ON HEWITT'S τ-MAXIMAL SPACES

MURRAY R. KIRCH*

(Received 25 May 1970; revised 5 August 1970)

Communicated by B. Mond

Let τ be any cardinal number. Edwin Hewitt [3] has defined a topological space $(X, \mathscr{T})$ to be *τ-maximal* if $\Delta(\mathscr{T}) \geqq \tau$ and $\Delta(\mathscr{T}') < \tau$ whenever $\mathscr{T}'$ is a topology for X which is strictly stronger that $\mathscr{T}$ (Δ denotes dispersion character, the least cardinality of a nonempty open set). The notion of an $\aleph_0$-maximal space was introduced independently by Katětov [4]. In this note we show that the concept of τ-maximality is essentially independent of τ by proving the following generalization of a result which appears in Bourbaki [2, 179–180].

THEOREM 1. *Let τ be any infinite cardinal number and let $(X, \mathscr{T})$ be a topological space which has dispersion character $\geqq \tau$. Then the following are equivalent*:

(a) *$(X, \mathscr{T})$ is τ-maximal*;

(b) *$(X, \mathscr{T})$ is extremally disconnected and every dense set in $(X, \mathscr{T})$ is open*:

(c) *Every dense-in-itself set in $(X, \mathscr{T})$ is open*;

(d) *Every topology for X which is strictly stronger than $\mathscr{T}$ has isolated points (i.e. $(X, \mathscr{T})$ is 2-maximal).*

We do not assume any separation axioms. In particular, by extremally disconnected we mean only that the closure of each open set is open. The closure and interior operators in the space $(X, \mathscr{T})$ will be denoted by $\mathrm{cl}_{\mathscr{T}}$ and $\mathrm{int}_{\mathscr{T}}$, respectively.

PROOF. Suppose $(X, \mathscr{T})$ is τ-maximal. There exists an extremally disconnected topology for X which is stronger than $\mathscr{T}$ and which has the same dispersion character as $\mathscr{T}$ [5]. The τ-maximality of $(X, \mathscr{T})$ implies that this extremally disconnected topology must be equal to $\mathscr{T}$. Using a technique suggested by [1] we now show that each dense set in $(X, \mathscr{T})$ is open. First observe that if S is a subset of X such that card $(X - S) < \tau$ then $\Delta(\mathscr{T}_S) = \Delta(\mathscr{T})$ and hence $\mathscr{T}_S = \mathscr{T}$. (Here $\mathscr{T}_S$ denotes the simple extension of $\mathscr{T}$ by S [6], i.e. the topology for X which has $\mathscr{T} \cup \{S\}$ as a subbase.) Thus S must be open in $(X, \mathscr{T})$. It follows that

* Partially supported by a SUNY-Research Foundation Faculty Research Fellowship.

every subset of X with cardinality strictly less than τ must be closed in $(X,\mathscr{T})$; therefore no dense set in a τ-maximal space can have cardinality strictly less than τ. Next we observe that τ-maximality is inherited by open subspaces. Now let D be dense in $(X,\mathscr{T})$ and let G be a nonempty open set in $(X,\mathscr{T}_D)$. Then there exist sets E and F which are open in $(X,\mathscr{T})$ such that $G = E \cup (F \cap D)$. Clearly if E is nonempty then

$$\mathrm{card}(G) \geqq \mathrm{card}(E) \geqq \Delta(\mathscr{T}) \geqq \tau.$$

If E is empty then $F \cap D$ must be nonempty. But in this case $F \cap D$ is dense in the τ-maximal space consisting of F with the relative topology of $(X,\mathscr{T})$ and hence card $(F \cap D) \geqq \tau$. Thus, in either case, $\mathrm{card}(G) \geqq \tau$ and hence $\Delta(\mathscr{T}_D) \geqq \tau$. But $\mathscr{T}_D$ is stronger than the τ-maximal topology $\mathscr{T}$; therefore $\mathscr{T}_D = \mathscr{T}$ and D is open in $(X,\mathscr{T})$. Hence we have shown that condition (a) implies condition (b).

Now assume that condition (b) holds. Let A be any dense-in-itself set in $(X,\mathscr{T})$ and let

$$K = A - \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}A.$$

Clearly $\mathrm{int}_{\mathscr{T}}K$ is empty. Hence K is closed in $(X,\mathscr{T})$ since its complement is dense. Similarly every subset of K is closed, i.e. K is totally isolated. Let x be any point in K. Then there exists an open set W in $(X,\mathscr{T})$ such that $W \cap K = \{x\}$. Thus

$$W \cap (X - \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}A)$$

is open in $(X,\mathscr{T})$ and its intersection with A is simply $\{x\}$, contradicting the assumption that A is dense-in-itself. Therefore K must be empty and hence $A \subset \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}A$. This implies that

$$A = (\mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}A) \cap (A \cup (X - \mathrm{cl}_{\mathscr{T}}A)).$$

But $\mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}A$ is open since $(X,\mathscr{T})$ is extremally disconnected and $A \cup (X - \mathrm{cl}_{\mathscr{T}}A)$ is open since it is dense. Thus A is open in $(X,\mathscr{T})$ and we have shown that condition (b) implies condition (c).

The validity of the remaining implications is clear.

The following result improves upon a theorem of Hewitt on the existence of τ-maximal spaces.

THEOREM 2. *Let $(X,\mathscr{T})$ be a topological space with infinite dispersion character. Then there exists a $\Delta(\mathscr{T})$-maximal topology $\mathscr{T}'$ for X which is stronger than $\mathscr{T}$ and such that $\mathrm{cl}_{\mathscr{T}}S = \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}\mathrm{cl}_{\mathscr{T}}S$ whenever S is open in $(X,\mathscr{T}')$.*

PROOF. Let $\mathscr{F}_0 = \{E \subset X : \mathrm{card}(X - E) < \Delta(\mathscr{T})\}$. An application of Zorn's lemma enables one to obtain a filter $\mathscr{F}$ of sets which are dense in $(X,\mathscr{T})$ such that $\mathscr{F}_0 \subset \mathscr{F}$ and $\mathscr{F}$ is not properly contained in any other filter of $\mathscr{T}$-dense sets. Let $\mathscr{T}''$ be the topology for X which has $\mathscr{T} \cup \mathscr{F}$ as a subbase and define

$$\mathscr{B} = \{G \cap D \colon G \in \mathscr{T} \text{ and } D \in \mathscr{F}\}.$$

Since $\mathscr{B}$ is clearly closed under finite intersections and $\mathscr{T} \cup \mathscr{F} \subset \mathscr{B} \subset \mathscr{T}''$, it follows that $\mathscr{B}$ is a base for $\mathscr{T}''$. We shall prove, in fact that

$$(1) \qquad \mathscr{T}'' = \mathscr{B}$$

First observe that every member of $\mathscr{F}$ meets every member of $\mathscr{B}$ and hence must be dense as well as open in $(X, \mathscr{T}'')$. Thus every $\mathscr{T}''$-dense set must meet each member of $\mathscr{F}$ in a $\mathscr{T}''$-dense (and hence $\mathscr{T}$-dense) set. The maximality of $\mathscr{T}$ implies that each $\mathscr{T}''$-dense set is a member of $\mathscr{F}$ and hence

$$(2) \qquad \text{each } \mathscr{T}''\text{-dense set is } \mathscr{T}''\text{-open.}$$

Let S be any $\mathscr{T}''$-open set and let $x \in X - \mathrm{cl}_{\mathscr{T}''} S$. Since $\mathscr{B}$ is a base for $\mathscr{T}''$, there exist $G \in \mathscr{T}$ and $D \in \mathscr{F}$ such that $x \in G \cap D$ and $G \cap D \cap S$ is empty. Thus $G \cap S$ must be empty since it is a $\mathscr{T}''$-open set which does not meet the $\mathscr{T}''$-dense set D. But G is a $\mathscr{T}$-neighborhood of x and therefore $x \notin \mathrm{cl}_{\mathscr{T}} S$. Thus we have shown that $\mathrm{cl}_{\mathscr{T}''} S = \mathrm{cl}_{\mathscr{T}} S$ for each $S \in \mathscr{T}''$. Hence for each $S \in \mathscr{T}''$ we have

$$\begin{aligned} X - \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} S &= \mathrm{cl}_{\mathscr{T}} \mathrm{int}_{\mathscr{T}} (X - S) = \mathrm{cl}_{\mathscr{T}''} \mathrm{int}_{\mathscr{T}} (X - S) \subset \mathrm{cl}_{\mathscr{T}''} \mathrm{int}_{\mathscr{T}''} (X - S) \\ &= X - \mathrm{int}_{\mathscr{T}''} \mathrm{cl}_{\mathscr{T}''} S \end{aligned}$$

and therefore we have

$$\mathrm{int}_{\mathscr{T}''} \mathrm{cl}_{\mathscr{T}''} S \subset \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} S.$$

Hence

$$S = \mathrm{int}_{\mathscr{T}''} S \subset \mathrm{int}_{\mathscr{T}''} \mathrm{cl}_{\mathscr{T}''} S \subset \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} S.$$

Define $D' = S \cup (X - \mathrm{cl}_{\mathscr{T}''} S)$. Clearly D' is $\mathscr{T}''$-dense and hence, as we have shown, must be a member of $\mathscr{F}$. But $S = D' \cap \mathrm{int}_{\mathscr{T}} \mathrm{cl}_{\mathscr{T}} S$ and therefore S is a member of $\mathscr{B}$. Thus condition (1) has been established. One can easily show that $\mathscr{T}$ and $\mathscr{T}''$ have the same dispersion character by using condition (1) and the fact that $\mathscr{F}_0 \subset \mathscr{F}$. Let $\mathscr{T}'$ be an extremally disconnected topology for X such that the following hold (see [5]):

$$(3) \qquad \mathscr{T}' \text{ is stronger that } \mathscr{T}'',$$

$$(4) \qquad S \subset \mathrm{cl}_{\mathscr{T}''} \mathrm{int}_{\mathscr{T}''} S \text{ whenever } S \text{ is open in } (X, \mathscr{T}').$$

Conditions (3) and (4) imply that $\mathscr{T}''$ and $\mathscr{T}'$ have the same dispersion character. Conditions (2) and (3) imply that each $\mathscr{T}'$-dense set is $\mathscr{T}'$-open. Thus, by Theorem 1, $\mathscr{T}'$ is a $\Delta(\mathscr{T})$-maximal topology which is stronger than $\mathscr{T}$. Now let S be any open set in $(X, \mathscr{T}')$. By condition (1) we may select $G \in \mathscr{T}$ and $D \in \mathscr{F}$ such that $\mathrm{int}_{\mathscr{T}''} S = G \cap D$. Then we obtain

$$S \subset \mathrm{cl}_{\mathscr{T}''}\mathrm{int}_{\mathscr{T}''}S = \mathrm{cl}_{\mathscr{T}''}(G \cap D) \subset \mathrm{cl}_{\mathscr{T}}(G \cap D) = \mathrm{cl}_{\mathscr{T}}G = \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}G$$
$$\subset \mathrm{cl}_{\mathscr{T}}\ \mathrm{int}_{\mathscr{T}}\mathrm{cl}_{\mathscr{T}}G \subset \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}\mathrm{cl}_{\mathscr{T}}S.$$

Thus $S \subset \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}\mathrm{cl}_{\mathscr{T}}S$ which is equivalent to $S = \mathrm{cl}_{\mathscr{T}}\mathrm{int}_{\mathscr{T}}\mathrm{cl}_{\mathscr{T}}S$.

References

[1] D. R. Anderson, 'On connected irresolvable Hausdorff spaces', *Proc. Amer. Math. Soc.* 16 (1965), 463—466.
[2] N. Bourbaki, *Éléments de Mathématique*, Topologie Générale Book III, Chap. 1 (Hermann, Paris, 1961).
[3] E. Hewitt, 'A problem of set-theoretic topology', *Duke Math. J.* 10 (1943), 309–333.
[4] M. Katětov, 'On topological spaces containing no disjoint dense sets', *Mat. Sb.* (N.S.) 21 (1947), 3–12.
[5] M. R. Kirch, 'Weakly equivaent topologies', *Bull. Acad. Polon. Sci.* (to appear).
[6] N. Levine, 'Simple extensions of topologies', *Amer. Math. Monthly* 71 (1964), 22–25.

State University of New York
Buffalo
U.S.A.