *For all $g',m' \in \mathbb{N}$ such that $2g'-2+m' > 0$, the equality $\mathbb{Q}^\ell_{g',m'} = \mathbb{Q}^\ell_{0,3}$ holds.*

Proof. For every $g',m' \in \mathbb{N}$ such that $2g'-2+m' > 0$, there are $g,m \in \mathbb{N}$ and a compatible abstract Hurwitz data $\underline{kr}$ such that $\mathcal{M}_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}}$ is nonempty and (g',m') is the quotient data. This nonemptiness assertion is obtained by Proposition 3.7 of [4].

$$\begin{array}{ccccccccc}
\mathbb{Q}^\ell_{g,m} & \hookrightarrow & \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})_{\underline{kr}} & \hookrightarrow & \mathbb{Q}^\ell_{g,[m]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}} & \hookrightarrow & \mathbb{Q}^\ell_{g,[m+s]}(\mathbb{Z}/\ell\mathbb{Z})^{\nu}_{\underline{kr}^{\mathrm{et}}} & \hookrightarrow & \mathbb{Q}^\ell_{0,3} \\
\updownarrow & & & & & & \uparrow & & \\
\mathbb{Q}^\ell_{0,3} & \hookrightarrow & \mathbb{Q}^\ell_{g',m'} & & \hookrightarrow & & \mathbb{Q}^\ell_{g',m'+s'} & &
\end{array} \tag{5.1}$$

From Proposition 2.5, there is a nonempty stack $\mathcal{M}_{g,[m+s]}(G)_{\underline{kr}^{\mathrm{et}}}$ for some $s \geq 0$ with $\underline{kr}^{\mathrm{et}}$ of étale type by construction, and such that the quotient space is $\mathcal{M}_{g',m'+s'}$ for some $s' \geq 0$. By Theorem 2.11, we obtain the inclusion $\mathbb{Q}^\ell_{g',m'+s'} \subset \mathbb{Q}^\ell_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{\mathrm{et}}}$; then $\mathbb{Q}^\ell_{g,[m+s]}(G)^{\nu}_{\underline{kr}^{\mathrm{et}}} \subset \mathbb{Q}^\ell_{0,3}$ by Theorem 5.3. The conclusions follow by [34] and [31] which gives the inclusion $\mathbb{Q}^\ell_{g',m'} \subset \mathbb{Q}^\ell_{g',m'+s'}$ with $s' \geq 1$, and finally by the inclusion $\mathbb{Q}^\ell_{0,3} \subset \mathbb{Q}^\ell_{g',m'}$ which is again Theorem A of [23]; see Diagram 5.1 for a summary. □

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References

[1] M. P. Anderson, Exactness properties of profinite completion functors. *Topology* **13** (1974), 229–239. ISSN 0040-9383. doi: https://doi.org/10.1016/0040-9383(74)90016-0.

[2] J. Bertin and M. Romagny, Hurwitz stacks. *Mém. Soc. Math. Fr., Nouv. Sér.* **125–126** (2011), 219. doi: 10.24033/msmf.437.

[3] B. Collas, Action of a Grothendieck-Teichmüller group on torsion elements of full Teichmüller modular groups of genus one. *Int. J. Number Theory* **8**(3) (2012), 763–787. doi: 10.1142/S1793042112500431.

https://doi.org/10.1017/S1474748024000409 Published online by Cambridge University Press

[4] B. Collas and S. Maugeais, Composantes irréductibles de lieux spéciaux d'espaces de modules de courbes, action galoisienne en genre quelconque. *Ann. Inst. Fourier* **65**(1) (2015), 245–276. doi: 10.5802/aif.2930.

[5] B. Collas and S. Maugeais, On Galois action on the inertia stack of moduli spaces of curves. *Publ. Res. Inst. Math. Sci.* **59**(4) (2023), 731–758. https://doi.org/10.4171/prims/59-4-2.

[6] T. Ekedahl, Boundary behaviour of Hurwitz schemes. In *The Moduli Space of Curves. Proceedings of the Conference Held on Texel Island, Netherlands During the Last Week of April 1994*. (Basel: Birkhäuser, 1995), 173–198. https://doi.org/10.1007/978-1-4612-4264-2_7.

[7] G. González-Diez and R. A. Hidalgo, Conformal versus topological conjugacy of automorphisms on compact Riemann surfaces. *Bull. Lond. Math. Soc.* **29**(3) (1997), 280–284. doi: 10.1112/S0024609396002640.

[8] A. Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.* (**4**) (1960), 228.

[9] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. *Inst. Hautes Études Sci. Publ. Math.* (**17**) (1963), 137–223.

[10] A. Grothendieck and J. P. Murre, *The Tame Fundamental Groups of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme*, vol. 208. Lect. Notes Math. New York : Springer. https://doi.org/10.1007/BFb0069608.

[11] Y. Hoshi, Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero. *Nagoya Math. J.* **203** (2011), 47–100. doi: 10.1215/00277630-1331863.

[12] Y. Hoshi, On a problem of Matsumoto and Tamagawa concerning monodromic fullness of hyperbolic curves: genus zero case. *Tohoku Math. J. (2)* **65**(2) (2013), 231–242. doi: 10.2748/tmj/1372182723.

[13] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations. *Hiroshima Math. J.* **41**(3) (2011), 275–342. doi: 10.32917/hmj/1323700038.

[14] Y. Ihara, Profinite braid groups, Galois representations and complex multiplications. *Ann. of Math. (2)* **123**(1) (1986), 43–106. doi: 10.2307/1971352.

[15] Y. Ihara, The Galois representation arising from $\mathbb{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree. In *Galois Groups Over* $\mathbb{Q}$ *(Berkeley, CA, 1987)*, vol. 16. (Math. Sci. Res. Inst. Publ. New York: Springer, 1989), 299–313. doi: 10.1007/978-1-4613-9649-9_4.

[16] Y. Ihara, On the embedding of Gal(Q/Q) intô GT. In *The Grothendieck Theory of Dessins D'enfants (Luminy, 1993)*, vol. 200. London Math. Soc. Lecture Note Ser. (Cambridge: Cambridge Univ. Press, 1994), 289–321. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak.

[17] Y. Ihara and H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda's problem. *J. Reine Angew. Math.* **487** (1997), 125–151. https://doi.org/10.1515/crll.1997.487.125.

[18] F. F. Knudsen, The projectivity of the moduli space of stable curves. II: The stacks Mg,n. *Math. Scand.* **52** (1983), 161–199. http://eudml.org/doc/166839.

[19] K. Magaard (2002), The locus of curves with prescribed automorphism group. *Sūrikaisekikenkyūsho Kōkyūroku* T. Shaska, S. Shpectorov and H. Völklein, (1267), 112–141. Communications in arithmetic fundamental groups (Kyoto, 1999/2001).

[20] M. Matsumoto, Galois representations on profinite braid groups on curves. *J. Reine Angew. Math.* **474** (1996), 160–219. https://doi.org/10.1515/crll.1996.474.169.

[21] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces. *Hokkaido Math. J.* **37**(1) (2008), 75–131. doi: 10.14492/hokmj/1253539588.

[22] S. Morita, The extension of Johnson's homomorphism from the Torelli group to the mapping class group. *Invent. Math.* **111**(1) (1993), 197–224. doi: 10.1007/BF01231286.

[23] H. Nakamura, Coupling of universal monodromy representations of Galois- Teichmüller modular groups. *Math. Ann.* **304**(1) (1996), 99–119. doi: 10.1007/BF01446287.

[24] H. Nakamura, Tangential base points and Eisenstein power series. In *Aspects of Galois Theory. Papers from the Conference on Galois theory, Gainesville, FL, USA, October 14–18, 1996.* (Cambridge: Cambridge University Press, 1999), 202–217.

[25] H. Nakamura, N. Takao and R. Ueno, Some stability properties of Teichmüller modular function fields with pro-l weight structures. *Math. Ann.* **302**(2) (1995), 197–213. doi: 10.1007/BF01444493.

[26] B. Noohi, Fundamental groups of algebraic stacks. *J. Inst. Math. Jussieu* **3**(1) (2004), 69–103. doi: 10.1017/S1474748004000039.

[27] T. Oda, The universal monodromy representations on the pro-nilpotent fundamental groups of algebraic curves. *Mathematische Arbeitstagung (Neue Serie), Max-Planck-Institut preprint MPI/93-57*, 9-15, 6.

[28] C. Rasmussen and A. Tamagawa, Arithmetic of abelian varieties with constrained torsion. *Trans. Amer. Math. Soc.* **369**(4) (2017), 2395–2424. doi: 10.1090/tran/6790.

[29] M. Romagny, Connected and irreducible components in families. *Manuscr. Math.* **136** (1–2) (2011), 1–32. doi: 10.1007/s00229-010-0424-7.

[30] A. Seyama, On the curves of genus g with automorphisms of prime order 2g + 1. *Tsukuba J. Math.* **6** (1982), 67–77. https://www.jstor.org/stable/43685303.

[31] N. Takao, Braid monodromies on proper curves and pro-ℓ Galois representations. *J. Inst. Math. Jussieu* **11**(1) (2012), 161–181. doi: 10.1017/S1474748010000241.

[32] N. Takao, Some remarks on field towers arising from pro-nilpotent universal monodromy representations. *RIMS Kôkyûroku Bessatsu* **B51** (2014), 55–70.

[33] H. Tsunogai, Some new-type equations in the Grothendieck-Teichmüller group arising from geometry M0,5. In *Primes and Knots. Proceedings of an AMS Special Session, Baltimore, MD, USA, January 15–16, 2003 and the 15th JAMI (Japan-US Mathematics Institute) Conference, Baltimore, MD, USA, March 7–16, 2003. Providence*, (RI: American Mathematical Society, 2006), 263–284. https://doi.org/10.1090/conm/416.

[34] R. Ueno, A Stability of the Galois Kernel Arising from Pro-ℓ Monodromy Representation. Master thesis, (RIMS, Kyoto University 1994).