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SOME IMBEDDING THEOREMS FOR SOBOLEV SPACES

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1. Introduction. We shall be concerned throughout this paper with the Sobolev space $W^{m,p}(G)$ and the existence and compactness (or lack of it) of its imbeddings (i.e. continuous inclusions) into various L^p spaces over G, where G is an open, not necessarily bounded subset of n-dimensional Euclidean space E_n. For each positive integer m and each real $p \geqq 1$ the space $W^{m,p}(G)$ consists of all u in $L^p(G)$ whose distributional partial derivatives of all orders up to and including m are also in $L^p(G)$. With respect to the norm

$$(1.1) \qquad ||u||_{m,p,G} = \left\{ \sum_{0 \leq |\alpha| \leq m} \int_G |D^\alpha u(x)|^p \, dx \right\}^{1/p}$$

$W^{m,p}(G)$ is a Banach space. It has been shown by Meyers and Serrin [**9**] that the set of functions in $C^m(G)$ which, together with their partial derivatives of orders up to and including m, are in $L^p(G)$ forms a dense subspace of $W^{m,p}(G)$. Here, as usual, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers; $|\alpha| = \alpha_1 + \ldots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_j = \partial/\partial x_j$. Consistent with (1.1), $|| \cdot ||_{0,p,G}$ denotes the norm in $L^p(G)$.

The domain G is said to satisfy the cone condition if there exists a finite cone C (the intersection of an open ball in E_n centred at the origin, with a set of the form $\{\lambda x : x \in B, \lambda > 0\}$ where B is an open ball not containing the origin) with the property that each x belonging to the boundary ∂G of G is the vertex of a finite cone C_x contained in G and congruent to C.

For any G, bounded or unbounded, we have the natural imbedding

$$(1.2) \qquad W^{m,p}(G) \to L^p(G).$$

If G satisfies the cone condition there exist (the Sobolev imbedding theorem; e.g., see [**7**]) imbeddings of the form

$$(1.3) \qquad W^{m,p}(G) \to L^q(G)$$

for $p \leqq q \leqq np/(n - mp)$ if $n > mp$, or for $p \leqq q < \infty$ if $n \leqq mp$, and, if G also has finite volume, for $1 \leqq q < p$ as well. If G is bounded and satisfies the cone condition a well-known theorem of Rellich and Kondrachov [**8**; **10**] asserts that (1.3) is compact for all the above values of q excepting only $q = np/(n - mp)$ if $n > mp$.

The compactness of these imbeddings is a useful tool (especially in the case $p = 2$) for developing existence and spectral theory for partial differen-

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tial operators on G (especially elliptic operators with boundary conditions of Neumann, third, or mixed type). The question therefore naturally arises (e.g., see the final remarks in [**5**]) as to whether the Rellich-Kondrachov theorem possesses extensions to unbounded domains G. For the subspace $W_0^{m,p}(G)$, defined as the closure in $W^{m,p}(G)$ of the set of infinitely differentiable functions with compact support in G, this question has been rather thoroughly investigated (e.g., see [**1**; **2**; **3**; **6**]) and this study has resulted in the formulation in [**3**] of an analytic condition on (unbounded) G which is necessary and sufficient for the compactness of the imbedding

$$W_0^{m,p}(G) \to L^p(G). \tag{1.4}$$

It is clear that no imbedding of type (1.2), (1.3) or (1.4) for unbounded G can be compact if G contains infinitely many disjoint congruent balls, and, in particular, if it satisfies the cone condition. In [**6**], C. W. Clark notes that for the case $m = 1$, $p = 2$ and G contained in a cylinder of finite cross-section, that (1.2) cannot be compact if G has infinite volume. In sections 2 and 3 below we generalize this, showing, for any G, that (1.3) cannot be compact for any $q \geqq p$ unless G has finite volume and, in fact, unless the volume of G outside the ball of radius k with centre the origin tends to zero faster than any geometric sequence as k tends to infinity. We show also that if $q < p$ then $W^{m,p}(G) \subset L^q(G)$ if and only if G has finite volume. In this case the inclusion is, in fact, a compact imbedding.

Though necessary, the finiteness of the volume of G is not sufficient for the compactness of (1.3) for $q \geqq p$ (though it is sufficient for the compactness of (1.4); see [**3**]) as is shown by the following example.

Example 1. Let G be the union of infinitely many disjoint balls B_j of radius r_j. Define u_j on G by

$$u_j(x) = \begin{cases} 0 & \text{if } x \notin B_j \\ (\text{vol. } B_j)^{-1/q} & \text{if } x \in B_j. \end{cases}$$

For $q \geqq p$, $\{u_j\}$ is bounded in $W^{m,p}(G)$ provided $\{r_j\}$ is bounded. However $\{u_j\}$ is not precompact in $L^q(G)$ no matter how rapidly r_j tends to zero as j tends to infinity. Together with Theorem 2 below the method of this example can be used to show that (1.3) cannot be compact if G has infinitely many components.

In section 4 we establish (see Theorem 5) a sufficient condition for the compactness of (1.2) for suitably regular G. Theorem 5 is almost a converse of the necessary condition for compactness obtained in section 3. The method involves construction of nonstationary flows in G in terms of which the volume decay of G at infinity can be conveniently expressed. Theorem 5 generalizes the Rellich-Kondrachov theorem to many bounded domains (not satisfying the cone condition) as well as unbounded domains.

2. Finite volume. In this section we show that if there is a compact imbedding

$$W^{m,p}(G) \to L^q(G) \tag{2.1}$$

for some q, then G has finite volume. For $q < p$ we show that there is not even an inclusion (2.1) unless G has finite volume.

Consider a tesselation of E_n by n-cubes of side h. K will always denote a cube in the tesselation under discussion. $N(K)$, called the neighbourhood of K, will be the cube of side $3h$ concentric with K and having its faces parallel to those of K. $F(K)$, called the fringe of K, will be the shell $N(K) \sim K$. Let μ denote n-dimensional Lebesgue measure and let $\lambda > 0$.

Definition. K will be called λ-*fat* if

$$\mu(K \cap G) > \lambda\mu(F(K) \cap G).$$

If K is not λ-fat it will be called λ-*thin*.

THEOREM 1. *Suppose that there is a compact imbedding of the form* (2.1) *for some* $q \geqq p$. *Then for each* $\lambda > 0$ *every tesselation of* E_n *by cubes of fixed size contains only finitely many* λ-*fat cubes.*

Proof. Suppose for some $\lambda > 0$ that we have a tesselation of E_n by cubes of side h containing an infinite sequence $\{K_j\}^{\infty}{}_{j=1}$ of λ-fat cubes. Passing to a subsequence, if necessary, we can arrange that the neighbourhoods $N(K_j)$ are disjoint. Clearly for each K there is a function ϕ_K in $C_0^{\infty}(N(K))$ with $|\phi_K(x)| \leqq 1$ for all x, and

$$\phi_K(x) = 1, \qquad \text{for all } x \in K,$$
$$|D^{\alpha}\phi_K(x)| \leqq M, \quad \text{for all } x \in E_n,\ 0 \leqq |\alpha| \leqq m,$$

where M is a constant depending on n, m, and h, but not on K. Let $\psi_j = c_j \cdot \phi_{K_j}$, where the positive constant c_j is chosen so that

$$\int_G |\psi_j(x)|^q\,dx \geqq (c_j)^q \int_{K_j \cap G} |\phi_{K_j}(x)|^q\,dx$$
$$= (c_j)^q \mu(K_j \cap G) = 1.$$

But then

$$||\psi_j||^p{}_{m,p,G} = (c_j)^p \sum_{0<|\alpha|\leqq m} \int_{N(K_j) \cap G} |D^{\alpha}\phi_{K_j}(x)|^p\,dx$$
$$\leqq \text{const. } (c_j)^p \mu(N(K_j) \cap G).$$

Since K_j is λ-fat

$$\mu(N(K_j) \cap G) = \mu(K_j \cap G) + \mu(F(K_j) \cap G)$$
$$< \left(1 + \frac{1}{\lambda}\right) \mu(K_j \cap G) = \left(1 + \frac{1}{\lambda}\right) (c_j)^{-q}.$$

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Hence $||\psi_j||^p{}_{m,p,G} \leqq \text{const.}\ (c_j)^{p-q}$. Because $q \geqq p$ and $c_j \geqq h^{-n/q}$ for all j we have that $\{\psi_j\}$ is bounded in $W^{m,p}(G)$ and bounded away from zero in $L^q(G)$. Since the functions ψ_j have disjoint supports $\{\psi_j\}$ cannot be precompact in $L^q(G)$. This contradicts the assumption that (2.1) is compact. Hence there can be no such sequence of λ-fat cubes. This completes the proof.

Remark 1. This method also shows that if there is a continuous imbedding of the form (2.1) for some $q > p$ then for any $\lambda > 0$ and any tesselation of E_n by cubes of fixed side there is $\epsilon > 0$ such that $\mu(K \cap G) \geqq \epsilon$ for all λ-fat cubes K. For suppose to the contrary that there is a sequence $\{K_j\}^\infty{}_{j=1}$ of λ-fat cubes with $\mu(K_j \cap G) \to 0$ as j tends to infinity. If c_j is defined as in the above proof we have $c_j \to \infty$ and $||\psi_j||_{m,p,G} \to 0$ as $j \to \infty$. But $\{\psi_j\}$ is bounded away from zero in $L^q(G)$, contradicting the continuity of (2.1).

It follows that if there is such a continuous imbedding then either

(a) there is a tesselation of E_n by cubes of fixed side and an $\epsilon > 0$ so that $\mu(K \cap G) \geqq \epsilon$ for infinitely many cubes K in the tesselation, or

(b) for every $\lambda > 0$ and every tesselation of E_n by cubes of fixed side there are only finitely many λ-fat cubes.

In case (a) the volume of G is infinite; indeed $\mu\{x \in G : N \leqq |x| \leqq N + 1\}$ does not tend to zero as N tends to infinity. In case (b), as we shall see in theorems 2 and 4, G has finite volume and $\mu\{x \in G : N \leqq |x| \leqq N + 1\}$ tends more rapidly to zero as N tends to infinity than any geometric progression. Clearly many domains fall between these cases and for such domains there is no continuous imbedding of the form (2.1) for any $q > p$.

If an unbounded domain satisfies the cone condition then the Sobolev imbedding theorem provides imbeddings of the form (2.1) for some values of $q > p$. Such domains come under case (a). We have no examples of unbounded domains falling under case (b) for which there is an imbedding of the form (2.1) for some $q > p$, but our methods do not rule out this possibility.

THEOREM 2. *Suppose that there is a compact imbedding of the form* (2.1) *for some $q \geqq p$. Then G has finite volume.*

Proof. Tesselate E_n by cubes of side 1 and let $\lambda = [2(3^n - 1)]^{-1}$. Let P be the union of the finitely many λ-fat cubes in the tesselation. Clearly $\mu(P \cap G) < \infty$. Let K be a λ-thin cube. Let K_1 be a cube in $F(K)$ for which $\mu(K_1 \cap G)$ is maximal. Then

$$\begin{aligned}\mu(K \cap G) &\leqq \lambda\mu(F(K) \cap G)\\ &\leqq \lambda(3^n - 1)\,\mu(K_1 \cap G) = \tfrac{1}{2}\mu(K_1 \cap G)\end{aligned}$$

because $F(K)$ contains only $3^n - 1$ cubes. If K_1 is also λ-thin, select $K_2 \subset F(K_1)$ with $\mu(K_1 \cap G) \leqq \frac{1}{2}\mu(K_2 \cap G)$.

Suppose that an infinite chain $\{K, K_1, K_2, \ldots\}$ of λ-thin cubes can be constructed in the above manner. Then for each j

$$\mu(K \cap G) \leqq \frac{1}{2^j}\,\mu(K_j \cap G) \leqq \frac{1}{2^j}$$

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so that $\mu(K \cap G) = 0$. Let P_∞ denote the union of the cubes K for which such an infinite chain starting at K can be constructed. Then $\mu(P_\infty \cap G) = 0$.

Let P_j denote the union of the cubes K for which some such chain ends on the jth step (i.e., K_j is λ-fat). For each $K \subset P_j$ select a particular chain of length j starting at K and ending at some λ-fat K_j. For how many K in P_j can a particular λ-fat cube K' occur as the end K_j? Any such K must lie in the cube of side $2j + 1$ centred on K'. Hence there are at most $(2j + 1)^n$ such cubes. Therefore

$$\begin{aligned}\mu(P_j \cap G) &= \sum_{K \subset P_j} \mu(K \cap G) \\ &\leqq \frac{1}{2^j} \sum_{K \subset P_j} \mu(K_j \cap G) \\ &\leqq \frac{(2j+1)^n}{2^j} \sum_{K' \subset P} \mu(K' \cap G) \\ &= \frac{(2j+1)^n}{2^j} \mu(P \cap G).\end{aligned}$$

Hence $\sum^\infty{}_{j=1} \mu(P_j \cap G) < \infty$. Since $E_n = P \cup P_\infty \cup P_1 \cup P_2 \cup \ldots$ we have $\mu(G) < \infty$. This completes the proof.

THEOREM 3. *Suppose* $W^{m,p}(G) \subset L^q(G)$ *for some* $q < p$. *Then* G *has finite volume. In particular* G *has finite volume if there is a (continuous) imbedding of type* (2.1) *for some* $q < p$.

Proof. Again tesselate E_n by cubes of unit side and let $\lambda = [2(3^n - 1)]^{-1}$. Let P be the union of the λ-fat cubes in the tesselation. We claim that $\mu(P \cap G) < \infty$. If not, there is a sequence $\{K_i\}^\infty{}_{i=1}$ of λ-fat cubes with $\sum^\infty{}_{i=1} \mu(K_i \cap G) = \infty$. We want the neighbourhoods $N(K_i)$ to be disjoint. This can be arranged as follows. Let L be the lattice of centres of cubes in the tesselation. Break up L into 3^n disjoint sublattices $\{L_j\}^{3^n}{}_{j=1}$ with each L_j having period 3 in every coordinate direction. For each j let T_j be the set of cubes in the tesselation with centres in L_j. For some j we have clearly

$$\sum_{\lambda\text{-fat } K \in T_j} \mu(K \cap G) = \infty.$$

Let $\{K_i\}$ be an enumeration of the λ-fat cubes in T_j. Then $\{K_i\}$ has the desired properties.

Choose an integer i_1 so that

$$2 \leqq \sum_{i=1}^{i_1} \mu(K_i \cap G) < 4.$$

Recall the functions ϕ_K used in the proof of Theorem 1, and let

$$\psi_1 = 2^{-1/p} \sum_{i=1}^{i_1} \phi_{K_i}.$$

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Because the sets $N(K_i)$ are disjoint,

$$\int_G |\psi_1(x)|^p\,dx = \tfrac{1}{2}\sum_{i=1}^{i_1} \int_G |\phi_{K_i}(x)|^p\,dx.$$

Because K_i is λ-fat,

$$\int_G |\phi_{K_i}(x)|^p\,dx \leqq \mu(N(K_i) \cap G) < \left(1 + \frac{1}{\lambda}\right) \mu(K_i \cap G).$$

Thus

$$\int_G |\psi_1(x)|^p\,dx < \tfrac{1}{2}\left(1 + \frac{1}{\lambda}\right)\sum_{i=1}^{i_1} \mu(K_i \cap G) < 2\left(1 + \frac{1}{\lambda}\right).$$

Similarly, for $|\alpha| \leqq m$

$$\int_G |D^\alpha \psi_1(x)|^p\,dx < 2\left(1 + \frac{1}{\lambda}\right) M^p.$$

On the other hand

$$\int_G |\psi_1(x)|^q\,dx \geqq (\tfrac{1}{2})^{q/p}\sum_{i=1}^{i_1} \mu(K_i \cap G) \geqq 2^{1-q/p}.$$

Now choose i_2 so that

$$2^2 \leqq \sum_{i=i_1+1}^{i_2} \mu(K_i \cap G) < 2^3$$

and let

$$\psi_2 = (\tfrac{1}{2})^{2/p}(2^2)^{-1/p} \sum_{i=i_1+1}^{i_2} \phi_{K_i}.$$

As above, we have for $|\alpha| \leqq m$

$$\int_G |D^\alpha \psi_2(x)|^p\,dx < \frac{2M^p(1 + 1/\lambda)}{2^2},$$

and also

$$\int_G |\psi_2(x)|^q\,dx \geqq (\tfrac{1}{2})^{2q/p} 2^{2(1-q/p)}.$$

Proceeding in this fashion we obtain a sequence $\{\psi_j\}^\infty{}_{j=1}$ of C^∞ functions with disjoint supports such that for $|\alpha| \leqq m$

$$\int_G |D^\alpha \psi_j(x)|^p\,dx < \frac{2M^p(1 + 1/\lambda)}{j^2}$$

and

$$\int_G |\psi_j(x)|^q\,dx \geqq \left(\frac{1}{j}\right)^{2q/p} 2^{j(1-q/p)}.$$

Then $\psi = \sum^\infty{}_{j=1} \psi_j \in W^{m,p}(G)$ but $\psi \notin L^q(G)$.

This contradicts our assumption that $W^{m,p}(G) \subset L^q(G)$. Therefore $\mu(P \cap G) < \infty$, and, by the argument of Theorem 2, $\mu(G) < \infty$. This completes the proof.

Of course if $\mu(G) < \infty$ then for all $q < p$, there is a continuous imbedding of the form (2.1). Moreover the usual proof of the compactness theorem for the case of bounded G uses only the property $\mu(G) < \infty$. So for $q < p$ we have the circle of implications:

$$\mu(G) < \infty \Rightarrow (2.1) \text{ is compact} \Rightarrow (2.1) \text{ is continuous} \Rightarrow (2.1) \text{ exists} \Rightarrow \mu(G) < \infty.$$

3. Rapid decay. Suppose that there is a compact imbedding of the form (2.1) for some $q \geqq p$. By Theorem 2, G has finite volume. In this section we show that $\mu\{x \in G : |x| \geqq R\}$ tends very rapidly to zero as R tends to infinity.

First we extend the notions of neighbourhood and fringe introduced in the previous section. Fix a tesselation of E_n and let Q be a union of cubes K in the tesselation. Define

$$N(Q) = \bigcup_{K \subset Q} N(K),$$

$$F(Q) = N(Q) \sim Q.$$

Given $\delta > 0$, let $\lambda = \delta[3^n(1 + \delta)]^{-1}$. Suppose that all the cubes K in Q are λ-thin. As K runs through the cubes inside Q the sets $F(K)$ are contained in $N(Q)$ and cover $N(Q)$ at most 3^n times. Therefore

$$\begin{aligned}\mu(Q \cap G) &= \sum_{K \subset Q} \mu(K \cap G) \\ &\leqq \lambda \sum_{K \subset Q} \mu(F(K) \cap G) \\ &\leqq 3^n \lambda \mu(N(Q) \cap G) \\ &= 3^n \lambda[\mu(Q \cap G) + \mu(F(Q) \cap G)].\end{aligned}$$

Since $\mu(G) < \infty$ we can transpose and get

$$\mu(Q \cap G) \leqq \frac{3^n \lambda}{1 - 3^n \lambda} \mu(F(Q) \cap G) = \delta \mu(F(Q) \cap G).$$

For any set $S \subset E_n$ let Q be the union of the cubes K of our tesselation whose interiors intersect S and define $F(S) = F(Q)$. If S is at a positive distance from the finitely many λ-fat cubes, then all of the cubes in Q are λ-thin and

$$\mu(S \cap G) \leqq \mu(Q \cap G) \leqq \delta \mu(F(S) \cap G). \tag{3.1}$$

THEOREM 4. *Suppose that there is a compact imbedding of the form* (2.1) *for some* $q \geqq p$. *For each* $r \geqq 0$ *let* $G_r = \{x \in G : |x| > r\}$ *and let* S_r *be the surface* $\{x \in G : |x| = r\}$. *Let* A_r *denote the* $n - 1$ *dimensional surface area of* S_r. *Then*

(a) *given* $\epsilon, \delta > 0$ *there exists an* R *so that for* $r \geqq R$,

$$\mu(G_r) \leqq \delta \mu\{x \in G : r - \epsilon \leqq |x| \leqq r\},$$

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(b) *if* A_r *is positive and ultimately non-increasing as* r *tends to infinity, for each* $\epsilon > 0$, $A_{r+\epsilon}/A_r$ *tends to zero as* r *tends to infinity.*

Proof. Given $\epsilon > 0$ tesselate E_n by cubes of side $h = \epsilon/2\sqrt{n}$. Then any cube K whose interior intersects G_r is contained in $G_{r-\frac{1}{2}\epsilon}$ and

$$F(G_r) \subset \{x \in G : r - \epsilon \leqq |x| \leqq r\}.$$

Given $\delta > 0$, define λ as above and take R large enough that the finitely many λ-fat cubes are all contained in the ball of radius $R - \frac{1}{2}\epsilon$ centred at the origin. Then for $r \geqq R$ all the cubes K whose interiors intersect G_r are λ-thin, and (a) follows from (3.1).

For (b) choose R_0 so that A_r is non-increasing in $[R_0, \infty)$. Fix $\epsilon', \delta > 0$ and let $\epsilon = \frac{1}{2}\epsilon'$. Let R be as in (a). If $r \geqq \max\{R_0 + \epsilon', R\}$ then

$$\begin{aligned} A_{r+\epsilon'} &\leqq \frac{1}{\epsilon}\int_{r+\epsilon}^{r+2\epsilon} A_s\,ds \leqq \frac{\mu(G_{r+\epsilon})}{\epsilon} \\ &\leqq \frac{\delta}{\epsilon}\mu\{x \in G : r \leqq |x| \leqq r + \epsilon\} \\ &= \frac{\delta}{\epsilon}\int_r^{r+\epsilon} A_s\,ds \leqq \delta A_r. \end{aligned}$$

Since ϵ' and δ are arbitrary, (b) follows. This completes the proof.

COROLLARY. *If there is a compact imbedding of the form* (2.1) *for some* $q \geqq p$ *then for all* k

$$\lim_{r\to\infty} e^{kr}\mu(G_r) = 0.$$

Proof. From (a), we have that $\mu(G_{r+1}) \leqq \delta\mu(G_r)$ for given $\delta > 0$ and all sufficiently large r. Thus $\mu(G_r)$ tends to zero more rapidly than e^{-kr} for any k.

Remark 2. The argument used in the proof of (a) works for any norm ρ on E_n. For (b), we need in addition to have that A_r is well defined and that

$$\mu\{x \in G : r \leqq \rho(x) \leqq r + \epsilon\} = \int_r^{r+\epsilon} A_s\,ds.$$

This is true, for instance, when $\rho(x) = \max |x_i|$.

Remark 3. For the proof of (b), it is sufficient that A_r have an equivalent, positive, non-increasing majorant. That is, there should exist a positive, non-increasing function f and a constant $M > 0$ so that for all sufficiently large r, $A_r \leqq f(r) \leqq MA_r$. Indeed, if there is such a majorant then (a) and (b) are equivalent.

It is easier to determine whether a domain satisfies the conclusions of Theorem 4 than it is to determine whether it satisfies those of Theorem 1. We now show, however, that Theorem 1 is sharper than Theorem 4.

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Example 2. (A horn) Let $f \in C^1([0, \infty))$ be positive and non-increasing, with bounded derivative f'. Let G be the horn-shaped domain in E_3 given by $G = \{(r, \theta, z) : z > 0, r < f(z)\}$ where (r, θ, z) are cylindrical polar coordinates. Let ρ be the supremum norm on E_3, i.e.,

$$\rho(x, y, z) = \max(|x|, |y|, |z|).$$

Then for all sufficiently large s, $A_s = \pi[f(s)]^2$. Clearly G satisfies conclusion (b) of Theorem 4 if and only if

$$\lim_{s\to\infty} \frac{f(s + \epsilon)}{f(s)} = 0 \quad \text{for all } \epsilon > 0. \tag{3.2}$$

The monotonicity of f implies that conclusion (a) of Theorem 4 also holds if and only if (3.2) does.

We shall see later that, for domains of this type, the natural imbedding

$$W^{m,p}(G) \to L^p(G) \tag{3.3}$$

is compact. (In fact the techniques of [**4**] can be used to show this.)

Example 3. (A bihorn) Let f be as in example 2 above, satisfying (3.2) and also $f'(0) = 0$. Choose a positive, non-increasing function $g \in C^1([0, \infty))$ satisfying

(a) $g(0) = f(0)/\sqrt{2}$, $g'(0) = 0$,
(b) $g(s) \leqq f(s)$ for all $s \geqq 0$,
(c) g is constant on infinitely many disjoint intervals of unit length.

Let $h = \sqrt{(f^2 - g^2)}$. Consider the domain

$$G = \{(r, \theta, z) : r < g(z) \quad \text{if } z \geqq 0, \quad r < h(-z) \text{ if } z < 0\}.$$

Once again we have that $A_s = \pi[f(s)]^2$ for all sufficiently large s, and that G satisfies the conclusions of Theorem 4.

Tesselate E_n by cubes of side $\frac{1}{4}$ with faces parallel to the coordinate planes, one of the cubes being centred at the origin. There are infinitely many λ-fat cubes with centres on the positive z-axis, for $\lambda < \frac{1}{2}$. By Theorem 1 the natural imbedding (3.3) is not compact.

Theorem 4 fails to reveal this fact because its conclusions are global conditions and the compactness of (3.3) seems to depend on the local properties of G.

4. Flows. In this section we prove that the natural imbedding

$$W^{m,p}(G) \to L^p(G) \tag{4.1}$$

is compact for a class of domains which includes the horn of example 2 (when (3.2) is satisfied). We need a sequence $\{H_N\}^\infty{}_{N=1}$ of subdomains of G for which the imbedding is known to be compact, and we require the volume of G to decay very rapidly in each branch of G_N, the complement of H_N in G. Another way to state this second property is that the volume should increase rapidly as we flow towards the origin through G_N.

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Definition. By a *flow* on G we mean a C^1 map $\phi : U \to G$, where U is an open set in $G \times E_1$ containing $G \times \{0\}$, and where $\phi(x, 0) = x$ for all x in G.

For fixed x in G the curve $t \mapsto \phi(x, t)$ is called a streamline of the flow. For fixed t the map $\phi_t : x \mapsto \phi(x, t)$ sends a subset of G into G. We shall be concerned with the Jacobian $\det \phi_t'(x)$ of this map (where ϕ_t' denotes the Frechet derivative). Sometimes it is required of a flow that $\phi_{s+t} = \phi_s \circ \phi_t$, but we do not need this property and so do not assume it.

Example 4. Take G to be the horn of example 2 with (3.2) satisfied. Define

$$\phi(r, \theta, z, t) = \left(r \frac{f(z-t)}{f(z)}, \theta, z - t\right) \text{ for } t < z.$$

The flow is toward the plane $z = 0$ and the streamlines diverge as the domain widens. This indicates that ϕ_t is a local magnification for $t > 0$. Indeed

$$\det \phi_t'(r, \theta, z) = \left(\frac{f(z-t)}{f(z)}\right)^2 \to \infty \text{ as } z \to \infty.$$

In this case the magnification is just A_{z-t}/A_z because the speed of the flow in the z direction is constant. Thus the local properties of the flow reflect the global behaviour of the volume and cross-sectional area of G.

For $N = 1, 2, \ldots$ let $H_N = \{(r, \theta, z) \in G : 0 < z < N + 1\}$. The natural imbeddings $W^{m,p}(H_N) \to L^p(H_N)$ are known to be compact because these sets are bounded and satisfy the cone condition. This compactness together with the above properties of the flow are sufficient to force the compactness of (4.1) for our horn.

THEOREM 5. *Let G be an open set in E_n for which*

(a) *there is a sequence $\{H_N\}^\infty_{N=1}$ of open subsets of G such that for all N the imbedding $W^{1,p}(H_N) \to L^p(H_N)$ is compact;*

(b) *there is a flow $\phi : U \to G$ such that if $G_N = G \sim H_N$ then*

(i) *$G_N \times [0, 1] \in U$ for each N,*

(ii) *ϕ_t is one-to-one for all t,*

(iii) $\left|\frac{\partial}{\partial t} \phi(x, t)\right| \leqq M$ *for all (x, t) in U;*

(c) *the functions $d_N(t) = \sup_{x \in G_N} |\det \phi_t'(x)|^{-1}$ satisfy*

(i) *$d_N(1) \to 0$ as $N \to \infty$, and,*

(ii) $\int_0^1 d_N(t)\, dt \to 0$ *as $N \to \infty$.*

Then the imbedding (4.1) *is compact.*

Proof. Let $\psi \in C^1(G)$. We want to estimate $\int_{G_N} |\psi(x)| dx$. For each x in G_N we have

$$\psi(x) = \psi(\phi_1(x)) - \int_0^1 \frac{\partial}{\partial t} \psi(\phi_t(x))\, dt.$$

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Now

$$\int_{G_N} |\psi(\phi_1(x))|\,dx \leqq d_N(1) \int_{G_N} |\psi(\phi_1(x)|\,|\det \phi_1'(x)|\,dx$$
$$\leqq d_N(1) \int_{\phi_1(G_N)} |\psi(y)|\,dy$$
$$\leqq d_N(1) \int_G |\psi(y)|\,dy.$$

And

$$\int_{G_N} \left| \int_0^1 \frac{\partial}{\partial t} \psi(\phi_t(x))\,dt \right| dx$$
$$\leqq \int_{G_N} dx \int_0^1 |\nabla\psi(\phi_t(x))| \left| \frac{\partial}{\partial t} \phi_t(x) \right| dt$$
$$\leqq \int_0^1 dt \int_{G_N} |\nabla\psi(\phi_t(x))| M\,dx$$
$$\leqq M \int_0^1 d_N(t)\,dt \int_{G_N} |\nabla\psi(\phi_t(x))|\,|\det \phi_t'(x)|\,dx$$
$$\leqq M \left\{ \int_0^1 d_N(t)\,dt \right\} \left\{ \int_G |\nabla\psi(y)|\,dy \right\}.$$

Letting $\delta_N = \max(d_N(1), M\int_0^1 d_N(t)dt)$ we have

$$\int_{G_N} |\psi(x)|\,dx \leqq \delta_N \int_G \{|\psi(x)| + |\nabla\psi(x)|\}\,dx \leqq \delta_N ||\psi||_{1,1,G}$$

and $\delta_N \to 0$ as $N \to \infty$.

Now suppose that u is real-valued and belongs to $C^1(G) \cap W^{1,1}(G)$. The distributional partial derivatives of $|u|^p$ are

$$D_j(|u|^p) = p \cdot |u|^{p-1} \cdot (\text{sgn } u) \cdot D_j u.$$

By Hölder's inequality

$$\int_G |D_j(|u(x)|^p)|\,dx \leqq p||D_j u||_{0,p,G} ||u||^{p-1}{}_{0,p,G} \leqq p||u||^p{}_{1,p,G}.$$

Therefore $|u|^p \in W^{1,1}(G)$ and by the theorem of Meyers and Serrin [**9**] there is a sequence $\{\psi_j\}^\infty{}_{j=1}$ of functions in $C^1(G) \cap W^{1,1}(G)$ so that $\psi_j \to |u|^p$ in $W^{1,1}(G)$. Then

$$\int_{G_N} |u(x)|^p\,dx = \lim_{j\to\infty} \int_{G_N} \psi_j(x)\,dx$$
$$\leqq \limsup_{j\to\infty} \delta_N ||\psi_j||_{1,1,G}$$
$$= \delta_N || \, |u|^p ||_{1,1,G}$$
$$\leqq \text{const. } \delta_N ||u||^p{}_{1,p,G}.$$

For complex-valued u in $C^1(G) \cap W^{1,p}(G)$ we can apply the above argument to the real and imaginary parts of u to obtain the Poincaré inequality

$$\int_{G_N} |u(x)|^p \, dx \leqq \text{const. } \delta_N ||u||^p{}_{1,p,G}. \tag{4.2}$$

Since $C^1(G) \cap W^{1,p}(G)$ is dense in $W^{1,p}(G)$, inequality (4.2) holds for all u in $W^{1,p}(G)$.

Finally, let $\{u_j\}^\infty{}_{j=1}$ be bounded in $W^{1,p}(G)$. To show that $\{u_j\}$ is precompact in $L^p(G)$ it suffices, by a diagonalization argument, to prove that

(i) the sequence $\{u_j|_{H_N}\}^\infty{}_{j=1}$ is precompact in $L^p(H_N)$, for all N, and

(ii) for every $\epsilon > 0$ there exists N such that $||u_j||_{0,p,G_N} < \epsilon$ for all j.

But (i) is true by assumption (a) of the theorem, and (ii) follows from the Poincaré inequality and the fact that $\delta_N \to 0$ as $N \to \infty$. Thus the imbedding $W^{1,p}(G) \to L^p(G)$ is compact. For $m > 1$ the imbedding $W^{m,p}(G) \to W^{1,p}(G)$ is continuous and so the composition $W^{m,p}(G) \to L^p(G)$ is compact. This completes the proof.

Remark 4. We note that in example 4

$$d_N(t) = \sup_{s \geqq N+1} \left\{ \frac{f(s-t)}{f(s)} \right\}^{-2} \leqq 1 \text{ for all } t \geqq 0.$$

Also

$$\lim_{N \to \infty} d_N(t) = 0 \text{ if } t > 0.$$

By dominated convergence

$$\lim_{N \to \infty} \int_0^1 d_N(t) \, dt = 0.$$

The assumption that f' is bounded guarantees that the speed

$$\left| \frac{\partial}{\partial t} \phi(x, t) \right|$$

is bounded. Theother hypotheses of Theorem 5 are easily verified, so that the imbedding (4.1) is compact.

Remark 5. It is easy to imagine more general domains to which Theorem 5 applies, although it may be difficult to specify a suitable flow. There appears to be such a flow, for instance, in a connected domain with infinitely many horn-like branches, as long as the volume decays rapidly enough in each branch. Indeed, for unbounded domains in which the volume decays monotonically in each branch, Theorem 5 is essentially the converse to Theorem 4. That is, we can apply the argument of Theorem 4 separately to each branch and show that, if the natural imbedding (4.1) is compact, then for all $\epsilon > 0$

$$\lim_{r \to \infty} \frac{A_{r+\epsilon}}{A_r} = 0, \tag{4.3}$$

where A_r is the cross-sectional area of the branch. If the domain is sufficiently regular it will carry a flow, such that, as in example 4, the magnification in each branch will imitate the behaviour of A_{r-t}/A_r. Since A_r is monotonic and (4.3) holds, Theorem 5 applies and the natural imbedding is compact.

Remark 6. Theorem 5 can also be applied to bounded domains. Consider, for example, a domain like the horn of example 4 except that it is centred on a bounded spiral rather than the z-axis. Such a domain carries a flow much like the one in example 4, and by Theorem 5 the natural imbedding (4.1) is compact. The usual compactness theorem does not apply to this domain, however, because it does not satisfy the cone condition.

As another example consider a bounded domain which satisfies the cone condition except in neighbourhoods of one point where it has a cusp. Imagine a flow out of this cusp and let H_N consist of all points in the domain distant at least $1/N$ from the tip of the cusp. Again Theorem 5 can be used to show that the natural imbedding is compact.

Remark 7. The proof of Theorem 5 can be modified to work with weaker hypotheses than (b) and (c). There are domains which appear not to satisfy (b) and (c) but for which some modified argument works. For instance, there are horns for which the natural imbedding is compact although f fails to be monotonic or even to have an equivalent non-increasing majorant. Most of the changes are fairly obvious, however, and we omit the details.

Remark 8. Finally, are there compact imbeddings of the sort $W^{m,p}(G) \to L^q(G)$ for G unbounded and $q > p$? By Theorem 2 such a domain would have to have finite volume and could not satisfy the cone condition. It does not even appear to be known whether there are any such domains for which there is a continuous imbedding of the above sort with $q > p$. If, however, such a continuous imbedding exists, then a standard interpolation argument shows that the imbedding $W^{m,p}(G) \to L^r(G)$ is compact for all $r < q$.

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