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AN INVESTIGATION INTO THE EFFECTS OF HEAT TRANSFER ON THE MOTION OF A SPHERICAL BUBBLE

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Abstract

This paper investigates the effect of heat transfer on the motion of a spherical bubble in the vicinity of a rigid boundary. The effects of heat transfer between the bubble and the surrounding fluid, and the resulting loss of energy from the bubble, can be incorporated into the simple spherical bubble model with the addition of a single extra ordinary differential equation. The numerical results show that for a bubble close to an infinite rigid boundary there are significant differences in both the radius and Kelvin impulse of the bubble when the heat transfer effects are included.

1. Introduction

Over the past couple of decades there has been considerable interest in studying the motion of a gas or vapour bubble in a liquid. Typically these bubbles occur in applications such as cavitation and under-water explosions and can be responsible for considerable damage to nearby structures immersed in the liquid. The mathematical models for predicting the motion of a bubble fall into two categories. The first category consists of full numerical models, such as the boundary integral method, which completely determine the fluid motion and can predict phenomena such as the re-entrant jet which forms as the bubble collapses, but which are computationally expensive. The alternative is to use a simplified model where the bubble is assumed to remain spherical throughout its growth and collapse phases and then use techniques such as the Kelvin impulse to determine the direction of the bubble jet. However, in most previous work the relationship between the volume and pressure of the gas inside the bubble has ignored the effect of heat transfer to or from the bubble and this results

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in a simple polytropic-type model for computing the internal pressure of the bubble for a given volume [1–3]. However, the effects of the heat transfer can be incorporated into the model by using an additional differential equation for the internal temperature [8–10], although more sophisticated thermal models have been proposed [7]. In the case of a pulsating cavitation bubble, the damping of the pulsation is due to the loss of energy in the bubble. This study investigates the effect of energy loss due to heat transfer between the bubble and the surrounding water.

This paper describes a simple spherical bubble model which has been modified to include the effects of the heat exchange between the bubble gas and the surrounding water. The vapour or gas inside the bubble is assumed to follow the gas law. It is also true that the Biot number of a cavitation bubble is much smaller than unity. Note that the Biot number is the product of the heat transfer coefficient, h, and the bubble radius, R, divided by the thermal conductance of water, k, and is much smaller than unity, that is, $2hR/k \ll 1$. Under this condition the "lumped-mass" approximation can be incorporated into the model and hence the temperature distribution inside the bubble can be represented by the bulk temperature in the analysis (see [11]). Whilst this may not always be true for an explosion bubble, the method presented here still gives some worthwhile insight into the motion of the bubble for very little computational cost.

2. Mathematical model

Assume that the fluid is incompressible, inviscid and irrotational. Then the velocity at any point in the fluid can be expressed as the gradient of a scalar potential ϕ which in turn satisfies Laplace's equation

$$\nabla^2\phi = 0.$$

The basic simplifying assumptions of the bubble models employed here are that the bubble remains spherical throughout its lifetime, and that its centre of mass cannot move, allowing the velocity potential due to the bubble to be represented by a point-source located at the bubble's centre of mass. That is, the potential due to the bubble at any point x in the fluid is given by [6]

$$\phi(x) = \frac{m(t)}{|\boldsymbol{x} - \boldsymbol{x}_b|},$$

where $m(t)$ is the time-dependent source-strength of the point source and $\boldsymbol{x}_b$ is the position vector of the centre of mass of the bubble. Equating the rate of change of the bubble radius to the normal velocity of the fluid at the bubble surface leads to

$$m(t) = -R^2\dot{R},$$

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where R is the bubble radius and an overdot denotes differentiation with respect to time.

For a bubble close to an infinite rigid boundary the method of images can be used to modify the velocity potential to include the contribution from the image of the bubble in the rigid boundary. Further, if the velocity potential on the surface of the bubble due to the image is approximated by the velocity potential due to the image at the centroid of the bubble, then the total velocity potential can be expressed in the form

$$\phi(\boldsymbol{x}) = -R^2\dot{R}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_b|}+\mu\right), \quad \boldsymbol{x} \in S_b,$$

where $\mu = 1/|\boldsymbol{x}_b - \boldsymbol{x}_b^*|$ and $\boldsymbol{x}_b^*$ is the image of the point $\boldsymbol{x}_b$ in the rigid boundary. The kinetic energy of the fluid is given by [6]

$$K = \iint_S \phi \frac{\partial \phi}{\partial \boldsymbol{n}}\, dS,$$

where $\boldsymbol{n}$ is the unit normal to S_b directed into the liquid. The rate of change of the kinetic energy must be equal to the rate of work being done. This leads to the ordinary differential equation

$$(R+\mu R^2)\ddot{R} + (3/2 + 2\mu R)\dot{R}^2 = \frac{p_b(t) - p_\infty}{\rho} + g z_b, \tag{2.1}$$

where p_∞ is the far-field pressure in the $z = 0$ plane, $p_b(t)$ is the pressure of the gas inside the bubble, ρ is the density of the liquid, z_b is the z-coordinate of the bubble's centre and g is the acceleration due to gravity which is assumed to be directed along the negative z-axis.

The bubble is assumed to be filled with an ideal, non-condensable gas which satisfies a gas law of the form

$$\frac{p_b(t)\, V(t)}{T(t)} = k,$$

where $V(t)$ is the volume of the bubble, $T(t)$ is the temperature of the gas inside the bubble and k is a constant. Hence the pressure inside the bubble at any time is given by

$$p_b(t) = p_0\left(\frac{V_0}{V(t)}\right)\left(\frac{T(t)}{T_0}\right), \tag{2.2}$$

where p_0, V_0 and T_0 denote the initial internal pressure, volume and temperature of the bubble respectively.

Let $\dot{Q}$ denote the rate of heat energy being transferred into the bubble. At any instant, this must equal the sum of the work done on the bubble and the change in the internal energy of the bubble. Thus

$$\dot{Q} = p_b(t)\dot{V}(t) + m_g\dot{E}, \tag{2.3}$$

where E denotes the internal energy of the bubble and m_g denotes the mass of the compressible gas inside the bubble. As the gas is assumed to be non-condensable the rate of change of the internal energy is proportional to the rate of change of the temperature of the bubble. Hence

$$\dot{E} = C_v\dot{T}, \tag{2.4}$$

where C_v is the specific heat of the gas for constant volume. Further, the rate of heat energy transferred into the bubble is proportional to the product of the surface area of the bubble and the temperature difference between the bubble and the surrounding liquid (here assumed to be water), yielding

$$\dot{Q} = hA(t)(T_w - T(t)), \tag{2.5}$$

where h is the heat transfer constant, T_w is the temperature of the surrounding liquid which is assumed to remain constant and $A(t)$ is the surface area of the bubble. Substituting (2.4) and (2.5) into (2.3) gives

$$hA(t)(T_w - T(t)) = p_b(t)\dot{V} + m_gC_v\dot{T}, \tag{2.6}$$

which is a differential equation for the temperature of the gas inside the bubble.

Substituting (2.2) into (2.1) and (2.6), and since the bubble is assumed to be spherical, it is possible to express its surface area and volume in terms of its radius to yield

$$(R + \mu R^2)\ddot{R} + (3/2 + 2\mu R)\dot{R}^2 = \frac{1}{\rho}\left(p_0\left(\frac{R_0}{R}\right)^3\left(\frac{T}{T_0}\right) - p_\infty + \rho g z\right), \tag{2.7}$$

$$m_gC_v\dot{T} = 4\pi R^2 h(T_w - T) - 4\pi p_0\left(\frac{R_0}{R}\right)^3\left(\frac{T}{T_0}\right)R^2\dot{R}, \tag{2.8}$$

which is a system of coupled differential equations which, given suitable initial conditions, can be solved for the bubble radius R and the temperature T.

It is possible to consider an adiabatic gas by taking $h = 0$ in (2.8). If the process is also assumed to be isentropic, solving (2.8) leads to the relationship

$$\frac{T}{T_0} = \left(\frac{R_0}{R}\right)^{4\pi R^3 p_0/m_g C_v T_0} = \left(\frac{R_0}{R}\right)^{3(\gamma-1)},$$

where γ is the ratio of specific heats. This will allow a comparison with earlier work where the thermodynamic processes inside the bubble were assumed to be adiabatic.

The Kelvin impulse of the bubble is given by

$$\boldsymbol{I}(t) = \int_{S_b} \phi \boldsymbol{n}\, dS. \tag{2.9}$$

Best and Blake [1] show that this can be written as

$$\boldsymbol{I}(t) = \boldsymbol{I}(0) + \int_0^t \boldsymbol{F}(\tau)\, d\tau, \tag{2.10}$$

where $\boldsymbol{F}(t)$ is the force applied to the bubble and it is usual to take $\boldsymbol{I}(0) = \boldsymbol{0}$. Further, it can be shown that [1]

$$\boldsymbol{F}(t) = -\frac{\pi \rho R^4 \dot{R}^2}{r^2} \boldsymbol{n}_p + \frac{4}{3} \pi \rho g R^3 \boldsymbol{e}_z, \tag{2.11}$$

where $\boldsymbol{e}_z$ is the unit vector directed along the positive z-axis, $\boldsymbol{n}_p$ is the unit vector perpendicular to the rigid plane, directed into the fluid region and r is the perpendicular distance of the centre of the bubble from the rigid plane. Thus it is possible to calculate the Kelvin impulse of the bubble at the end of the first collapse using (2.10) and (2.11).

In general the above equations cannot be solved analytically and have to be solved numerically. By expressing (2.7) in terms of two first-order differential equations it is possible to obtain the following system:

$$\begin{aligned}
\dot{R} &= U, \\
\dot{U} &= \frac{1}{R + \mu R^2} \left(\frac{p_0}{\rho} \left(\frac{R_0}{R} \right)^3 \frac{T}{T_0} - \frac{p_\infty}{\rho} + g z_b - (3/2 + 2\mu R) U^2 \right), \\
\dot{T} &= \frac{4\pi R^2}{m_g C_v} \left[h(T_w - T) - p_0 \left(\frac{R_0}{R} \right)^3 \frac{T}{T_0} U \right],
\end{aligned} \tag{2.12}$$

which can be integrated through time using a fourth-order Runge Kutta scheme.

The initial conditions for the system depend on the type of problem under consideration. For the bubbles considered here the initial conditions are that the initial radius of the bubble is small, the initial rate of change of the radius is zero and there is a large internal pressure, relative to the surrounding liquid, which causes the bubble to expand. The initial temperature is usually taken to be higher than that of the surrounding liquid.

Once the bubble's radius and surface velocity have been determined, the Kelvin impulse can be computed by evaluating the integral appearing in (2.10) with a quadrature rule, such as the trapezium rule, which uses the points in time at which the differential equations (2.12) have been solved as its nodal points.

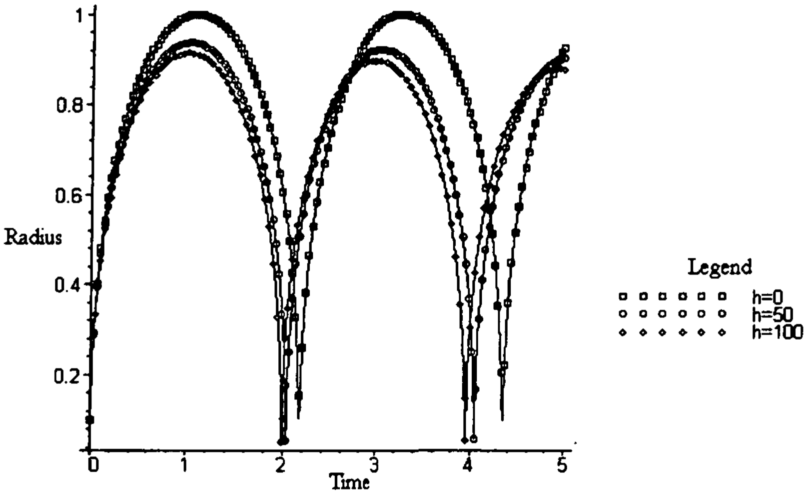

FIGURE 1. The radius of a bubble 1.5 units from a horizontal rigid boundary against time for different heat transfer constants h.

3. Numerical results

The above equations can be solved numerically in non-dimensional form. The length scale is the maximum bubble radius for an isolated adiabatic bubble in the plane $z = 0$, whilst pressure is scaled with respect to the hydrostatic pressure in the plane $z = 0$ and temperatures are scaled to the temperature of the water. The densities are scaled with respect to the density of the surrounding liquid. This leads to the following implicit scaling: time: $(1/R_m)\sqrt{p_\infty/\rho}$, specific heat: $p_\infty/(T_w\rho)$, mass of gas in bubble: $1/(R_m^3\rho)$, where R_m denotes the maximum bubble radius. All calculations in this paper are carried out in terms of these non-dimensional variables. Here the initial conditions are that the initial radius of the bubble is 0.1, the internal pressure is 303.76 and the initial internal temperature is 5.

The first problem considered is that of a bubble close to a horizontal rigid boundary in the plane $z = 0$. Since the acceleration due to gravity is acting perpendicular to this plane this is an axisymmetric problem which could be efficiently solved using a suitable boundary integral method, although that has not been done here.

Figure 1 shows the results of computing the bubble radius for a bubble 1.5 units from the rigid boundary with $h = 0$, $h = 50$ and $h = 100$. Clearly this shows that increasing the value of h (and thereby increasing the rate of heat exchange) has an effect on both the bubble period and the bubble radius.

These results show that as the heat transfer constant increases the bubble period becomes shorter and the maximum radius of the bubble decreases.

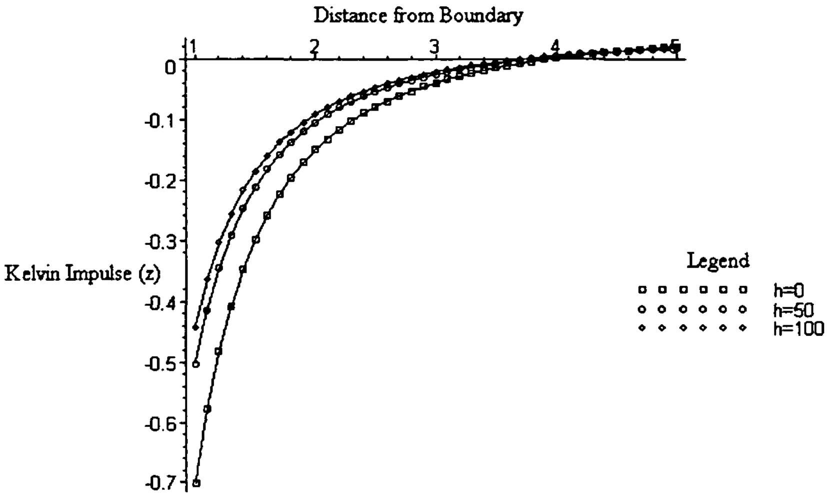

FIGURE 2. The vertical component of the Kelvin impulse against the distance from a horizontal rigid boundary for different heat transfer constants h.

Figure 2 shows the effect that changing the heat exchange parameter has on the vertical component of the Kelvin impulse of each of the bubbles shown in Figure 1. We note that in this case only the vertical component of the Kelvin impulse is non-zero.

Although there is little change in the distance from the boundary at which the Kelvin impulse changes sign, there is quite a dramatic effect on the magnitude of the Kelvin impulse for bubbles relatively close to the rigid boundary. These results indicate that if the heat transfer effects are included then the magnitude of the Kelvin impulse is reduced which suggests that the magnitude of a bubble's jet is reduced and a bubble's general migration towards the boundary will be smaller.

The second problem considered here involves the motion of a bubble close to a vertical rigid boundary in the plane $y = 0$. This problem is not axisymmetric and would require a fully three-dimensional boundary integral method for its solution which would be computationally very expensive. In this case $\mu = 1/(2y_b)$, where y_b is the coordinate of the bubble's centre.

Figure 3 shows the bubble radius for a bubble 1.5 units from a vertical rigid boundary for different values of the heat transfer constant. Similar to the bubble close to a horizontal rigid boundary we can see that the bubble's period gets shorter as the heat transfer constant is increased, and that the maximum radius (and hence volume) of the bubble also decreases as the heat transfer constant increases.

Figures 4 and 5 show the horizontal and vertical components of the Kelvin impulse respectively for each of the bubbles shown in Figure 3. Clearly the horizontal component is tending towards zero, as expected, as the bubble gets further away from the

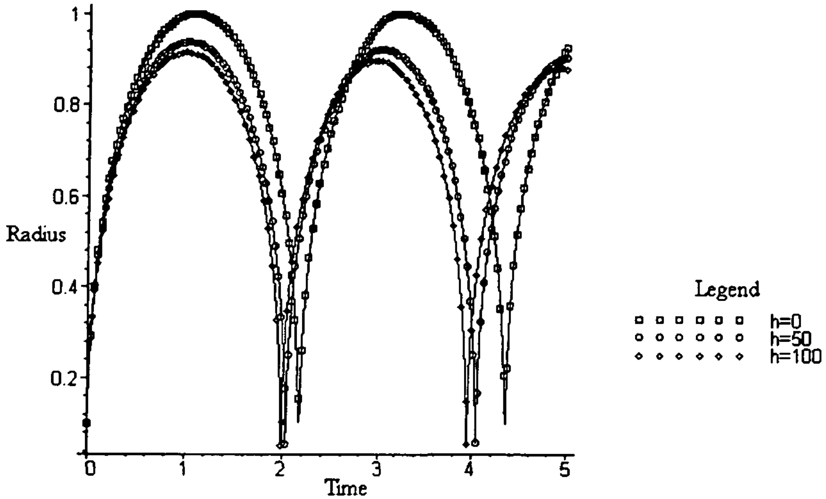

FIGURE 3. The radius of a bubble 1.5 units from a vertical rigid boundary against time for different heat exchange constants h.

boundary and the influence of the boundary is diminished. The vertical component is clearly tending towards the values that are obtained for an isolated bubble away from any boundaries. Again, in each case the effect of including the heat transfer effects is to diminish the components of the Kelvin impulse.

Figures 6 and 7 show the magnitude and direction, respectively, of the Kelvin impulse for different distances from the vertical rigid boundary. In all cases it can be seen that the Kelvin impulse rapidly decreases as the bubble moves away from the boundary. Further, the magnitude of the Kelvin impulse gets smaller as the heat transfer constant increases, which is what we would expect in light of the results presented in Figures 4 and 5. Figure 7 shows that as the bubble moves away from the boundary, the direction of the Kelvin impulse changes from near horizontal and towards the boundary (corresponding to an angle of 90°) to almost vertically upwards (corresponding to an angle of 0°). The size of the heat transfer has a much smaller effect on the direction of the Kelvin impulse, particularly when the bubble is close to the boundary.

4. Conclusions

This paper has shown that the heat transfer between a bubble and the surrounding water can have a significant effect on the motion of the bubble. This can clearly be seen in the differences in the bubble radius between the adiabatic case ($h = 0$) and the heat transfer case ($h > 0$). These changes in the motion of the bubble affect the

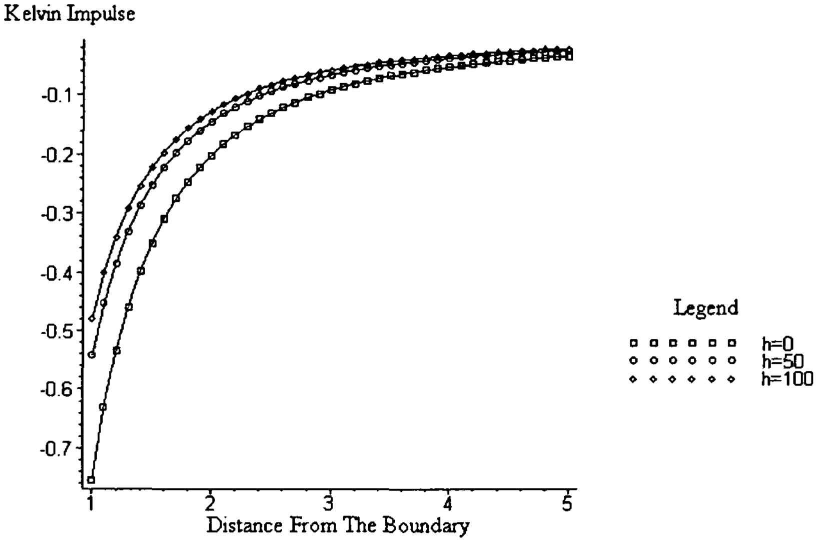

FIGURE 4. The horizontal component of the Kelvin impulse against the distance from a vertical rigid plane for different heat transfer constants h.

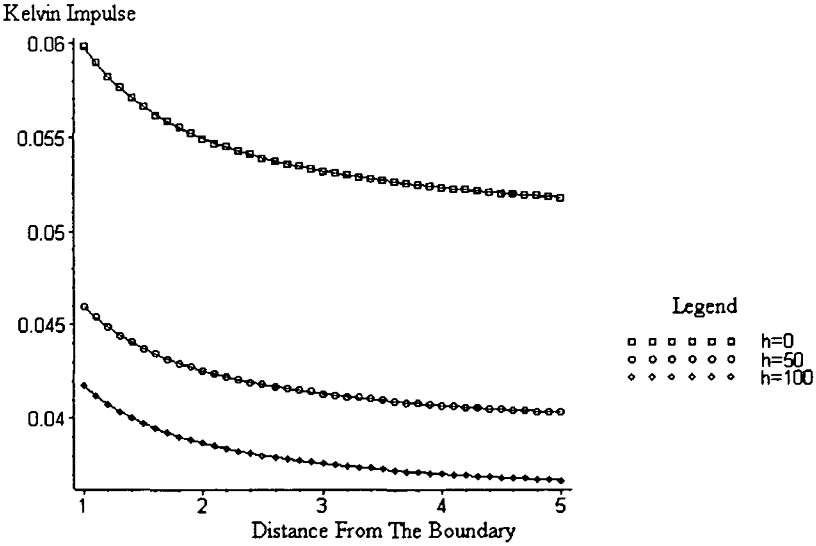

FIGURE 5. The vertical component of the Kelvin impulse against the distance from a vertical rigid plane for different heat transfer constants h.

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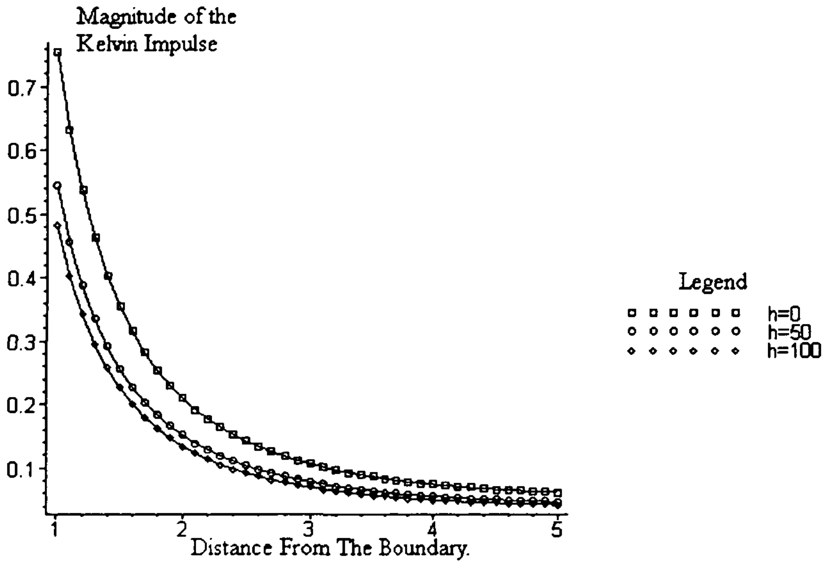

FIGURE 6. The magnitude of the Kelvin impulse against the distance from a vertical rigid plane for different heat transfer constants h.

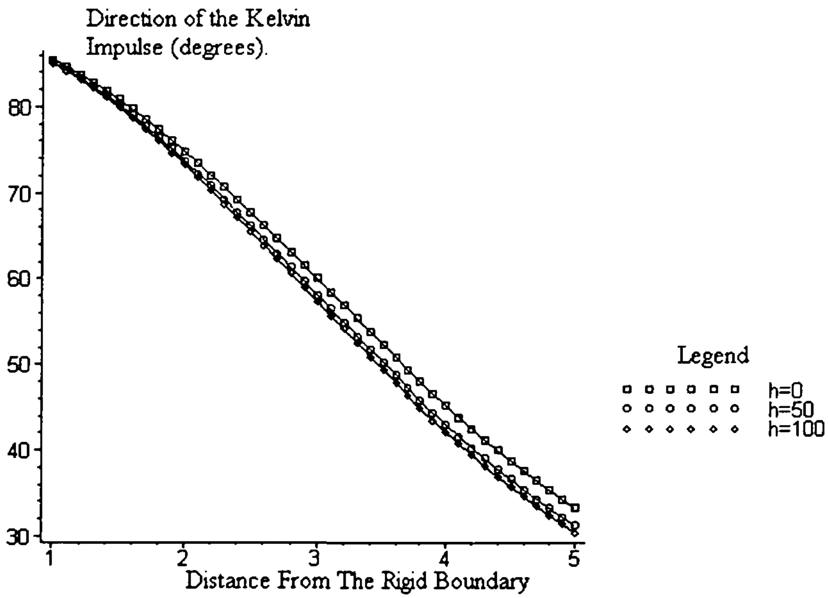

FIGURE 7. The direction of the Kelvin impulse against the distance from a vertical rigid plane for different heat transfer constants h.

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way in which the bubble interacts with other structures in the fluid domain. Further, this work seems to indicate that the heat transfer effects are a major influence on the magnitude of the Kelvin impulse for bubbles which are close to the boundary. The results seems to indicate that the adiabatic case will over-estimate the magnitude of the Kelvin impulse and this implies that the jet predicted by more sophisticated methods, such as the boundary integral method [3,4], will be larger and faster than is the case when heat transfer is included. However, the results seem to indicate that for bubbles close to vertical boundaries the changes in the direction of the Kelvin impulse when heat transfer effects are included are relatively small.

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