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A NOTE ON THE CONJUGACY OF CARTAN SUBALGEBRAS

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Abstract

The conjugacy of Cartan subalgebras of a Lie algebra $\mathbf{L}$ over an algebraically closed field under the connected automorphism group G of $\mathbf{L}$ is inherited by those G-stable ideals $\mathbf{B}$ for which $\mathbf{B}/\mathbf{C}_i$ is restrictable for some hypercenter $\mathbf{C}_i$ of $\mathbf{B}$. Consequently, if $\mathbf{L}$ is a restrictable Lie algebra such that $\mathbf{L}/\mathbf{C}_i$ is restrictable for some hypercenter $\mathbf{C}_i$ of $\mathbf{L}$, and if the Lie algebra of Aut $\mathbf{L}$ contains ad $\mathbf{L}$, then the Cartan subalgebras of $\mathbf{L}$ are conjugate under G. (The techniques here apply in particular to Lie algebras of characteristic 0 and classical Lie algebras, showing how the conjugacy of Cartan subgroups of algebraic groups leads quickly in these cases to the conjugacy of Cartan subalgebras.)

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The conjugacy of Cartan subalgebras over algebraically closed fields of characteristic 0 does not carry over to characteristic $p>0$, since there are instances of a Lie algebra having Cartan subalgebras with different dimensions. However, we do have the following theorem for algebraic Lie algebras $\mathbf{G} = \operatorname{Lie} G$ over an algebraically closed field k of characteristic $p \geqslant 0$.

THEOREM 1 (Humphreys (1967)). *The Cartan subalgebras of the Lie algebra* $\mathbf{G} = $ *Lie* G *of a connected algebraic group* G *are all conjugate under* $\operatorname{Ad} G$.

The purpose of this note is to show that Theorem 1 together with Theorem 2 (below) can be used to establish the conjugacy of Cartan subalgebras of a Lie

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algebra in a fairly general context which, in particular, covers the cases of Lie algebras of characteristic 0 and classical Lie algebras of characteristic $p>3$.

Throughout the paper, the ground field is an algebraically closed field k of characteristic $p \geqslant 0$.

DEFINITION 1. A Lie algebra $\mathbf{L}$ over k is *restrictable* if either $p=0$ or $p>0$ and $(\mathrm{ad}_{\mathbf{L}}\mathbf{L})^p \subset \mathrm{ad}_{\mathbf{L}}\mathbf{L}$. An ideal $\mathbf{B}$ of $\mathbf{L}$ is a *restrictable ideal* of $\mathbf{L}$ if $\mathbf{B}$ is restrictable as a Lie algebra.

Restrictable Lie algebras are just those Lie algebras which can be given the structure of a restricted Lie algebra (Jacobson (1968)). The condition 'restrictable ideal' used here is much weaker than the condition 'restricted ideal' in restricted Lie algebras.

THEOREM 2 (Winter (1970)). *Let* $\mathbf{L}$ *be a restrictable Lie algebra with restrictable ideal* $\mathbf{B}$. *Then every Cartan subalgebra of* $\mathbf{B}$ *is the Fitting null space* $\mathbf{B}_0(\mathrm{ad}(\mathbf{H}\cap\mathbf{B}))$ *in* $\mathbf{B}$ *of* $\mathrm{ad}(\mathbf{H}\cap\mathbf{B})$ *for some Cartan subalgebra* $\mathbf{H}$ *of* $\mathbf{L}$.

DEFINITION 2. The ith *hypercenter* $\mathbf{C}_i$ of $\mathbf{L}$ is defined recursively by $\mathbf{C}_0=\{0\}$ and $\mathbf{C}_{i+1}=\{x\in\mathbf{L} \mid [x,\mathbf{L}]\subset\mathbf{C}_i\}$ for $i=1,2,\ldots$.

PROPOSITION 1. *Every Cartan subalgebra* $\mathbf{H}$ *of a Lie algebra* $\mathbf{L}$ *over* k *contains all of the hypercenters of* $\mathbf{L}$.

PROOF. We can assume that the center $\mathbf{C}_1$ of $\mathbf{L}$ is nonzero (otherwise the hypercenters are all 0). Since $\mathbf{H}$ contains $\mathbf{C}_1$ and $\mathbf{H}/\mathbf{C}_1$ is a Cartan subalgebra of $\mathbf{L}/\mathbf{C}_1$ (for example, see Theorem 3 below), $\mathbf{H}/\mathbf{C}_1$ contains the hypercenters of $\mathbf{L}/\mathbf{C}_1$, by induction, so that $\mathbf{H}$ contains the hypercenters of $\mathbf{L}$.

The following theorem, due to Barnes and, later, Block, is proved in Winter (1972), p. 127.

THEOREM 3. *Let* $\varphi\colon \mathbf{L}_1\to\mathbf{L}_2$ *be a surjective homomorphism of Lie algebras over* k. *Then for every Cartan subalgebra* $\mathbf{H}$ *of* $\mathbf{L}_1$, $\varphi(\mathbf{H})$ *is a Cartan subalgebra of* $\mathbf{L}_2$. *Moreover, every Cartan subalgebra of* $\mathbf{L}_2$ *is of the form* $\varphi(\mathbf{H})$ *for some Cartan subalgebra* $\mathbf{H}$ *of* $\mathbf{L}_1$.

THEOREM 4. *Let* $\mathbf{L}$ *be a restrictable Lie algebra,* $\mathbf{B}$ *an ideal of* $\mathbf{L}$ *such that* $\mathbf{B}/\mathbf{C}_i$ *is restrictable for some hypercenter* $\mathbf{C}_i$ *of* $\mathbf{B}$. *Then if* G *is a group of automorphisms of* $\mathbf{L}$ *stabilizing* $\mathbf{B}$ *and if any two Cartan subalgebras of* $\mathbf{L}$ *are conjugate under* G, *then any two Cartan subalgebras of* $\mathbf{B}$ *are conjugate under* G.

PROOF. Define $\mathbf{L}_j, \mathbf{B}_j$ recursively by $\mathbf{L}_0 = \mathbf{L}$, $\mathbf{B}_0 = \mathbf{B}$ and

$$\mathbf{L}_{j+1} = \mathrm{ad}_{\mathbf{B}_j}\mathbf{L}_j = \{\mathrm{ad}\, x|_{\mathbf{B}_j} \mid x \in \mathbf{L}_j\}, \quad \mathbf{B}_{j+1} = \mathrm{ad}_{\mathbf{B}_j}\mathbf{B}_j = \{\mathrm{ad}\, x|_{\mathbf{B}_j} \mid x \in \mathbf{B}_j\}.$$

Then $\mathbf{L}_i$ is restrictable. And, since the ideal $\mathbf{B}_i$ of $\mathbf{L}_i$ is isomorphic to $\mathbf{B}/\mathbf{C}_i$, $\mathbf{B}_i$ is also restrictable. Thus, any Cartan subalgebra of $\mathbf{B}_i$ has the form $\mathbf{B}_{i0}(\mathrm{ad}\,(\mathbf{H} \cap \mathbf{B}_i))$ for some Cartan subalgebra $\mathbf{H}$ of $\mathbf{L}_i$ (Theorem 2). Since any two Cartan subalgebras of $\mathbf{L}_i$ are conjugate under the induced action of G on $\mathbf{L}_i$ (as one easily sees using Theorem 3), it follows that any two Cartan subalgebras of $\mathbf{B}_i$ are conjugate under the induced action of G on $\mathbf{B}_i$. But then any two Cartan subalgebras of $\mathbf{B}$ are conjugate under G, since the Cartan subalgebras of $\mathbf{B}$ contain $\mathbf{C}_i$ (by Proposition 1) and $\mathbf{B}_i$ is isomorphic to $\mathbf{B}/\mathbf{C}_i$ (see Theorem 3).

COROLLARY 1. *Let $\mathbf{G} = \mathrm{Lie}\ G$ where G is a connected algebraic group. Let $\mathbf{B}$ be an Ad G-stable ideal of $\mathbf{G}$ such that $\mathbf{B}/\mathbf{C}_i$ is restrictable for some hypercenter $\mathbf{C}_i$ of of $\mathbf{B}$. Then the Cartan subalgebras of $\mathbf{B}$ are conjugate under $\mathrm{Ad}\, G$.*

PROOF. By Theorem 1, this follows immediately from Theorem 4.

COROLLARY 2. *Let $\mathbf{L}$ be a Lie algebra such that $\mathrm{ad}\mathbf{L}$ is contained in the Lie algebra of the connected automorphism group G of $\mathbf{L}$. Suppose that $\mathbf{L}/\mathbf{C}_i$ is restrictable for some hypercenter $\mathbf{C}_i$ of $\mathbf{L}$. Then the Cartan subalgebras of $\mathbf{L}$ are conjugate under G.*

PROOF. Apply Corollary 1 to $\mathbf{G} = \mathrm{Lie}\ G$ and $\mathrm{ad}\,\mathbf{L}$ to show that the Cartan subalgebras of $\mathrm{ad}\,\mathbf{L}$ are conjugate under $\mathrm{Ad}\, G$, whence the Cartan subalgebras of $\mathbf{L}$ are conjugate under G.

The conjugacy of the Cartan subalgebras of a Lie algebra of characteristic 0 follows immediately from Corollary 2.

To illustrate Theorem 4 and to show that it is more general than Theorem 1, we now give a simple proof of the conjugacy of Cartan subalgebras of classical Lie algebras based on Corollary 1.

THEOREM 5 (Seligman (1957)). *Let $\mathbf{L}$ be a classical Lie algebra of characteristic $p > 3$. Then the Cartan subalgebras of $\mathbf{L}$ are conjugate under the connected automorphism group of $\mathbf{L}$.*

PROOF. $\mathbf{L}$ has the form $\mathbf{L} = [\mathbf{G}, \mathbf{G}]/\mathbf{C}_1$ where $\mathbf{C}_1$ is the center of $[\mathbf{G}, \mathbf{G}]$ (for example see Humphreys (1967), p. 22). Since $\mathbf{L}$ is restrictable by Seligman (1967), p. 48, Corollary 1 applies to $\mathbf{B} = [\mathbf{G}, \mathbf{G}]$. Thus, the Cartan subalgebras of $[\mathbf{G}, \mathbf{G}]$ are conjugate under G, whence the Cartan subalgebras of $\mathbf{L} = [\mathbf{G}, \mathbf{G}]/\mathbf{C}_1$ are conjugate under the connected automorphism group of $\mathbf{L}$.

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