

# A general framework for predicting mean profiles in compressible turbulent boundary layers with established scaling laws

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A prediction framework for the mean quantities in a compressible turbulent boundary layer (TBL) with given Reynolds number, free-stream Mach number and wall-to-recovery ratio as inputs is proposed based on the established scaling laws regarding the velocity transformations, skin-friction coefficient and temperature–velocity (TV) relations. The established velocity transformations that perform well for collapsing the compressible mean profiles onto incompressible ones in the inner layer are used for the scaling of such inner-layer components of mean velocity, while the wake velocity scaling is determined such that self-consistency is achieved under the scaling law for the skin-friction coefficient. A total of 44 compressible TBLs from six direct numerical simulations databases are used to validate the proposed framework, with free-stream Mach numbers ranging from 0.5 to 14, friction Reynolds numbers ranging from 100 to 2400, and wall-to-recovery ratios ranging from 0.15 to 1.9. When incorporated with the scaling laws for velocity transformation from Griffin *et al.* (2021, *Proc. Natl Acad. Sci.*, vol. 118, e2111144118), the skin-friction coefficient from Zhao & Fu (2025, *J. Fluid Mech.*, vol. 1012, R3) and the TV relation from Duan & Martín (2011, *J. Fluid Mech.*, vol. 684, pp. 25–59), the prediction errors in the mean velocity and temperature profiles remain within 4.0 % and 6.0 %, respectively, across all tested cases. Correspondingly, the skin-friction and wall-heat-transfer coefficients are also accurately predicted, with root mean square prediction errors of approximately 3 %. When adopting different velocity transformation methods that are valid for inner-layer scaling, the root mean square prediction errors in the mean velocity and temperature profiles remain below 2.3 % and 3.6 %, respectively, which further highlights the universality of the proposed framework.

**Key words:** boundary layers, compressible boundary layers, turbulence modelling

## 1. Introduction

The high-speed turbulent boundary layer (TBL) is known to significantly affect the surface drag and heat transfer, the accurate prediction of which is thus of great importance for reliable vehicle design and flow control (Bradshaw 1977). Numerous studies conducted are engaged in uncovering physical insights in the TBL, based on which fruitful mathematical models in describing the physical properties are extracted (e.g. van Driest 1951; Spalding & Chi 1964; Huang, Bradshaw & Coakley 1993; Duan, Beekman & Martin 2010; Chen, Gan & Fu 2024). As an idealised simplification of the compressible TBL, the incompressible counterpart exhibits distinct universal laws in the mean quantities, such as those for the mean velocity profile in the wall-normal direction (Johnson & King 1985; Kawai & Larsson 2012). However, due to the variations of mean properties such as density and viscosity in the wall-normal direction, the prediction of the mean properties of compressible TBLs with non-negligible Mach numbers is still a challenging task and needs further refinement.

Inspired by Morkovin (1962), where it is hypothesised that the compressible wall-bounded flows can be mapped onto incompressible counterparts by accounting for the variations in mean properties, the established scaling laws for incompressible TBLs can be applied in compressible ones with appropriate transformations. Over decades, studies on velocity transformations have been actively conducted (e.g. van Driest 1951; Zhang *et al.* 2012; Trettel & Larsson 2016; Volpiani *et al.* 2020; Griffin, Fu & Moin 2021*b*; Hasan *et al.* 2023). The pioneer work by van Driest (1951) (denoted as vD) built upon the mixing length assumption performs well in high-speed adiabatic flows but deteriorates for diabatic conditions. On the other hand, the total-stress-based transformation by Griffin *et al.* (2021*b*) (denoted as GFM), which is parameter-free, demonstrates exceptional performance in collapsing the mean streamwise velocity profiles of various flow types, including turbulent channel flows, pipe flows and TBLs even with strong heat transfer, into the incompressible counterparts in the inner layers. In addition to GFM, the transformations proposed by Volpiani *et al.* (2020) (termed Volpiani), which is data-driven, and Hasan *et al.* (2023) (termed HLPP), by interpreting intrinsic compressibility effects, also perform well in the inner layers of compressible TBLs.

Besides the velocity transformation, the scaling of the skin-friction coefficient in compressible TBLs is a related but different topic. Accounting for the variations in density and viscosity, van Driest (1951) proposes a scaling law for the skin-friction coefficient that is applicable in both compressible and incompressible TBLs. Spalding & Chi (1964) further improve the theory of van Driest (1951) by including the impacts of free-stream Mach number and temperature. However, neither of these two theories provides accurate predictions on the skin-friction coefficient with a very cold wall (Bradshaw 1977). Recently, Zhao & Fu (2025) developed a general scaling law for the skin-friction coefficient based on physical and asymptotic analyses, which precisely predicts the transformed skin-friction coefficient with wide ranges of free-stream Mach numbers and wall-to-recovery ratios. The above-mentioned established scaling laws for velocity transformation and skin-friction coefficient, in combination with the well-established temperature–velocity (TV) relations (e.g. Duan & Martín 2011; Zhang *et al.* 2014), provide abundant theoretical foundations for modelling the mean quantities of compressible TBLs in the current study.

To predict the mean profiles in compressible TBLs, Huang *et al.* (1993) apply inverse vD transformation to a modelled incompressible velocity profile. Griffin *et al.* (2021*a*) compute the velocity and temperature profiles of compressible wall-bounded turbulence with inverse GFM transformation and TV relation (Zhang *et al.* 2014). Such a strategy is further developed for evaluating the mean profiles, wall shear stress and wall heat

flux for wall-modelled large-eddy simulations (Griffin, Fu & Moin 2023). Kumar & Larsson (2022) also evaluate the entire profile within the boundary layer based on the inverse Volpiani transformation. Despite these advancements, given that the current velocity transformations (e.g. vD, GFM, HLPP and Volpiani) are developed based on the physical properties in the inner layer, the determination of the entire velocity profile depending solely on one scaling law is somehow questionable. To address this, Hasan *et al.* (2024) separately model the inner and wake components with inverse HLPP and vD transformations, where the scaling factor for the wake component is modelled based on data fitting. However, the scaling factor expressed as a function of the momentum-thickness-based Reynolds number  $Re_\theta$  still results in non-negligible data scatter, indicating potential error sources in the predicted results. On the other hand, Chen *et al.* (2024) propose to predict the mean profiles in the inner and outer layers with inverse GFM and vD transformations, respectively, and splice them at a certain matching point between the two layers. While these approaches show promise, a more reliable framework is needed for appropriate scalings of the entire mean profiles, especially for that in the outer layer.

In our current study, a universal framework for predicting mean profiles of compressible TBLs is proposed based on established scaling laws regarding the velocity transformation, skin-friction coefficient and TV relation, by which the inner and wake components of the mean profiles are properly scaled. No additional data fitting operations are introduced in this framework, allowing it to be feasible to incorporate any velocity transformations with validity in the inner layer. The prediction framework is derived in § 2, with prediction results for the mean quantities presented in § 3. Concluding remarks are provided in § 4.

## 2. Methodology

The mean-profile-prediction framework to be derived includes the incompressible model for mean velocity, velocity transformation, general scaling law for skin-friction coefficient and TV relation, as elaborated in §§ 2.1, 2.2, 2.3 and 2.4, respectively. They are leveraged to iteratively compute the mean velocity and temperature profiles until the results converge, as summarised in § 2.5.

### 2.1. Modelling the mean profiles of the incompressible TBL

The mean velocity profile for the incompressible TBL can be expressed by (Coles 1956)

$$U^{inc}(y) = U_{inner}^{inc} \left( \frac{y}{\delta_v} \right) + \Pi U_{wake}^{inc} \left( \frac{y}{\delta_e} \right), \quad (2.1)$$

where  $y$  is the wall-normal distance,  $\delta_e$  is the boundary layer thickness where the mean streamwise velocity is 99 % of the free-stream velocity,  $\delta_v = \nu/u_\tau$  is the viscous length,  $\nu$  is the molecular kinetic viscosity,  $u_\tau = \sqrt{\tau_w/\rho_w}$  is the friction velocity,  $\tau_w = (\rho \partial_y U)_w$  is the wall shear stress,  $\rho$  is the mean density and the subscript  $w$  denotes the quantities at the wall. In this study, the velocities and lengths with superscripts  $+$  denote those normalised by  $u_\tau$  and  $\delta_v$ . For instance,  $U^{inc,+} = U^{inc}/u_\tau$  and  $y^+ = y/\delta_v$ . Assuming equilibrium, the inner component of the mean velocity profile can be expressed with (Kawai & Larsson 2012)

$$\frac{dU_{inner}^{inc}(y/\delta_v)}{dy} = \frac{u_\tau}{\delta_v} \frac{1}{1 + \mu_t/\mu}, \quad (2.2)$$

where  $\mu = \rho\nu$  is the molecular dynamic viscosity, and  $\mu_t$  is the eddy dynamic viscosity. The Johnson–King model (Johnson & King 1985) that is built upon the arguments of

the mixing length model (van Driest 1956) is adopted to describe the eddy dynamic viscosity, i.e.

$$\mu_t = \kappa \rho y \sqrt{\frac{\tau_w}{\rho}} \mathcal{D}, \quad \mathcal{D} = [1 - \exp(-y^+/A^+)]^2, \quad (2.3)$$

with  $\kappa = 0.41$  and  $A^+ = 17$ . On the other hand, Coles' wake function (Coles 1956) is adopted to describe  $U_{wake}^{inc}(y/\delta_e)$ , which is the normalised shape of the wake component of the mean velocity profile, as expressed by

$$\frac{dU_{wake}^{inc}(y/\delta_e)}{dy} = \frac{u_\tau}{\delta_e} \frac{\pi}{\kappa} \sin\left(\pi \frac{y}{\delta_e}\right). \quad (2.4)$$

According to Coles' law of the wake (Coles 1956), the wake scaling factor  $\Pi$  for a zero-pressure-gradient incompressible TBL can be approximately treated as a constant 0.55. In our proposed framework, Coles' law of the wake is further extended to the compressible TBL by determining  $\Pi$  based on the self-consistency criterion regarding the general scaling law for  $C_f$  (Zhao & Fu 2025), as illustrated in the next subsection.

The inner component  $U_{inner}^{inc}$  in (2.2) and normalised wake component  $U_{wake}^{inc}$  in (2.4) of the incompressible TBL introduced in this subsection will be utilised for constructing the compressible counterparts, as derived in the following.

## 2.2. Shaping the mean profiles of compressible TBLs with velocity transformations

In this study, the mean streamwise velocity profile of a compressible TBL is reconstructed with

$$U = U_{inner} + \Pi U_{wake}, \quad (2.5)$$

where  $U_{inner}$  and  $U_{wake}$  are inversely transformed from  $U_{inner}^{inc}$  and  $U_{wake}^{inc}$ , respectively. Given the validity of the established velocity transformations (e.g. GFM, HLPP and Volpiani) in the inner layer,  $U_{inner}^{inc}$  can be readily transformed to the compressible counterpart with these methods. With inverse GFM transformation applied, for example, the mean velocity in an incompressible TBL compared to that in a compressible TBL with the same  $Re_\tau^*$  is expressed by

$$\frac{dU_{inner}^+}{dy^*} = \frac{\frac{dU_{inner}^{inc,+}}{dy^*}}{\frac{1}{\mu^+} - \frac{1}{\mu^+} \frac{dU_{inner}^{inc,+}}{dy^*} + \mu^+ \frac{dU_{inner}^{inc,+}}{dy^+}}, \quad (2.6)$$

where  $y^* = y/\delta_v^*$ ,  $Re_\tau^* = \delta_e/\delta_v^*(y = \delta_e)$ ,  $\delta_v^*(y) = \nu(y)/u_\tau^*(y)$  is the semi-local length scale,  $u_\tau^*(y) = \sqrt{\tau_w/\rho(y)}$  is the semi-local velocity scale, and  $\mu^+ = \mu/\mu_w$ . The derivations of the inverse GFM transformation (2.6) are provided in Appendix A. In the rest of this paper, the inverse GFM transformation is the default method for inner scalings unless otherwise stated.

On the other hand, most of the currently available velocity transformations are fundamentally based on the physical characteristics of the inner layer, which results in a less robust foundation for outer scaling when compared to the well-established inner scaling. As a practical compromise without sacrificing generality, the inverse vD transformation that provides fair scaling of  $U_{wake}$  (Duan, Beekman & Martin 2011; Chen *et al.* 2024) is employed for outer scaling in the tests conducted in this study:

$$\frac{dU_{wake}^+}{dy^*} = \sqrt{\frac{1}{\rho^+}} \frac{dU_{wake}^{inc,+}}{dy^*}, \quad (2.7)$$

where  $\rho^+ = \rho/\rho_w$ . Note that the transformation for outer scaling can be replaced with more reliable methods developed in future studies.

### 2.3. Determining the wake scaling factor with the general scaling law for $C_f$

To determine the wake scaling factor  $\Pi$ , the general scaling law for  $C_f$  recently proposed by Zhao & Fu (2025) is introduced. In their theory, the redefined skin friction coefficient  $C_{f,i}$  closely matches the predicted value as a function of the redefined momentum-thickness-based Reynolds number  $Re_{\theta,i}$  in actual compressible TBLs within a fairly wide range of flow conditions. The redefined skin-friction coefficient is described with

$$\left(\frac{2}{C_{f,i}}\right)^{1/2} = \frac{1}{\kappa_f} \ln Re_{\theta,i} + C, \quad (2.8)$$

with  $\kappa_f = 0.344$  and  $C = 1.770$ . Here,  $C_{f,i}$  and  $Re_{\theta,i}$  are defined by

$$C_{f,i} = F_{C^*} C_f, \quad Re_{\theta,i} = F_{\theta^*} Re_{\theta^*}, \quad (2.9)$$

with  $F_{C^*} = (\rho_\infty/\rho_w)F^{-2}$ ,  $F_{\theta^*} = (\rho_w\mu_\infty/(\rho_\infty\mu_w))F$  and  $F = U_\infty^{inc,+}/U_\infty^+$ . Here, the subscript  $\infty$  denotes the free-stream quantities, and  $U_\infty^{inc,+}$  is obtained from the forward GFM transformation that is formulated by

$$\frac{dU_\infty^{inc,+}}{dy^*} = \frac{\frac{1}{\mu^+} \frac{dU^+}{dy^*}}{1 + \frac{1}{\mu^+} \frac{dU^+}{dy^*} - \mu^+ \frac{dU^+}{dy^+}}. \quad (2.10)$$

In (2.9),  $Re_{\theta^*}$  is expressed as  $Re_{\theta^*} = U_\infty \theta^* / \mu_\infty$ , with

$$\theta^* = \int_0^{\delta_e} \frac{\rho}{\rho_\infty} \frac{U_\infty^{inc,+}}{U_\infty^{inc,+}} \left(1 - \frac{U_\infty^{inc,+}}{U_\infty^{inc,+}}\right) d(y^* \delta_\nu). \quad (2.11)$$

Based on (2.8)–(2.11), the skin-friction coefficient can be predicted with

$$C_f = 2 \left/ \left[ F_{C^*} \left( \frac{1}{\kappa_f} \ln Re_{\theta,i} + C \right)^2 \right] \right. \quad (2.12)$$

Since the above theory accurately describes the skin-friction coefficient for TBLs within a vast range of flow conditions, it provides an important criterion to judge the self-consistency of the predicted results from the to-be-proposed framework. Hence we propose to determine the wake scaling factor  $\Pi$  such that the discrepancy between the predicted  $C_f$  directly from definition, i.e.  $C_f = \tau_w / (0.5\rho_\infty U_\infty^2) = 2/(\rho_\infty^+ U_\infty^{+2})$ , and that from (2.12) is lower than  $10^{-5}$ , which can be derived to be

$$|\epsilon| = \left| \rho_\infty^+ U_\infty^{+2} \left/ \left[ F_{C^*} \left( \frac{1}{\kappa_f} \ln Re_{\theta,i} + C \right)^2 \right] - 1 \right| \leq 10^{-5}. \quad (2.13)$$

An *a priori* test is conducted here by reconstructing the mean velocity profile with actual mean temperature from the direct numerical simulations (DNS) such that the performance of the proposed self-consistency criterion (2.13) in determining the wake scaling factor is scrutinised individually. The prediction error of  $U^+(y)$  is investigated in this test, which is defined by

$$\mathcal{E}(U^+) = \sqrt{\frac{1}{\delta_e} \int_0^{\delta_e} (U_P^+(\eta) - U_{DNS}^+(\eta))^2 d\eta} \left/ \left( \frac{1}{\delta_e} \int_0^{\delta_e} U_{DNS}^+(\eta) d\eta \right) \right., \quad (2.14)$$

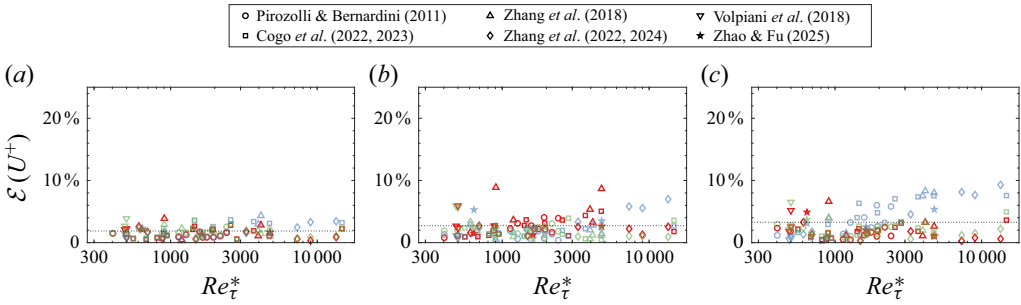


Figure 1. A *priori* test for prediction errors of the mean velocity profile in the results from (a) present framework, (b) empirical formulation in Hasan *et al.* (2024), and (c) direct inverse transformation as in Kumar & Larsson (2022) for inner-layer scaling. The black dotted lines denote the root mean square values of the prediction errors for all the considered cases. The red, blue and green symbols denote the results from GFM, HLPP and Volpiani.

where the subscripts *P* and *DNS* denote the predicted and DNS results, respectively. The prediction errors based on a total of 44 compressible TBLs from six DNS databases (Pirozzoli & Bernardini 2011; Volpiani, Bernardini & Larsson 2018; Zhang, Duan & Choudhari 2018; Cogo *et al.* 2022, 2023; Zhang *et al.* 2022, 2024; Zhao & Fu 2025) are summarised in figure 1. Detailed flow parameters of these 44 test cases are summarised in Appendix B. Note that all of these cases will be further used for comprehensive validations of the to-be-proposed framework in the following parts of this study. For comparison, the results from the empirical formulation of the compressible mean profile in Hasan *et al.* (2024) and direct inverse transformation of the incompressible counterparts as proposed by Kumar & Larsson (2022) are also included in figure 1. In all the tested cases, the prediction errors from the presented framework are lower than 5.0 % regardless of the specific choice of the velocity transformation method among GFM, HLPP and Volpiani for inner scaling with root mean square values equal to 1.9 %, which is much lower than those from the other two frameworks. The validity of the self-consistency criterion regarding the general scaling law of  $C_f$  in determining the wake scaling factor is thus demonstrated.

To further investigate how the wake scaling factor  $\Pi$  varies with increasing free-stream Mach number  $M_\infty$ , the results of  $\Pi$  determined from the criterion (2.13) with actual mean temperature from DNS are summarised in figure 2. For subsonic flow with  $M_\infty = 0.5$  from Zhang *et al.* (2022), the value of  $\Pi$  is 0.48, which is close to the recommended value 0.55 for incompressible flows (Coles 1956). As  $M_\infty$  increases,  $\Pi$  tends to decrease. In particular, for all hypersonic cases with  $M_\infty \geq 5.0$ ,  $\Pi$  falls below 0.35. This dependence of the wake scaling factor  $\Pi$  on the free-stream Mach number underscores the importance of accounting for compressibility effects when modelling mean profiles in high-speed wall-bounded turbulent flows.

#### 2.4. Determining mean temperature with TV relation

The mean temperature profile can be evaluated from the established TV relationships. In the following test cases in this study, the TV relation established by Duan & Martín (2011) is adopted to determine the mean temperature profile from the velocity, which is formulated as

$$\frac{T}{T_e} = \frac{T_w}{T_e} + \frac{T_r - T_w}{T_e} \left[ (1 - sPr) \left( \frac{U}{U_e} \right)^2 + sPr \left( \frac{U}{U_e} \right) \right] + \frac{T_e - T_r}{T_e} \left( \frac{U}{U_e} \right)^2, \quad (2.15)$$

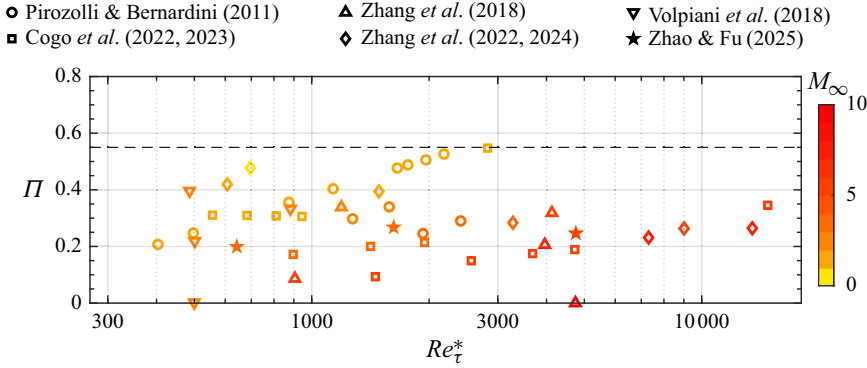


Figure 2. Values of  $\Pi$  for compressible TBLs with GFM for inner-layer scaling. The black dashed line denotes  $\Pi = 0.55$  as suggested by Coles (1956) for incompressible TBLs.

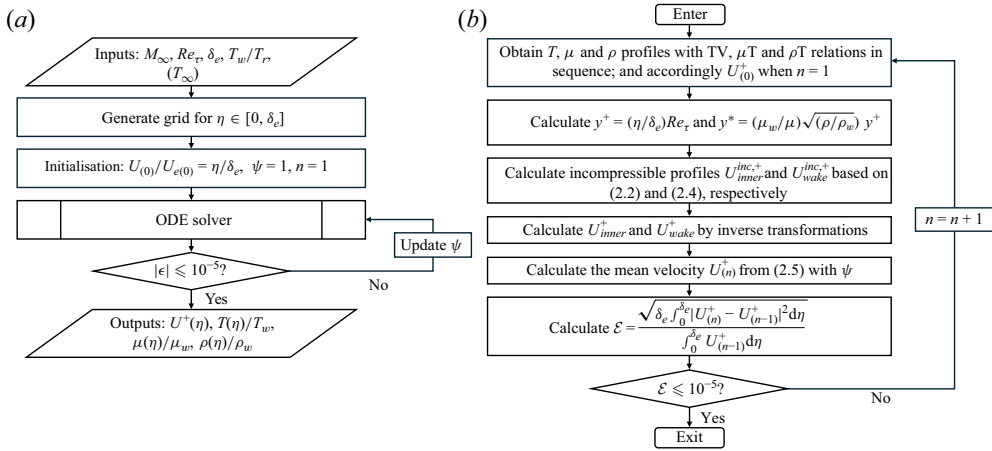


Figure 3. Program chart of the prediction framework: (a) main program, (b) ODE solver.

with the optimal value of  $sPr$  equal to 0.8 (Zhang *et al.* 2014). Here, the subscript  $e$  denotes the quantities at  $y = \delta_e$ , and  $T_r$  is the recovery temperature. Also,  $T_e$  is obtained from the adiabatic wall temperature relation

$$T_r = T_e \left[ 1 + r \frac{\gamma - 1}{2} Ma_e^2 \right], \quad (2.16)$$

with specific heat ratio  $\gamma = 1.4$ , recovery factor  $\gamma = 0.9$  and boundary-layer-edge Mach number  $Ma_e = 0.99 Ma_\infty$ .

### 2.5. Summary of the prediction framework

The prediction framework is summarised in figure 3. The inputs are free-stream Mach number  $M_\infty$ , friction Reynolds number  $Re_\tau = \delta_e/\delta_v$ ,  $\delta_e$  and  $T_w/T_r$ , with optionally  $T_\infty$  when the Sutherland's law is applied for the viscosity–temperature ( $\mu T$ ) relation. To numerically solve  $\Pi$  that satisfies the iteration tolerance ( $|\epsilon| \leq 10^{-5}$ ), the Newton–Raphson method is applied. The outputs include wall-normal profiles of  $U^+$ ,  $T/T_w$ ,  $\mu/\mu_w$  and  $\rho/\rho_w$ . Here, the convergence threshold  $10^{-5}$  is demonstrated to provide converged results, as discussed in Appendix B in detail. The number of iterations required to

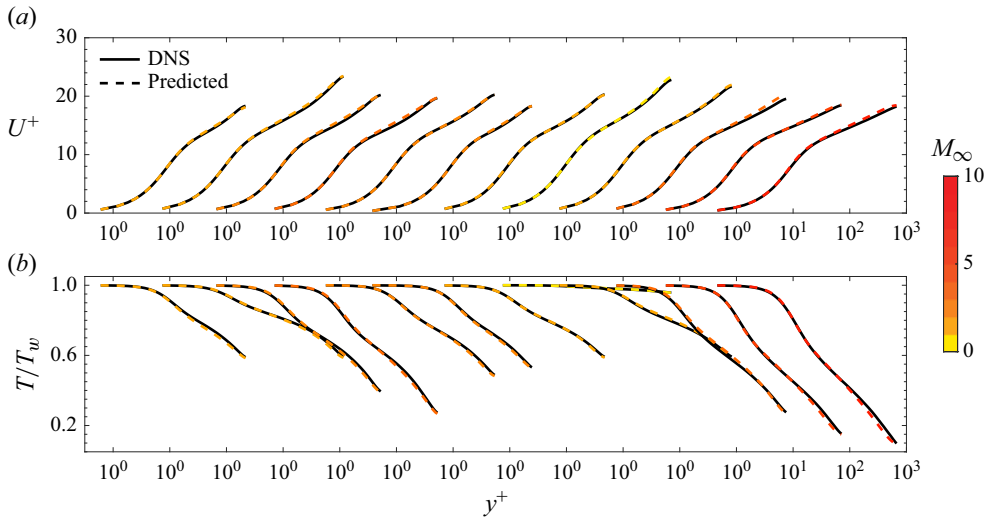


Figure 4. Predicted and DNS results for (a) the mean streamwise velocity, and (b) the temperature, for adiabatic boundary layers. The presented results include, from left to right:  $M_\infty = 2.0$  with  $Re_\tau = 204$ ,  $M_\infty = 2.0$  with  $Re_\tau = 1106$ ,  $M_\infty = 3.0$  with  $Re_\tau = 502$ ,  $M_\infty = 4.0$  with  $Re_\tau = 501$  (Pirozzoli & Bernardini 2011);  $M_\infty = 2.5$  with  $Re_\tau = 505$  (Zhang *et al.* 2018);  $M_\infty = 2.28$  with  $Re_\tau = 224$  (Volpiani *et al.* 2018);  $M_\infty = 2.0$  with  $Re_\tau = 444$  (Cogo *et al.* 2022, 2023);  $M_\infty = 0.5$  with  $Re_\tau = 660$ ,  $M_\infty = 2.0$  with  $Re_\tau = 701$ ,  $M_\infty = 4.0$  with  $Re_\tau = 709$ ,  $M_\infty = 6.0$  with  $Re_\tau = 667$ ,  $M_\infty = 8.0$  with  $Re_\tau = 626$  (Zhang *et al.* 2022, 2024). The colours (yellow to red) denote  $M_\infty$  (low to high).

reach convergence, and the corresponding computational costs, are also summarised in [Appendix B](#), where the computational robustness and efficiency of the algorithm are demonstrated. Typically, convergence is achieved within 6 iterations and 2.5 s even when the convergence threshold is selected as  $10^{-10}$ , as measured on a desktop computer with an Intel Core i7–8700 CPU @ 3.20 GHz and 32 GB of RAM, running MATLAB in single-threaded mode on Windows 10.

In addition to the algorithm that uses the friction Reynolds number  $Re_\tau$  as input, we also provide an alternative formulation based on the momentum-thickness Reynolds number  $Re_\theta$ , as described in [Appendix C](#), where the results indicate that the error distributions are similar under both input conditions regarding the types of Reynolds numbers.

### 3. Results

The predicted results for the mean profiles in adiabatic and diabatic TBLs are depicted in [figures 4](#) and [5](#), respectively. The presented results include the TBLs with lowest and highest values of  $M_\infty$ ,  $Re_\tau$  and  $T_w/T_r$ , respectively, in each of the six DNS databases. It is found that the predicted mean profiles for velocity and temperature both match well with the DNS results for all the depicted cases. To quantify the overall discrepancy of the velocity profile compared to the actual result, the prediction error  $\mathcal{E}(U^+)$  is defined in (2.14), and  $\mathcal{E}(T/T_w)$  is defined in the same way but replacing  $U^+$  with  $T/T_w$ . As in [figure 6\(a,b\)](#), the prediction errors for mean velocity and temperature are lower than 4.0 % and 6.0 % in all the tested cases. Here, the higher prediction error in mean temperature should stem from two aspects. First, the wake scaling factor  $\Pi$  is determined according to the self-consistency of the skin-friction coefficient calculated from the velocity profiles. Thus the framework is intrinsically more reliable for the prediction of velocities than that of temperature. Second, when determining the mean temperature profiles from the

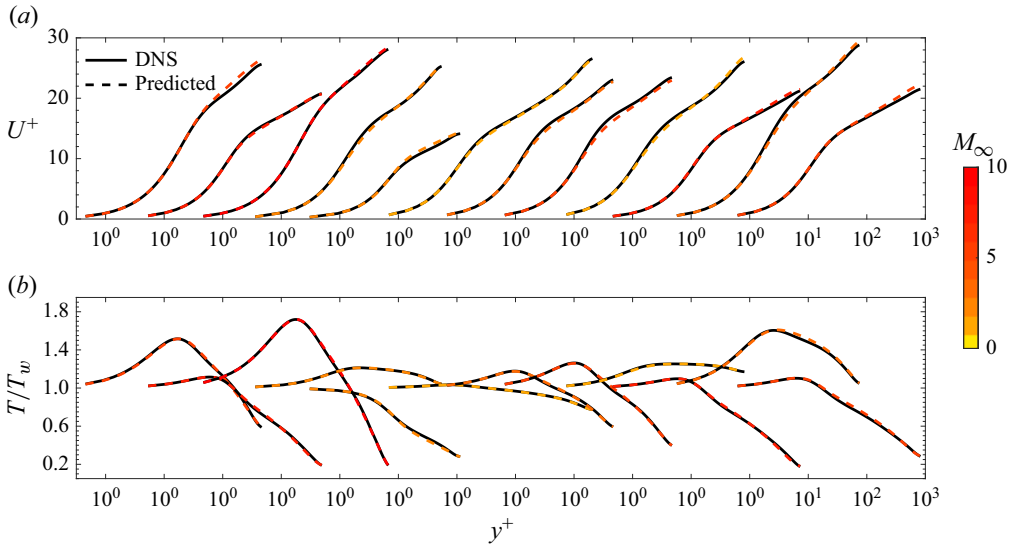


Figure 5. Predicted and DNS results for (a) the mean streamwise velocity, and (b) the temperature, for diabatic boundary layers. The presented results include, from left to right:  $M_\infty = 5.84$  with  $T_w/T_r = 0.25$  and  $Re_\tau = 436$ ,  $M_\infty = 7.87$  with  $T_w/T_r = 0.48$  and  $Re_\tau = 467$ ,  $M_\infty = 13.64$  with  $T_w/T_r = 0.18$  and  $Re_\tau = 634$  (Zhang *et al.* 2018);  $M_\infty = 2.28$  with  $T_w/T_r = 0.5$  and  $Re_\tau = 512$ ,  $M_\infty = 2.28$  with  $T_w/T_r = 1.9$  and  $Re_\tau = 100$  (Volpiani *et al.* 2018);  $M_\infty = 2.0$  with  $T_w/T_r = 0.76$  and  $Re_\tau = 1947$ ,  $M_\infty = 4.0$  with  $T_w/T_r = 0.44$  and  $Re_\tau = 444$ ,  $M_\infty = 6.0$  with  $T_w/T_r = 0.35$  and  $Re_\tau = 444$  (Cogo *et al.* 2022, 2023);  $M_\infty = 2.0$  with  $T_w/T_r = 0.5$  and  $Re_\tau = 757$ ,  $M_\infty = 8.0$  with  $T_w/T_r = 0.5$  and  $Re_\tau = 683$  (Zhang *et al.* 2022, 2024);  $M_\infty = 4.0$  with  $T_w/T_r = 0.25$  and  $Re_\tau = 706$ ,  $M_\infty = 6.0$  with  $T_w/T_r = 0.5$  and  $Re_\tau = 779$  (Zhao & Fu 2025).

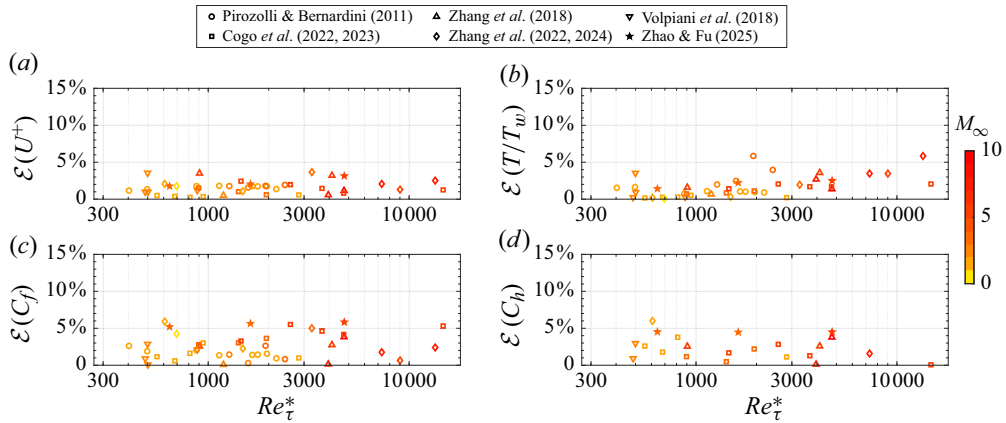


Figure 6. Prediction errors of (a) the mean velocity profile  $U^+$ , (b) the mean temperature profile  $T/T_w$ , and (c) the skin-friction coefficient  $C_f$ , for all the cases; and (d) wall-heat-transfer coefficient  $C_h$  for diabatic cases.

velocity, extra errors are introduced from the TV relation. This highlights the importance of the TV relation for the prediction accuracy of the present framework. In Appendix D, the sources of the prediction errors are further analysed by examining the performance of the underlying scaling laws in the prediction framework under different values of  $M_\infty$  and  $T_w/T_r$ . Further, the skin-friction coefficient  $C_f$  and wall-heat-transfer coefficient  $C_h$  that quantify the wall shear and heat transfer, respectively, are investigated. Here,  $C_h$  is defined by  $C_h = q_w / (c_p \rho_\infty U_\infty (T_w - T_r))$ , where  $q_w = -(c_p / Pr) \mu_w (\partial T / \partial y)_w$ , with

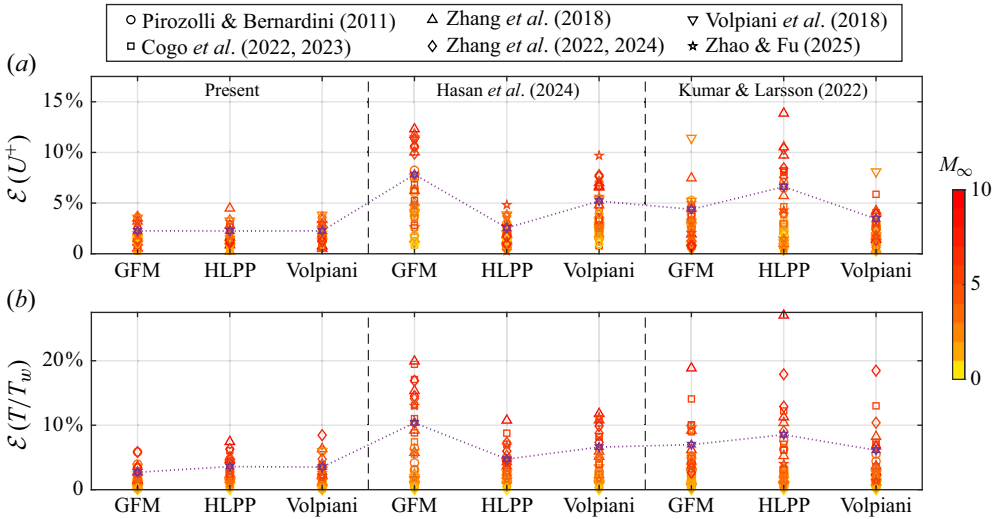


Figure 7. Prediction errors in (a) mean streamwise velocity and (b) temperature. The purple hexagons linked with dotted lines denote the root mean square prediction errors all across the considered TBL cases for each combination of mean-profile-prediction framework and velocity-transformation method.

$Pr$  the Prandtl number, and  $c_p$  the isobaric specific heat. The relative errors of  $C_f$  and  $C_h$  that are defined as  $\mathcal{E}(C_f) = |(C_{f,P}/C_{f,DNS}) - 1|$  and  $\mathcal{E}(C_h) = |(C_{h,P}/C_{h,DNS}) - 1|$  are summarised in figure 6(c,d), respectively, where figure 6(d) includes only diabatic cases with non-zero wall heat transfer. It is found that  $\mathcal{E}(C_f)$  and  $\mathcal{E}(C_h)$  keep relatively low values all across the considered cases, with root mean square values equal to 3.1 % and 2.9 %, respectively.

Finally, to test the universality of the proposed framework, HLPP (Hasan *et al.* 2023) and Volpiani (Volpiani *et al.* 2020), which also perform well in the inner layer, are incorporated into such a framework to replace GFM for inner-layer scaling. The corresponding results for  $\mathcal{E}(U^+)$  and  $\mathcal{E}(T/T_w)$  are depicted in figure 7. The root mean square prediction errors of all three prediction frameworks incorporating three inner-layer scaling laws are summarised in table 1. Upon applying any of the three considered velocity transformations, the prediction errors of the present framework for mean velocity and temperature are below 5.0 % and 10.0 %, respectively, with average values less than 2.3 % and 3.6 %. For comparison, the GFM, HLPP and Volpiani transformations are further incorporated in the frameworks proposed by Kumar & Larsson (2022) and Hasan *et al.* (2024). From figure 7, significant increases in the prediction errors are observed in the results from these two frameworks. Since the scaling factors for the wake region in the framework of Hasan *et al.* (2024) are fitted based on HLPP, this framework performs significantly better when incorporating HLPP than when incorporating GFM or Volpiani. However, even when using HLPP for inner-layer scaling, the prediction errors for mean velocity and temperature profiles with Hasan *et al.* (2024) are still 13.52 % and 32.42 % higher, respectively, than those obtained with the present framework incorporating HLPP, as indicated by table 1. Hence the validity and universality of the proposed framework are demonstrated.

In addition to the demonstrated validity and universality, it is also worth highlighting another important aspect implied by universality, i.e. future extensibility. The accuracy of the mean profiles predicted by the new framework is fundamentally determined by the four scaling laws mentioned above. As elaborated in Appendix D, the primary sources of

|                            |     | GFM                 | HLPP               | Volpiani           |
|----------------------------|-----|---------------------|--------------------|--------------------|
| Present                    | $U$ | 2.26 %              | 2.25 %             | 2.25 %             |
|                            | $T$ | 2.69 %              | 3.58 %             | 3.51 %             |
| Hasan <i>et al.</i> (2024) | $U$ | 7.83 % (+246.00 %)  | 2.55 % (+13.52 %)  | 5.20 % (+130.74 %) |
|                            | $T$ | 10.36 % (+284.67 %) | 4.75 % (+32.42 %)  | 6.60 % (+88.30 %)  |
| Kumar & Larsson (2022)     | $U$ | 4.38 % (+93.72 %)   | 6.62 % (+194.33 %) | 3.44 % (+52.52 %)  |
|                            | $T$ | 6.98 % (+159.27 %)  | 8.53 % (+138.06 %) | 6.12 % (+74.53 %)  |

Table 1. Comparisons of the root mean square prediction errors of the newly proposed framework and those proposed by Kumar & Larsson (2022) and Hasan *et al.* (2024). The percentages outside parentheses are the root mean square prediction errors, while those inside parentheses are the increasing ratios of the prediction errors compared to those from the present framework.

the prediction error are attributed to the TV relation and the scaling law for  $C_f$ . With the development of more advanced scaling laws in the future, the prediction accuracy of the present framework is expected to be further improved.

#### 4. Concluding remarks

In this study, a universal prediction framework for mean profiles in compressible TBL is proposed, leveraging the established scaling laws regarding velocity transformation,  $C_f$  and TV relations. The basic flow properties of Reynolds number, boundary layer thickness, free-stream Mach number and wall-to-recovery ratio are the inputs of such a framework. In the coupled solving procedure for the mean profile, the scalings of flow quantities in the inner layer are reliably described by the velocity transformation (e.g. Volpiani *et al.* 2020; Griffin *et al.* 2021*b*; Hasan *et al.* 2023), while the mean quantities in the wake region are well determined based on the self-consistency criterion regarding the general scaling law for  $C_f$  (Zhao & Fu 2025). The temperature, on the other hand, is obtained from the velocity with the TV relation (Duan & Martín 2011; Zhang *et al.* 2014). The scaling laws in these three aspects are leveraged in the prediction framework to iteratively refine the mean profiles until the results converge. The prediction framework is applied in compressible TBLs with a fairly wide range of flow conditions, demonstrating 11.9 % to 74.0 % lower root mean square prediction errors in the velocity and temperature profiles compared to existing mean-profile-prediction frameworks. Especially for all three velocity transformation methods used for inner-layer scaling, the root mean square prediction errors in the mean velocity and temperature profiles remain below 2.3 % and 3.6 %, respectively. Such robust validity of the present framework with different velocity transformations highlights its universality among the established and future scaling laws.

The four scaling laws underlying the proposed framework – i.e. inner-layer scaling, outer-layer scaling, the general scaling law for skin-friction coefficient and TV relation – are well established only for canonical equilibrium wall-bounded turbulence in the current stage. Moreover, as comprehensively tested in Bai, Griffin & Fu (2022), the existing velocity transformations do not deliver satisfying performance for non-canonical TBLs under high-enthalpy or supercritical conditions. Such facts indicate that the framework incorporating the currently established scaling laws is applicable only to canonical zero-pressure-gradient TBLs. On the other hand, the application of the framework in non-equilibrium flows would require the future development the scaling laws that are valid under the non-canonical conditions. It should be noted that our proposed framework is built upon the universal self-consistency criterion regarding the skin-friction coefficient

rather than tied to any specific scaling, which is thus inherently general. Therefore, with appropriate scaling laws for non-canonical TBLs developed in the future, the proposed framework is expected to be naturally extended to such cases.

The proposed prediction framework is of importance in fundamental research and engineering applications. For high-speed or high-Reynolds-number flows whose mean properties are expensive to obtain from experiments or DNS, the proposed prediction framework provides reliable predictions of their mean properties that are essential for evaluating the skin friction and heat transfer. Besides, the predicted mean profiles can also be applied for initialisation of the flow field and providing inflow conditions for the numerical simulations of a compressible TBL, which is expected to shorten the recovery distance and thus reduce the computational cost.

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**Data availability statement.** The data that support the findings of this study are available on request from the corresponding author, L.F.

**Declaration of interests.** The authors report no conflict of interest.

## Appendix A. Inverse GFM transformation

The forward GFM transformation (Griffin *et al.* 2021*b*) is expressed as

$$\frac{dU^{inc,+}}{dy^*} = \frac{\frac{1}{\mu^+} \frac{dU^+}{dy^*}}{1 + \frac{1}{\mu^+} \frac{dU^+}{dy^*} - \mu^+ \frac{dU^+}{dy^+}}, \quad (A1)$$

which can be rearranged to

$$\begin{aligned} \frac{1}{\mu^+} \frac{dU^+}{dy^*} &= \left( 1 + \frac{1}{\mu^+} \frac{dU^+}{dy^*} - \mu^+ \frac{dU^+}{dy^+} \right) \frac{dU^{inc,+}}{dy^*} \\ &= \frac{dU^{inc,+}}{dy^*} + \frac{1}{\mu^+} \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^*} - \mu^+ \frac{dU^+}{dy^+} \frac{dU^{inc,+}}{dy^*}. \end{aligned} \quad (A2)$$

According to the chain rule, (A2) is further derived to be

$$\begin{aligned} \frac{1}{\mu^+} \frac{dU^+}{dy^*} &= \frac{dU^{inc,+}}{dy^*} + \frac{1}{\mu^+} \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^*} - \mu^+ \frac{dU^+}{dy^+} \left( \frac{dy^+}{dy^*} \frac{dy^*}{dy^+} \right) \frac{dU^{inc,+}}{dy^*} \\ &= \frac{dU^{inc,+}}{dy^*} + \frac{1}{\mu^+} \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^*} - \mu^+ \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^+}. \end{aligned} \quad (A3)$$

Equation (A3) is further rearranged to

$$\begin{aligned}
 & \frac{1}{\mu^+} \frac{dU^+}{dy^*} - \frac{1}{\mu^+} \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^*} + \mu^+ \frac{dU^+}{dy^*} \frac{dU^{inc,+}}{dy^+} = \frac{dU^{inc,+}}{dy^*} \\
 \Rightarrow & \frac{dU^+}{dy^*} \left( \frac{1}{\mu^+} - \frac{1}{\mu^+} \frac{dU^{inc,+}}{dy^*} + \mu^+ \frac{dU^{inc,+}}{dy^+} \right) = \frac{dU^{inc,+}}{dy^*} \\
 \Rightarrow & \frac{dU^+}{dy^*} = \frac{\frac{dU^{inc,+}}{dy^*}}{\frac{1}{\mu^+} - \frac{1}{\mu^+} \frac{dU^{inc,+}}{dy^*} + \mu^+ \frac{dU^{inc,+}}{dy^+}}. \quad (A4)
 \end{aligned}$$

The inverse GFM transformation is thereby obtained in (A4), by which the incompressible velocity profile is transformed to its compressible counterpart with given temperature and viscosity profiles.

## Appendix B. Flow parameters and convergence test of the framework

The flow parameters of the 44 cases of compressible TBLs from six DNS databases (Pirozzoli & Bernardini 2011; Zhang *et al.* 2018, 2022, 2024; Volpiani *et al.* 2018; Cogo *et al.* 2022, 2023; Zhao & Fu 2025) are summarised in table 2, including information for free-stream Mach number  $Ma_\infty$  (0.5–13.64), wall-to-recovery ratio  $T_w/T_r$  (0.25–1.9), friction Reynolds number  $Re_\tau$  (100–1947), boundary-layer-thickness-based Reynolds number  $Re_{\delta_e}$  (10 216–1 343 863) and momentum-thickness-based Reynolds number  $Re_\theta$  (877–41 172).

Meanwhile, the number of iterations and execution time required for our proposed framework, incorporating GFM for inner-layer scaling, to reach the convergence thresholds  $10^{-5}$  and  $10^{-10}$  are also summarised in table 2. Here, the numbers of iterations refer to those of the outer loop sketched in figure 3(a), which updates the wake scaling factor  $\Pi$  at each step. In all the cases, the number of iterations is no more than 5 and 6 for convergence thresholds  $10^{-5}$  and  $10^{-10}$ , respectively, which indicates that the values of  $\Pi$  quickly converge during the iterations, and thus demonstrates the robustness of the algorithm. On the other hand, the total execution time is less than 2.5 seconds, with each iteration taking less than 0.4 seconds, as measured on a desktop computer with an Intel Core i7–8700 CPU @ 3.20 GHz and 32 GB of RAM, running MATLAB in single-threaded mode on Windows 10. Such computational costs are negligible compared to those of the DNS, which demonstrates the robustness and efficiency of the present framework.

To further test the impact of the convergence threshold on the prediction results, those with  $10^{-10}$  for both the outer loop and the ODE solver are compared with those with  $10^{-5}$  as adopted in the main text. To quantify their difference,  $\mathcal{D}(U^+)$  is defined here with

$$\mathcal{D}(U^+) = \sqrt{\frac{1}{\delta_e} \int_0^{\delta_e} \left( U_{(10^{-10})}^+(\eta) - U_{(10^{-5})}^+(\eta) \right)^2 d\eta} \bigg/ \left( \frac{1}{\delta_e} \int_0^{\delta_e} U_{(10^{-10})}^+(\eta) d\eta \right), \quad (B1)$$

where  $U_{(10^{-5})}^+$  and  $U_{(10^{-10})}^+$  are the predictions with thresholds  $10^{-5}$  and  $10^{-10}$ , respectively. We define  $\mathcal{D}(T/T_w)$  in the same way as  $\mathcal{D}(U^+)$  by replacing  $U^+$  with  $T/T_w$  in (B1). The values of  $\mathcal{D}(U^+)$  for all the tested cases are summarised in figure 8. For all the cases, the differences in the predicted results with different convergence thresholds are lower than  $10^{-5}$ , i.e. 0.001 %, which is negligible for the results. Thus the threshold  $10^{-5}$  is considered to yield converged results.

| Reference                        | $Ma_\infty$ | $T_w/T_r$ | $Re_\tau$ | $Re_{\delta_e}$ | $Re_\theta$ | Number of iterations | Execution time (s) |
|----------------------------------|-------------|-----------|-----------|-----------------|-------------|----------------------|--------------------|
| Pirozzoli & Bernardini (2011)    | 2.0         | 1.0       | 204       | 10 216          | 877         | 4 (5)                | 0.4040 (0.6746)    |
|                                  | 2.0         | 1.0       | 251       | 13 012          | 1131        | 4 (5)                | 0.3933 (0.5937)    |
|                                  | 2.0         | 1.0       | 445       | 24 792          | 2090        | 3 (5)                | 0.3838 (0.6538)    |
|                                  | 2.0         | 1.0       | 580       | 33 702          | 2890        | 4 (5)                | 0.4754 (0.6849)    |
|                                  | 2.0         | 1.0       | 838       | 51 312          | 4437        | 4 (5)                | 0.6417 (0.9278)    |
|                                  | 2.0         | 1.0       | 893       | 55 170          | 4760        | 4 (5)                | 0.6813 (0.9061)    |
|                                  | 2.0         | 1.0       | 992       | 62 125          | 5347        | 3 (5)                | 0.5378 (0.8822)    |
|                                  | 2.0         | 1.0       | 1106      | 70 513          | 6045        | 3 (5)                | 0.5435 (0.9103)    |
|                                  | 3.0         | 1.0       | 403       | 44 654          | 3013        | 4 (5)                | 0.4715 (0.7186)    |
|                                  | 3.0         | 1.0       | 502       | 57 893          | 3955        | 4 (5)                | 0.5107 (0.7672)    |
|                                  | 4.0         | 1.0       | 395       | 83 623          | 4713        | 4 (5)                | 0.5308 (0.8153)    |
| Zhang <i>et al.</i> (2018)       | 4.0         | 1.0       | 501       | 107 715         | 5943        | 4 (5)                | 0.5261 (0.8345)    |
|                                  | 2.50        | 1.0       | 510       | 36 944          | 2835        | 4 (4)                | 0.7286 (0.8225)    |
|                                  | 5.84        | 0.25      | 450       | 37 385          | 2121        | 5 (6)                | 1.1732 (1.2064)    |
|                                  | 5.86        | 0.76      | 453       | 242 029         | 9455        | 5 (6)                | 1.0748 (1.1391)    |
|                                  | 7.87        | 0.48      | 480       | 287 338         | 9714        | 5 (6)                | 1.1841 (1.2883)    |
| Volpiani <i>et al.</i> (2018)    | 13.64       | 0.18      | 646       | 701 454         | 14 408      | 5 (6)                | 1.9773 (2.2372)    |
|                                  | 2.28        | 0.5       | 512       | 11 997          | 1251        | 3 (5)                | 0.5665 (1.0361)    |
|                                  | 2.28        | 1.0       | 395       | 24 328          | 1989        | 4 (5)                | 0.6641 (0.9320)    |
|                                  | 2.28        | 1.0       | 224       | 12 628          | 1047        | 4 (5)                | 0.5539 (0.7967)    |
|                                  | 2.28        | 1.9       | 100       | 13 440          | 868         | 4 (6)                | 0.5160 (0.9741)    |
| Cogo <i>et al.</i> (2023)        | 2.0         | 0.76      | 1947      | 87 859          | 7954        | 4 (4)                | 1.2768 (1.4397)    |
|                                  | 2.0         | 0.69      | 444       | 13 399          | 1242        | 4 (5)                | 0.5288 (0.7561)    |
|                                  | 2.0         | 0.79      | 443       | 16 609          | 1486        | 4 (5)                | 0.5260 (0.7299)    |
|                                  | 2.0         | 0.90      | 443       | 19 971          | 1714        | 4 (5)                | 0.5357 (0.7433)    |
|                                  | 2.0         | 1.0       | 444       | 23 633          | 1981        | 4 (4)                | 0.5261 (0.6423)    |
|                                  | 4.0         | 0.44      | 444       | 27 001          | 1848        | 4 (5)                | 0.5780 (0.9278)    |
|                                  | 4.0         | 0.63      | 444       | 44 215          | 2749        | 4 (5)                | 0.5692 (0.8372)    |
|                                  | 4.0         | 0.81      | 444       | 63 551          | 3657        | 4 (5)                | 0.6367 (0.8536)    |
|                                  | 5.86        | 0.76      | 1947      | 996 276         | 41 172      | 4 (5)                | 0.1248 (1.7653)    |
|                                  | 6.0         | 0.35      | 444       | 56 610          | 2813        | 4 (5)                | 0.6430 (0.9729)    |
|                                  | 6.0         | 0.57      | 444       | 108 382         | 4841        | 5 (6)                | 0.7596 (1.1140)    |
|                                  | 6.0         | 0.78      | 444       | 166 073         | 6643        | 5 (6)                | 0.8024 (1.1388)    |
|                                  | 6.0         | 1.0       | 444       | 228 481         | 8401        | 5 (6)                | 0.7873 (1.1601)    |
| Zhang <i>et al.</i> (2022, 2024) | 0.5         | 1.0       | 660       | 16 335          | 1847        | 3 (5)                | 0.5619 (0.7109)    |
|                                  | 2.0         | 1.0       | 771       | 43 744          | 3790        | 4 (5)                | 0.7328 (0.9290)    |
|                                  | 2.0         | 0.5       | 757       | 14 865          | 1591        | 3 (5)                | 0.5727 (0.7762)    |
|                                  | 4.0         | 1.0       | 709       | 139 990         | 7653        | 4 (6)                | 0.8959 (0.1389)    |
|                                  | 6.0         | 1.0       | 667       | 592 097         | 21 797      | 5 (6)                | 1.0296 (1.3740)    |
|                                  | 8.0         | 1.0       | 626       | 1 343 863       | 35 628      | 5 (6)                | 1.1503 (1.8410)    |
|                                  | 8.0         | 0.5       | 683       | 524 923         | 17 858      | 5 (6)                | 1.1314 (1.4859)    |
| Zhao & Fu (2025)                 | 4.0         | 0.5       | 751       | 55 803          | 3716        | 4 (5)                | 0.8247 (0.9566)    |
|                                  | 4.0         | 0.25      | 706       | 19 231          | 1534        | 4 (5)                | 0.8241 (0.9387)    |
|                                  | 6.0         | 0.5       | 779       | 235 936         | 10 630      | 5 (6)                | 1.1367 (1.2571)    |

Table 2. Summary of flow parameters for the compressible TBLs in fully developed turbulent regions used in this study. The rightmost two columns show the number of iterations and execution times required for our framework with GFM-based inner-layer transformation to reach the convergence thresholds. Values outside parentheses correspond to convergence thresholds  $10^{-5}$  for both the outer loop and ODE solver; values inside parentheses correspond to thresholds  $10^{-10}$ .

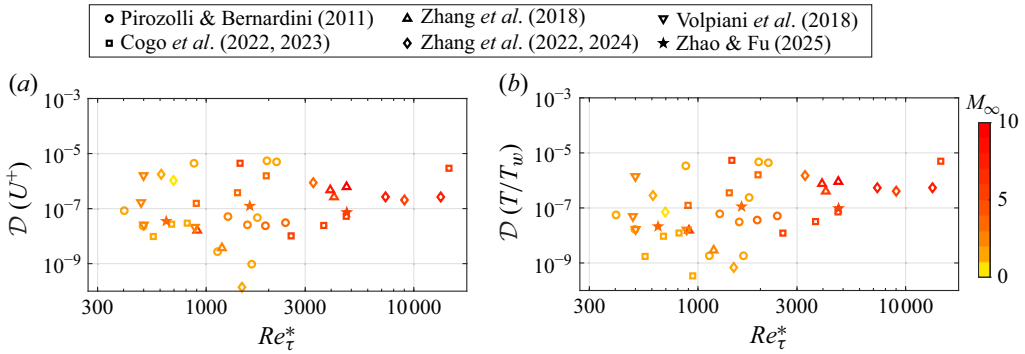


Figure 8. Differences of the prediction results with convergence thresholds  $10^{-5}$  and  $10^{-10}$  for (a) mean streamwise velocity and (b) mean temperature.

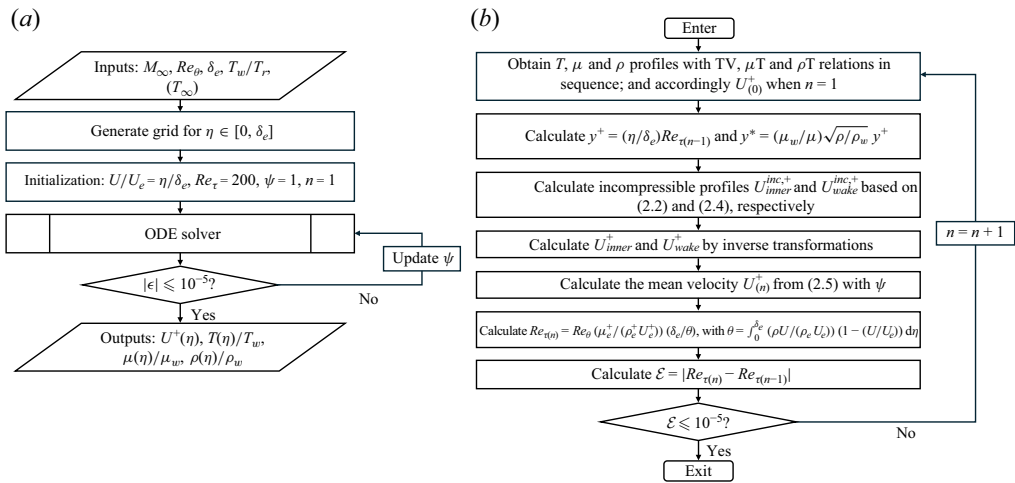


Figure 9. Program chart of the prediction framework with  $Re_\theta$  as input: (a) main program, (b) ODE solver.

## Appendix C. Prediction framework with the momentum-thickness-based Reynolds number as input

In the main text, the friction Reynolds number  $Re_\tau$  is utilised as an input of the prediction framework, which is summarised in figure 3. In this appendix, the alternative algorithm based on the momentum-thickness-based Reynolds number  $Re_\theta$  is presented, as in figure 9. Such an algorithm differs slightly from that with  $Re_\tau$  as input. Here, the unknown  $Re_\tau$  is updated at each step in the ODE solver until the results converge, while the core idea of the proposed framework, i.e. self-consistency regarding the skin-friction coefficient, is retained. Comparisons between the prediction results with  $Re_\tau$  and  $Re_\theta$  as inputs are depicted in figure 10. Although minor differences exist, the results show that the error distributions are similar under both input conditions. The consistency between the results with  $Re_\theta$  and  $Re_\tau$  as inputs further demonstrates the universality of the proposed framework.

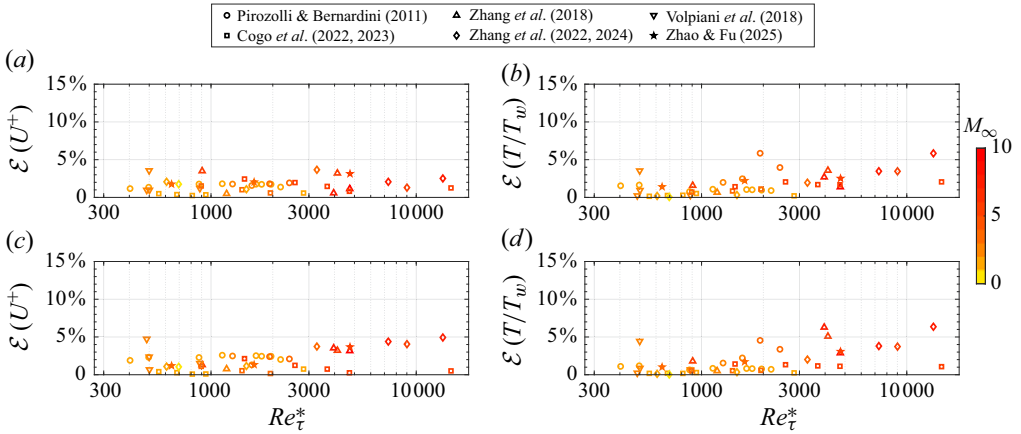


Figure 10. Prediction errors of (a,c) the mean velocity profile  $U^+$  and (b,d) the mean temperature profile  $T/T_w$  that are obtained from inputs (a,b)  $Re_\tau$  and (c,d)  $Re_\theta$ .

#### Appendix D. Discussions on the sources of prediction error

The prediction error in our framework comes from the four underlying scaling laws, i.e. the inner-layer velocity scaling, the outer-layer velocity scaling, the TV relation and the general scaling law for the skin-friction coefficient ( $C_f$ ). In Griffin *et al.* (2021b), it is demonstrated that the GFM transformation performs well for inner-layer velocity scaling with a wide range of flow conditions for canonical wall-bounded turbulence. Thus it is not considered to be the main source of prediction error in the canonical TBL cases tested in this study. On the other hand, while the outer-layer scaling shapes the wake region, its magnitude is determined by the scaling law for  $C_f$ . Thus the main sources of error are considered to be the TV relation and the scaling law for  $C_f$ . To clarify the contributions of each to the overall prediction error, we perform *a priori* analyses for each of these two types of scaling laws.

To quantify the prediction error of the TV relation (Duan & Martín 2011; Zhang *et al.* 2012), the predicted mean temperature profiles  $T^+ = T/T_w$  based on the actual mean velocity profiles from DNS with

$$\frac{T}{T_e} = \frac{T_w}{T_e} + \frac{T_r - T_w}{T_e} \left[ (1 - sPr) \left( \frac{U}{U_e} \right)^2 + sPr \left( \frac{U}{U_e} \right) \right] + \frac{T_e - T_r}{T_e} \left( \frac{U}{U_e} \right)^2 \quad (\text{D1})$$

are compared with the mean temperature directly from the DNS, where the prediction error  $\mathcal{E}(T/T_w)$  is defined in the same way as in (2.14), identical to that used in the main text. To investigate the impacts of  $M_\infty$  and  $T_r/T_w$  on the performance of the established TV relation, the values of  $\mathcal{E}(T/T_w)$  are depicted in figures 11(a) and 12(a) versus  $Re_\tau^*$ , where the symbols are coloured based on  $M_\infty$  and  $T_r/T_w$ , respectively. It is found that the prediction error of  $T/T_w$  notably increases as  $Re_\tau^*$  increases. On the other hand, the increases of  $M_\infty$  and  $T/T_w$  also appear to enlarge the prediction error, although the sole increase of each one of them does not show a unified effect. Considering that the increases of  $M_\infty$  or  $T/T_w$  at given  $T/T_w$  or  $M_\infty$  both enlarge the value of  $Re_\tau^*$ , we conclude that the performance of the current TV relation is negatively affected by both free-stream Mach number and wall heat transfer.

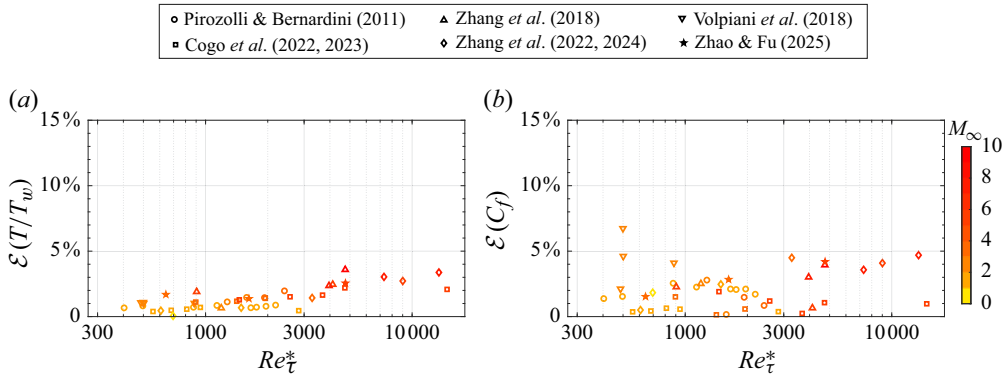


Figure 11. Prediction errors of (a) the TV relation (Duan & Martín 2011; Zhang *et al.* 2012) for mean temperature profile and (b) the general scaling law (Zhao & Fu 2025) for the skin-friction coefficient. The colours of the scattered symbols (yellow to red) denote the free-stream Mach number.

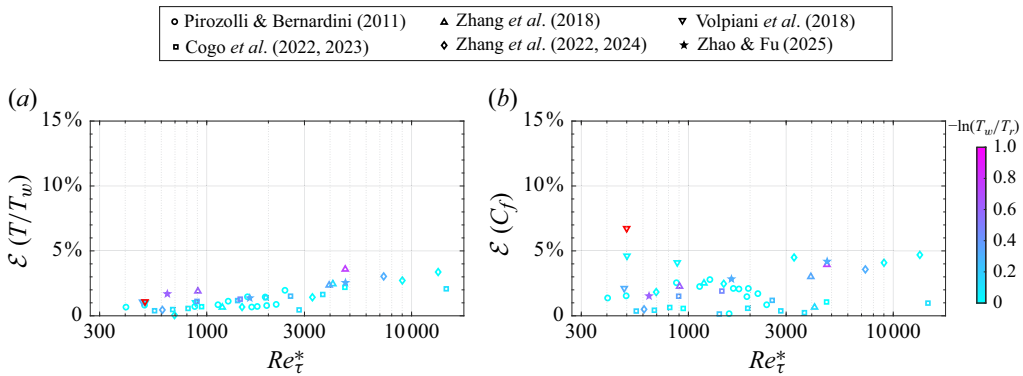


Figure 12. Prediction errors of (a) the TV relation (Duan & Martín 2011; Zhang *et al.* 2012) for mean temperature profile and (b) the general scaling law (Zhao & Fu 2025) for the skin-friction coefficient. The colours of the scattered symbols (cyan to magenta) denote  $-\ln(T_w/T_r)$ . The only symbol coloured with red denotes the hot-wall case from Volpiani *et al.* (2018) with  $M_\infty = 2.28$  and  $T_w/T_r = 1.9$ .

To analyse the performance of the general scaling law for  $C_f$  (Zhao & Fu 2025) as defined in (2.8), the prediction error  $\mathcal{E}(C_f)$  is defined by

$$\mathcal{E}(C_f) = \left| \left( \frac{2}{C_{f,i}} \right)^{1/2} - \left( \frac{1}{\kappa_f} \ln Re_{\theta,i} + C \right) \right| / \left( \frac{2}{C_{f,i}} \right)^{1/2}. \quad (\text{D2})$$

The values of  $\mathcal{E}(C_f)$  are shown in figures 11(b) and 12(b) as functions of  $Re_\tau^*$ . As with the mean temperature results, the data points are colour-coded according to  $M_\infty$  and  $T_w/T_r$ , respectively, to examine the effects of these parameters. In contrast to the TV relation, the scaling law for  $C_f$  does not display a significant dependence on either  $M_\infty$  or  $T_w/T_r$ . For example, cases from Volpiani *et al.* (2018) with  $M_\infty = 2.28$  exhibit prediction errors similar to those for  $M_\infty = 8.0$  from Zhang *et al.* (2024) – both approximately 5%. These results indicate that the performance of the established scaling law for  $C_f$  is not substantially affected by the free-stream Mach number or wall heat transfer, while the prediction results scatter within range approximately 5% relative to the DNS results.

According to the above discussions, the sources of prediction error are identified into two aspects. First, the increases of  $M_\infty$ ,  $T_w/T_r$  and  $Re_\tau^*$  enlarge the prediction errors of

the TV relation. Second, the prediction errors from the scaling law of  $C_f$  are not notably affected by  $M_\infty$  and  $T_w/T_r$ , but distribute from 0 to approximately 5 %. The improvement of the accuracy of the prediction framework relies on the further advancements of the TV relation and the scaling law of  $C_f$ , which are anticipated to be developed in future work.

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